4. Spline approximation

4.1. **Cubic spline interpolation.** We consider the problem of finding a C^2 piecewise cubic function $S(x)$ that satisfies $S(x_i) = f(x_i)$, $i = 0, \dots, n$ plus two additional conditions. These are usually taken to be either $S''(x_0) = f''(x_0)$ and $S''(x_n) = f''(x_n)$ or $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$. We will consider the first set of conditions.

We will obtain $S(x)$ by first obtaining $S''(x)$ and then integrating. Since $S''(x)$ is a continuous piecewise linear function, it is uniquely determined by the values $S''(x_i)$, $i = 0, \dots, n$. On the subinterval $[x_{i-1}, x_i]$, we can write it in the form

$$S''(x) = S_i''(x) = \frac{x_i - x}{h_i} S''(x_{i-1}) + \frac{x - x_{i-1}}{h_i} S''(x_i),$$

where $h_i = x_i - x_{i-1}$. Integrate twice on each subinterval to get

$$S_i(x) = \frac{h_i^2}{6}\left[\frac{x_i - x}{h_i}\right]^3 S''(x_{i-1}) + \frac{h_i^2}{6}\left[\frac{x - x_{i-1}}{h_i}\right]^3 S''(x_i) + A_i \frac{x_i - x}{h_i} + B_i \frac{x - x_{i-1}}{h_i}.$$

Note: Integrating twice introduces an arbitrary linear function that we represent as above.

Now $\quad f(x_{i-1}) = S_i(x_{i-1}) = \frac{h_i^2}{6} S''(x_{i-1}) + A_i, \qquad f(x_i) = S_i(x_i) = \frac{h_i^2}{6} S''(x_i) + B_i.$

Hence,

$$\begin{aligned} S_i(x) = {} & \frac{h_i^2}{6}\left[\frac{x_i - x}{h_i}\right]^3 S''(x_{i-1}) + \frac{h_i^2}{6}\left[\frac{x - x_{i-1}}{h_i}\right]^3 S''(x_i) \\ & + [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})]\frac{x_i - x}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)]\frac{x - x_{i-1}}{h_i}. \end{aligned}$$

We next determine the values of $S''(x_i)$ by the conditions that S' is continuous at each x_i, i.e., $S_i'(x_i) = S_{i+1}'(x_i)$, $i = 1, \dots, n-1$. Now

$$\begin{aligned} S_i'(x) = {} & -\frac{h_i}{2}\left[\frac{x_i - x}{h_i}\right]^2 S''(x_{i-1}) + \frac{h_i}{2}\left[\frac{x - x_{i-1}}{h_i}\right]^2 S''(x_i) \\ & - [f(x_{i-1}) - \frac{h_i^2}{6} S''(x_{i-1})]\frac{1}{h_i} + [f(x_i) - \frac{h_i^2}{6} S''(x_i)]\frac{1}{h_i}. \end{aligned}$$

Then

$$\begin{aligned} S_i'(x_i) &= \frac{h_i}{3} S''(x_i) + \frac{h_i}{6} S''(x_{i-1}) + \frac{1}{h_i}[f(x_i) - f(x_{i-1})], \\ S_{i+1}'(x_i) &= -\frac{h_{i+1}}{3} S''(x_i) - \frac{h_{i+1}}{6} S''(x_{i+1}) + \frac{1}{h_{i+1}}[f(x_{i+1}) - f(x_i)]. \end{aligned}$$

Equating these quantities to insure continuity, we get:

$$\frac{h_i}{6} S''(x_{i-1}) + \frac{h_i + h_{i+1}}{3} S''(x_i) + \frac{h_{i+1}}{6} S''(x_{i+1}) = f[x_i, x_{i+1}] - f[x_{i-1}, x_i].$$

Thus, the $n-1$ quantities $S''(x_1), \dots, S''(x_{n-1})$ are determined by solving the linear system

$$\begin{pmatrix} (h_1+h_2)/3 & h_2/6 & \cdots & \cdots \\ h_2/6 & (h_2+h_3)/3 & h_3/6 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & h_{n-1}/6 & (h_{n-1}+h_n)/3 \end{pmatrix} \begin{pmatrix} S''(x_1) \\ S''(x_2) \\ \cdots \\ S''(x_{n-1}) \end{pmatrix}$$
$$= \begin{pmatrix} f[x_1,x_2] - f[x_0,x_1] - h_1 f''(x_0)/6 \\ f[x_2,x_3] - f[x_1,x_2] \\ \cdots \\ f[x_{n-1},x_n] - f[x_{n-2},x_{n-1}] - h_n f''(x_n)/6 \end{pmatrix}$$

To check that the matrix is nonsingular, we can use the following result.

Theorem 2. *Let $A = (a_{ij})$ be an $N \times N$ matrix. If $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$, $i = 1, 2, \dots, N$, then A is nonsingular.*

Proof. Suppose A is singular. Then there is a vector $x \neq 0$ such that $\sum_{j=1}^{N} a_{ij} x_j = 0$ for $i = 1, \dots, N$. Let k satisfy $|x_k| \geq |x_j|$, $j \neq k$. Note $|x_k| \neq 0$. Since $a_{kk} x_k = -\sum_{j \neq k} a_{kj} x_j$, $|a_{kk}||x_k| \leq \sum_{j \neq k} |a_{kj}||x_j|$. Hence

$$|a_{kk}| \leq \sum_{j \neq k} |a_{kj}|(|x_j|/|x_k|) \leq \sum_{j \neq k} |a_{kj}|.$$

Contradiction. □

4.2. **Spline basis functions.** Any spline function of degree n can be expressed as a linear combination of B-splines (basis functions).

Given a set of knots $t_0, t_1, \dots t_m$, we show how to define a B-spline basis $B_{i,n}$, where n denotes the degree of the spline, and i denotes the associated knot.

The definition is by the following recursion formula (Cox-de Boor)

$$B_{i,0} := \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$
$$B_{i,k} := \frac{x - t_i}{t_{i+k} - t_i} B_{i,k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(x).$$

So when $k = 1$, we have

$$B_{i,1} = \frac{x - t_i}{t_{i+1} - t_i} B_{i,0}(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B_{i+1,0}(x)$$

If we insert the definitions of $B_{i,0}$ and $B_{i+1,0}$, we see that

$$B_{i,1} = \begin{cases} (x - t_i)/(t_{i+1} - t_i) & \text{if } t_i \leq x < t_{i+1} \\ (t_{i+2} - x)/(t_{i+2} - t_{i+1}) & \text{if } t_{i+1} \leq x < t_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$$

Note, $B_{i,1}$ is nonzero on only two subintervals.

To compute $B_{i,2}$, we need $B_{i,1}$ (which is non-zero only on the interval $[t_i, t_{i+2}]$) and $B_{i+1,1}$ (which is non-zero only on the interval $[t_{i+1}, t_{i+3}]$). Hence $B_{i,2}$ will be non-zero only on three subintervals. Continuing in this way, $B_{i,3}$ will be non-zero only on four subintervals.

To see how the recursion goes, we compute $B_{i,2}$.

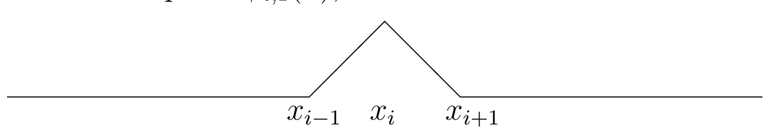
$$\begin{aligned}
B_{i,2}(x) &= \frac{x-t_i}{t_{i+2}-t_i}B_{i,1}(x) + \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}B_{i+1,1}(x) \\
&= \frac{x-t_i}{t_{i+2}-t_i}\Big[\frac{x-t_i}{t_{i+1}-t_i}B_{i,0}(x) + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}B_{i+1,0}(x)\Big] \\
&\quad + \frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}\Big[\frac{x-t_{i+1}}{t_{i+2}-t_{i+1}}B_{i+1,0}(x) + \frac{t_{i+3}-x}{t_{i+3}-t_{i+2}}B_{i+2,0}(x)\Big] \\
&= \frac{(x-t_i)^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)}B_{i,0} + \Big[\frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} + \frac{(x-t_{i+1})(t_{i+3}-x)}{(t_{i+2}-t_{i+1})(t_{i+3}-t_{i+1})}\Big]B_{i+1,0} \\
&\quad + \frac{(t_{i+3}-x)^2}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})}B_{i+2,0}.
\end{aligned}$$

Note that by the definition of $B_{j,0}$, the polynomial coefficient of $B_{j,0}$ gives the definition of $B_{i,2}(x)$ on the subinterval $t_j \le x < t_{j+1}$.

In this notation, we have associated the basis function $B_{i,k}$ to the knot t_i. For k odd, we often prefer to associate it to the middle knot. For example, given a mesh $a = x_0 < x_1 < \ldots < x_N = b$ we would define the basis function $\phi_{i,1}$ associated to the mesh point x_i by

$$\phi_{i,1} = \begin{cases} (x-x_{i-1})/(x_i - x_{i-1}) & \text{if } x_{i-1} \le x < x_i \\ (x_{i+1}-x)/(x_{i+1}-x_i) & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Graph of $\phi_{i,1}(x)$, called the *hat* function



When the mesh points are equally spaced, we can simplify the process. For example, to get a basis for the space of cubic splines defined on the partition $a = x_0 < \ldots < x_n = b$, where $x_{i+1} - x_i = h$, we first compute the cubic spline basis function B_3 on the knots $\{-2, -1, 0, 1, 2\}$ and then define for $i = -1, \ldots, n+1$, the B-spline $\phi_{i,3}(x) = B_3([x - x_i]/h)$, where we restrict these functions to the interval $[a, b]$. This gives the required $n + 3$ basis functions. Note that $\phi_{i.3}(x) \ge 0$ and one can show that $\sum \phi_{i,3}(x) = 1$,

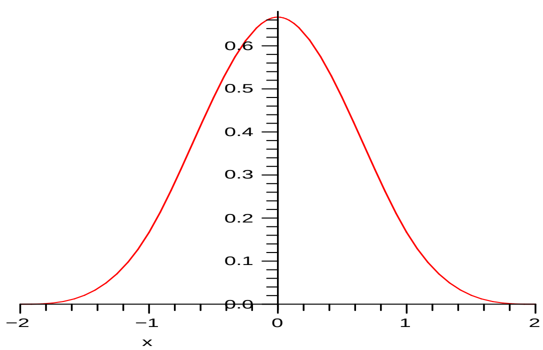


FIGURE 2. B-Spline B_3

4.3. **Error in cubic spline interpolation.** One can derive the following error estimates for cubic spline interpolation. Again, we consider only the case when $S''(a) = f''(a)$ and $S''(b) = f''(b)$.

Theorem 3. *There exists constants C_0, C_1, and C_2, independent of f, such that*

$$\max_{a\le x\le b} |f(x) - S(x)| \le C_0 h^4 M_4, \qquad \max_{a\le x\le b} |f'(x) - S'(x)| \le C_1 h^3 M_4,$$
$$\max_{a\le x\le b} |f''(x) - S''(x)| \le C_2 h^2 M_4,$$

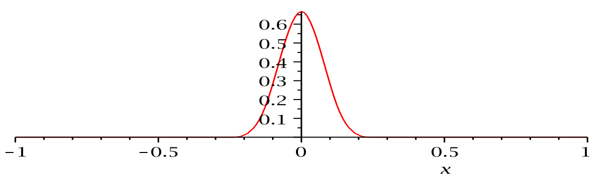
where $M_4 = \max_{a\le x\le b} |f^4(x)|$.

Splines have applications in computer graphics when smooth curves are desired. Considering the case of equally spaced points, we can use the B-spline basis defined previously to write any cubic spline $S(t)$ satisfying $S''(t_0) = S''(t_n) = 0$ in the form

$$S(t) = \sum_{i=0}^{n} p_i \phi_{i,3}(t),$$

where $\phi_{i,3}(t)$ is the cubic spline basis function centered at $t = t_i$. In this case, the values p_i, which are the degrees of freedom for $S(t)$, are called *control points*. Note that $S(t_i) \ne p_i$. An important aspect of this type of basis is that since each basis function $\phi_{i,3}(t)$ is nonzero only on the four subintervals $[t_{i-2}, t_{i-1}]$, $[t_{i-i}, t_i]$, $[t_i, t_{i+1}]$, $[t_{i+1}, t_{i+2}]$, if we change the value of the control point p_i, we only change the value of S on these four subintervals.

Consider the interval $[-1, 1]$ and an equally spaced mesh of width $1/8$. The following pictures show first a graph of a B-spline basis function centered at $t = 0$ (and hence nonzero only on the interval $[-1/4, 1/4]$ and then a graph of another choice of basis function, i.e., a cubic spline that is equal to one at $t = 0$, equal to zero at the other mesh points $\pm i/8, i = 1, 2, \ldots, 8$. Note that the second graph is different from zero on more than four subintervals. In fact, as seen in the final plot showing a magnified view of the second graph on the interval $[.5, 1]$, although the value of the this spline is small, it is not zero on any of the subintervals.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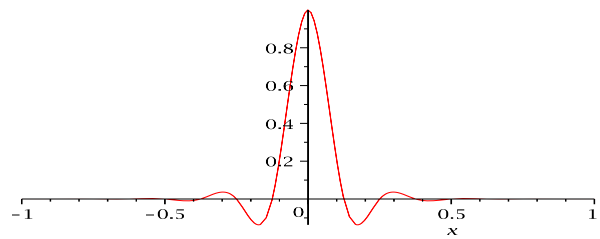
0.3
0.2
0.1
-1
-0.5
0
0.5
1
x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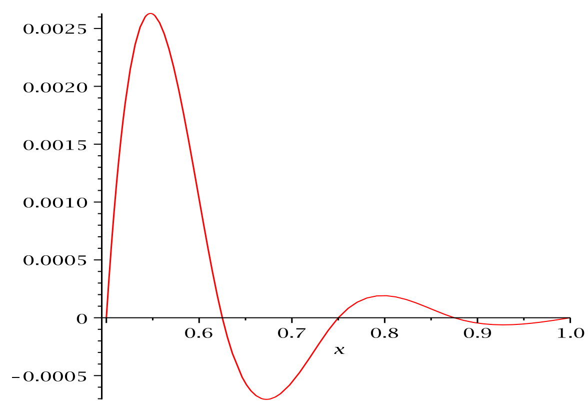
-0.5
0
0.5
1
x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0.0005
0
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0.6
0.7
0.8
0.9
1.0
x