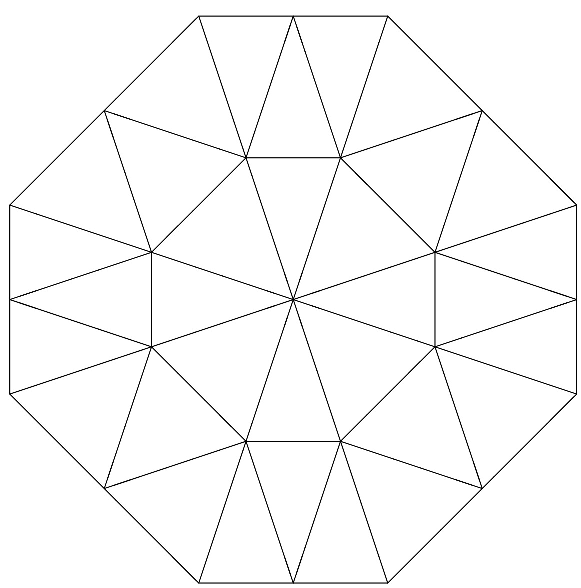
6. Piecewise polynomial approximation in two dimensions

We consider the approximation of a function $u(x, y)$, where u is defined on a polygon Ω. For each $0 < h < 1$, we let $\mathcal{T}_h$ be a triangulation of Ω into triangles T_i with the following properties:

(i) $\bar{\Omega} = \cup_i T_i$.
(ii) If T_i and T_j are distinct, then exactly one of the following holds: (a) $T_i \cap T_j = \emptyset$, (b) $T_i \cap T_j$ is a common vertex, (c) $T_i \cap T_j$ is a common side.
(iii) each $T_i \in \mathcal{T}_h$ has diameter $\leq h$.

Triangulation of a domain Ω



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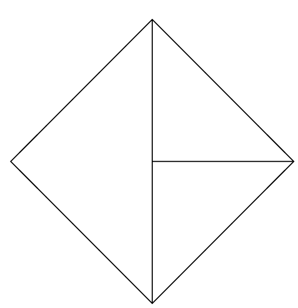

As in the case of one dimension, one can construct piecewise polynomial spaces with different amounts of continuity across elements. The case of discontinuous elements is straightforward. The space of polynomials in x and y of degree $\leq k$ has dimension $(k+1)(k+2)/2$.

As basis functions for the space of discontinuous, piecewise polynomials of degree $\leq k$, one can choose on each triangle, the monomials:

$$k=0: \quad 1, \qquad k=1: \quad 1, x, y, \qquad k=2: \quad 1, x, y, x^2, xy, y^2,$$
$$k=3: \quad 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3.$$

The case of continuous piecewise polynomials v_h of degree $\leq k$ is more difficult.

$k=0$. On each triangle T_i , $v_h = c_i$, a constant. But since $v_h \in C^0(\Omega)$, $c_i = c$ for all i, and so the space consists of a single constant. This is not a useful space, since there is only one degree of freedom.

$k=1$. On each triangle, $v_h|_T = c_0 + c_1 x + c_2 y$. The issue is how to find degrees of freedom so that the global function we construct is continuous. We shall show that the correct degrees of freedom are the values of v_h at the vertices of the triangulation. This involves establishing two facts. The first is to show that a linear function on a triangle is uniquely determined by its values at the vertices. The second is to observe that a piecewise linear function defined on two triangles sharing a common edge will be continuous everywhere on that edge if it is continuous at the vertices of that edge. To see this, consider an edge $\alpha x + \beta y = \gamma$, where we assume $\beta \neq 0$. Then, on that edge, we may write any linear polynomial

$$P_1(x, y) = c_0 + c_1 x + c_2 y = c_0 + c_1 x + c_2(\gamma - \alpha x)/\beta = c_0 + c_2\gamma/\beta + x(c_1 - c_2\alpha/\beta),$$

a linear function in only one variable. Since this function is uniquely determined by its values at the two endpoints of the interval, if v_h is continuous at the vertices, it will be continuous across the entire edge. To see why a linear function in two variables in uniquely determined by its values at the vertices of a triangle, recall the case of an interval $[a_1, a_2]$ in one dimension. Using the Lagrange form, we may write any linear polynomial

$$P_1(x) = P_1(a_1)\frac{x - a_2}{a_1 - a_2} + P_1(a_2)\frac{x - a_1}{a_2 - a_1}.$$

In this form, the degrees of freedom are the values $P_1(a_1)$ and $P_1(a_2)$, This is because, instead of taking 1 and x as the basis functions for linear polynomials, we are using the functions

$$\phi_1(x) = \frac{x - a_2}{a_1 - a_2}, \qquad \phi_2(x) = \frac{x - a_1}{a_2 - a_1}.$$

The key property is that $\phi_1(a_1) = 1$, $\phi_1(a_2) = 0$ and $\phi_2(a_1) = 0$, $\phi_2(a_2) = 1$.

We now try to follow an analogous procedure in two dimensions. First consider the special case of a triangle T with vertices $\boldsymbol{a}_1 = (1, 0)$, $\boldsymbol{a}_2 = (0, 1)$, and $\boldsymbol{a}_3 = (0, 0)$. Letting $\boldsymbol{x} = (x, y)$, we then seek linear basis functions $\lambda_i(\boldsymbol{x})$ satisfying $\lambda_i(\boldsymbol{a}_j) = 1$ if $i = j$ and $= 0$ if $i \neq j$. It is easy to check that the solution to this problem is given by: $\lambda_1 = x$, $\lambda_2 = y$, $\lambda_3 = 1 - x - y$. We can then write a general linear function on T in the form:

$$P_1(\boldsymbol{x}) = P_1(\boldsymbol{a}_1)\lambda_1(\boldsymbol{x}) + P_1(\boldsymbol{a}_2)\lambda_2(\boldsymbol{x}) + P_1(\boldsymbol{a}_3)\lambda_3(\boldsymbol{x}) = P_1(1, 0)x + P_1(0, 1)y + P_1(0, 0)(1 - x - y).$$

We now generalize this approach to an arbitrary triangle T with vertices $\boldsymbol{a}_i$. Again, we seek linear functions $\lambda_i(\boldsymbol{x})$ with the property that $\lambda_i(\boldsymbol{a}_j) = 1$ if $i = j$ and 0 if $i \neq j$. We now show that the solution is given by the barycentric coordinates of a point $\boldsymbol{x}$. To define the barycentric coordinates, we let $\boldsymbol{a}_j = (a_{1j}, a_{2j})$ be the vertices of a triangle T. Then

$$T = \{\boldsymbol{x} = \sum_{j=1}^{3} \lambda_j \boldsymbol{a}_j, \quad 0 \le \lambda_j \le 1, \quad 1 \le j \le 3, \quad \sum_{j=1}^{3} \lambda_j = 1\}.$$

The barycentric coordinates $\lambda_j = \lambda_j(\boldsymbol{x})$, $1 \le j \le 3$ of any point $\boldsymbol{x} \in \mathbb{R}^2$ with respect to the the vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are the unique solutions of the linear system:

$$\sum_{j=1}^{3} a_{ij}\lambda_j = x_i, \; i = 1, 2, \qquad \sum_{j=1}^{3} \lambda_j = 1,$$

i.e., we solve

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 1 & 1 & 1 \end{pmatrix}.$$

One can show that the matrix is nonsingular if the triangle is nondegenerate.

Observe that when $\boldsymbol{x} = \boldsymbol{a}_k$, i.e., $x_i = a_{ik}$, then $\lambda_1, \lambda_2, \lambda_3$ is the unique solution of

$$\sum_{j=1}^{3} a_{ij}\lambda_j = a_{ik}, \qquad \sum_{j=1}^{3} \lambda_j = 1.$$

The solution is given by $\lambda_j = \delta_{jk} = 1$, for $j = k$ and $= 0$ for $j \neq k$. So λ_j is the linear function that is equal to one at the vertex $\boldsymbol{a}_j$ and zero at the other two vertices.

Once we obtain the barycentric coordinates λ_i, we can again write any linear polynomial on T in the form:

$$P_1(\boldsymbol{x}) = P_1(\boldsymbol{a}_1)\lambda_1(\boldsymbol{x}) + P_1(\boldsymbol{a}_2)\lambda_2(\boldsymbol{x}) + P_1(\boldsymbol{a}_3)\lambda_3(\boldsymbol{x}).$$

Also observe that if $\boldsymbol{x} = (1-\theta)\boldsymbol{y} + \theta\boldsymbol{z}$, $0 \le \theta \le 1$, i.e., $\boldsymbol{x}$ lies on the line segment between $\boldsymbol{y}$ and $\boldsymbol{z}$, then

$$\begin{aligned} \lambda(\boldsymbol{x}) = A^{-1}\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} &= A^{-1}\begin{pmatrix} (1-\theta)\boldsymbol{y} + \theta\boldsymbol{z} \\ (1-\theta)1 + \theta 1 \end{pmatrix} \\ &= (1-\theta)A^{-1}\begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix} + \theta A^{-1}\begin{pmatrix} \boldsymbol{z} \\ 1 \end{pmatrix} = (1-\theta)\lambda(\boldsymbol{y}) + \theta\lambda(\boldsymbol{z}). \end{aligned}$$

Hence, on the edge joining $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, $\lambda_3(\boldsymbol{x}) = 0$. At the point midway between $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, $\lambda_3(\boldsymbol{x}) = 1/2$. We note that the barycenter of a triangle is the point of T for which all the $\lambda_i = 1/3$.

Next consider the degrees of freedom for higher order polynomials: Define $\boldsymbol{a}_{ij}$ to be the midpoint of the edge joining $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. We can show that we can write every polynomial of degree ≤ 2 defined on T in the form:

$$P_2(\boldsymbol{x}) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(\boldsymbol{a}_i) + \sum_{i<j} 4\lambda_i\lambda_j P(\boldsymbol{a}_{ij}).$$

To see why this is true, we apply the general formula of writing any polynomial $P(x)$ of degree $\leq k$ in the form

$$P(x) = \sum_i B_i(x)\phi_i(P),$$

where $\{\phi_i(P)\}$ are a unisolvent set of degrees of freedom for $P(x)$ and $B_i(x)$ is a polynomial of degree ≤ 2 which satisfies $\phi_i(B_j) = 0$ for $i \neq j$ and $\phi_i(B_i) = 1$. In this case, $\phi_i(P)$ are of the form $P(\boldsymbol{a}_i)$ or $P(\boldsymbol{a}_{ij})$. The corresponding basis function for the degree of freedom $P(\boldsymbol{a}_3)$ will be $\lambda_3(2\lambda_3 - 1)$, since $\lambda_3 = 0$ everywhere along the line segment joining $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (hence at $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_{12}$) and $(2\lambda_3 - 1) = 0$ when $\lambda_3 = 1/2$, which includes the midpoints $\boldsymbol{a}_{13}$ and $\boldsymbol{a}_{23}$. Furthermore $\lambda_3(2\lambda_3 - 1) = 1$ at $\boldsymbol{a}_3$. For the degree of freedom $P(a_{12})$, the corresponding basis function $4\lambda_1\lambda_2 = 0$ at all vertices and at $\boldsymbol{a}_{13}$ and $\boldsymbol{a}_{23}$ and equals 1 at the midpoint $\boldsymbol{a}_{12}$.

To represent cubic polynomials, we define for $i \neq j$, $\boldsymbol{a}_{iij} = (2/3)\boldsymbol{a}_i + (1/3)\boldsymbol{a}_j$ and $\boldsymbol{a}_{123} = (1/3)(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3)$ Then we can write every cubic polynomial in the form

$$P_3(\boldsymbol{x}) = \sum_{i=1}^{3}[\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)/2]P(\boldsymbol{a}_i) + \sum_{i\neq j}[9\lambda_i\lambda_j(3\lambda_i - 1)/2]P(\boldsymbol{a}_{iij}) + [27\lambda_1\lambda_2\lambda_3]P(\boldsymbol{a}_{123}).$$

In general, if T is an n-simplex with vertices $\boldsymbol{a}_j$, $1 \leq j \leq n+1$, then for a given integer $k \geq 1$, any polynomial $p \in P_k$ is uniquely determined by its values on the set:

$$L_k(T) = \{x = \sum_{j=1}^{n+1} \lambda_j \boldsymbol{a}_j, \quad \sum_{j=1}^{n+1} \lambda_j = 1, \quad \lambda_j \in \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}, 1 \leq j \leq n+1\}.$$