7.2. **Local convergence results.** In some cases, it can be difficult to verify the hypotheses of the convergence theorem. However, one may still be able to verify the hypotheses of the following theorem, that gives a *local convergence* result.

Theorem 24. *Suppose that (i) f and f' are continuous on the interval $I = [a, b]$ and (ii) the equation $x = f(x)$ has a solution $s \in (a, b)$ such that $|f'(s)| < 1$. Then there exists a number $d > 0$ such that the sequence $\{x_n\}$ determined by the iteration $x_{n+1} = f(x_n)$ converges to s for any initial guess x_0 satisfying $|x_0 - s| \leq d$.*

Note that since we do not know d explicitly, the theorem only says that fixed point iteration will converge if the initial guess x_0 is sufficiently close to the fixed point s.

Application to Newton's method: Let $f(x) = x - F(x)/F'(x)$. Suppose that $F(x) \in C^2[a, b]$, i.e., F, F', F'' are continuous on $[a, b]$ and that $F'(x) \neq 0$ for all $x \in [a, b]$. Suppose further that there is a point $s \in [a, b]$ such that $F(s) = 0$. (Note there can only be one such point since $F'(x) \neq 0$ so F is always increasing or always decreasing.) Then we can conclude that Newton's method will converge if the initial guess x_0 is sufficiently close to the root s. To see this, we apply the previous theorem. We first observe that the hypotheses on F imply that f and f' are continuous on the interval $I = [a, b]$. The root s of F is obviously a fixed point of f. Now

$$f'(x) = 1 - \frac{[F'(x)]^2 - F(x)F''(x)}{[F'(x)]^2} = \frac{F(x)F''(x)}{[F'(x)]^2}.$$

Hence $f'(s) = 0$, and so hypothesis (iii) above is satisfied. We therefore conclude that Newton's method is locally convergent.

7.3. **Order of convergence.** One way to compare methods is to compare their speed of convergence.

Definition: Suppose that $\{x_n\}$ converges to x^*. We say that convergence is of order p if there exists a positive constant α such that

$$\|x_{n+1} - x^*\| \leq \alpha \|x_n - x^*\|^p, \quad \text{for all } n \geq n_0.$$

If $p = 1$ (linear convergence), we also require that $\alpha < 1$. Finally, we say the convergence is *superlinear* if there is a sequence $\{\alpha_n\}$ converging to zero such that

$$\|x_{n+1} - x^*\| \leq \alpha_n \|x_n - x^*\|, \quad \text{for all } n \geq n_0.$$

Theorem 25. *Suppose the sequence $\{x_n\}$ is obtained by the iteration $x_{n+1} = f(x_n)$ and that $\lim_{n\to\infty} x_n = x^*$. Suppose $f^{(k)}(x)$ is continuous at x^* for $k = 0, 1, \dots, p$ and that*

$$x^* = f(x^*), \quad f'(x^*) = 0, \dots, f^{(p-1)}(x^*) = 0, \quad f^{(p)}(x^*) \neq 0.$$

Then the convergence is of order p.

Proof. We obtain this result by expanding in a Taylor series, i.e., for some ξ_n between x^* and x_n,

$$x_{n+1} - x^* = f(x_n) - f(x^*) = f'(x^*)(x_n - x^*) + \frac{f''(x^*)}{2}(x_n - x^*)^2 + \cdots + \frac{f^{(p-1)}(x^*)}{(p-1)!}(x_n - x^*)^{p-1} + \frac{f^{(p)}(\xi_n)}{p!}(x_n - x^*)^p.$$

Applying the hypotheses of the theorem, we get

$$|x_{n+1} - x^*| = \frac{|f^{(p)}(\xi_n)|}{p!}|(x_n - x^*)^p|.$$

This is not quite the desired result, since ξ_n depends on n and is not a constant, so we cannot just take $\alpha = |f^{(p)}(\xi_n)|/p!$. However, since $f^{(p)}(x)$ is continuous at x^*, given $\epsilon > 0$ (say $\epsilon = 1$), we can find $\delta > 0$ such that $|f^{(p)}(\xi_n) - f^{(p)}(x^*)| < \epsilon$ for $|\xi_n - x^*| < \delta$. But x_n converges to x^* as $n \to \infty$. Hence $|x_n - x^*| < \delta$ for $n \geq n_0$. This implies $|\xi_n - x^*| < \delta$ for $n \geq n_0$. Combining these results, we get that $|f^{(p)}(\xi_n)| \leq \epsilon + |f^{(p)}(x^*)|$ for $n \geq n_0$. We now choose $\alpha = [\epsilon + |f^{(p)}(x^*)|]/p!$ to finish the proof. □

If we apply this result to Newton's method, then we have already seen that if $F(x) \in C^2$ and $F'(x^*) \neq 0$, i.e., x^* is a simple root, then $x^* = f(x^*)$ and $f'(x^*) = 0$. Since $f'(x) = F(x)F''(x)/[F'(x)]^2$, a simple computation shows that

$$f''(x^*) = F''(x^*)/F'(x^*).$$

Hence, in general, we get that Newton's method is quadratically convergent.

If $F(x)$ has a root x^* of multiplicity $m > 1$, then we may write $F(x) = (x - x^*)^m G(x)$, where $G(x^*) \neq 0$. Then

$$F'(x) = (x - x^*)^{m-1}[(x - x^*)G'(x) + mG(x)],$$
$$F''(x) = (x - x^*)^{m-2}[(x - x^*)^2 G''(x) + 2m(x - x^*)G'(x) + m(m-1)G(x)].$$

Hence,

$$f'(x) = \frac{F(x)F''(x)}{[F'(x)]^2} = \frac{G(x)[(x - x^*)^2 G''(x) + 2m(x - x^*)G'(x) + m(m-1)G(x)]}{[(x - x^*)G'(x) + mG(x)]^2},$$

and so $f'(x^*) = (m-1)/m = 1 - 1/m$. Hence, the method is no longer quadratically convergent.

7.4. **The Dekker-Brent method.** This method combines the secant method and the method of bisection to produce an algorithm that has the advantages of both methods, i.e., it retains the root-bracketing property of the bisection method, but uses the secant method when appropriate to attain a higher rate of convergence. This method has been implemented in the Matlab routine *fzero* to find the root of a single nonlinear equation.