10. Two-point boundary value problems

We consider the approximation of problems of the form:

$$u'' = f(x, u, u'), \qquad u(a) = g_a, \quad u(b) = g_b.$$

We discuss three types of methods: (i) shooting method, (ii) finite difference method, (iii) finite element method.

10.1. Shooting method.

The main idea is to transform the boundary value problem into a sequence of initial value problems. To do this, we define for each value of a parameter γ, a function $u(x;\gamma)$ that solves the initial value problem

$$u'' = f(x, u, u'), \qquad u(a) = g_a, \quad u'(a) = \gamma.$$

The problem is then to find a value of γ such that $u(b;\gamma) = g_b$, i.e., we seek a root of the nonlinear equation $w(\gamma) = u(b;\gamma) - g_b = 0$. If we solve this equation by the secant method, for example, then we obtain the iteration:

$$\gamma_{n+1} = \gamma_n - w(\gamma_n)\frac{\gamma_n - \gamma_{n-1}}{w(\gamma_n) - w(\gamma_{n-1})}.$$

where γ_0 and γ_1 are initial approximations. Note that to evaluate $w(\gamma_n)$, we need to find $u(b, \gamma_n)$, which involves solving an initial value problem. Using the techniques discussed previously (last semester), we can do this by reformulating the problem as a first order system of odes by introducing a new variable $v = u'$. This gives the system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ f(x, u, v) \end{pmatrix}, \qquad u(a) = g_a, \quad v(a) = \gamma.$$

10.2. Finite difference method.

We first consider the case of a linear two-point boundary value problem of the form

$$L(u) \equiv -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \qquad u(a) = g_a, \quad u(b) = g_b,$$

where we assume that $p(x) \geq p_* > 0$ and $q(x) \geq 0$.

The idea now is to replace the derivatives in the equation by difference quotients, with the aim of producing an approximate problem whose unknowns represent approximations to the true solution at a finite set of points. These unknowns will then be determined by solving a finite dimensional linear system of equations. To do this, we place a mesh of width h on $[a, b]$, i.e., define mesh points $x_j = a + jh$, $j = 0, \ldots, N$, where $h = (b - a)/N$.

First, consider the special case when $p(x)$ is constant.

Using Taylor series expansions, we get

$$u(x \pm h) = u(x) \pm hu'(x) + \frac{h^2}{2}u''(x) \pm \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(\xi_\pm).$$

Hence, for some $\xi \in [x-h, x+h]$,

$$[u(x+h) - 2u(x) + u(x-h)]/h^2 = u''(x) + \frac{h^2}{24}[u^{(4)}(\xi_+) + u^{(4)}(\xi-)] = u''(x) + \frac{h^2}{12}u^{(4)}(\xi).$$

In the above we have used the Mean Value Theorem for sums, i.e.,

Theorem 27. *Let $F(x)$ be a continuous function on $[a,b]$, let $x_1, \cdots, x_n$ be points in $[a,b]$, and let $d_1, \cdots, d_n$ be real numbers all of one sign. Then $\sum_{i=1}^n F(x_i)d_i = F(\xi)\sum_{i=1}^n d_i$ for some $\xi \in [a,b]$.*

If we apply the above formula at the point $x = x_j$ and insert it into the differential equation, we get

$$\begin{aligned} 0 &= -pu''(x_j) + q(x_j)u(x_j) - f(x_j) \\ &= -p[u(x_{j+1}) - 2u(x_j) + u(x_{j-1})]/h^2 + q(x_j)u(x_j) - f(x_j) + p(h^2/12)u^{(4)}(\xi_j). \end{aligned}$$

We then define an approximation u_j to $u(x_j)$ by omitting the $O(h^2)$ error terms, i.e., we let $u_j, j = 1, \ldots, N-1$ satisfy:

$$L_h u_j \equiv p[-u_{j+1} + 2u_j - u_{j-1}]/h^2 + q(x_j)u_j = f(x_j).$$

Since the boundary conditions are given, we set $u_0 = g_a$ and $u_N = g_b$. Hence, we have $N-1$ linear equations for the $N-1$ unknowns $u_1, \ldots, u_{N-1}$.

For later use, it is useful to define the local truncation error of the method, given by

$$\tau_j = L_h u(x_j) - Lu(x_j).$$

This is basically how well the true solution satisfies the discrete equations. Using the Taylor expansions defined above, we see that

$$\tau_j = L_h u(x_j) - Lu(x_j) = -p(h^2/12)u^{(4)}(\xi_j).$$

We next consider the approximate discrete system in more detail. Multiplying our discrete equation by h^2 and collecting terms, we get

$$-pu_{j+1} + [2p + h^2 q(x_j)]u_j - pu_{j-1} = h^2 f(x_j).$$

Setting $a_j = 2p + h^2 q(x_j)$, we may write this system of linear equations in matrix form as:

$$\begin{pmatrix} a_1 & -p & 0 & 0 & \cdots & 0 \\ -p & a_2 & -p & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -p & a_{N-2} & -p \\ 0 & \cdots & 0 & 0 & -p & a_{N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f(x_1) \\ f(x_2) \\ \cdots \\ f(x_{N-2}) \\ f(x_{N-1}) \end{pmatrix} + \begin{pmatrix} pg_a \\ 0 \\ \cdots \\ 0 \\ pg_b \end{pmatrix}.$$

Thus, we need to solve the linear system $Au = F$, where A is a symmetric tridiagonal matrix. Note that if $\min_{[a,b]} q(x) \geq q_* > 0$, then A is strictly diagonally dominant and hence invertible.

Now let us return to the case when p depends on x. One possibility is to write

$$Lu = -pu'' - p'u' + qu$$

and approximate the first term at $x = x_j$ (as above) by

$$p(x_j)[-u_{j+1} + 2u_j - u_{j-1}]/h^2.$$

To get an approximation to u' that is also second order, we could use the Taylor series to write

$$u(x \pm h) = u(x) \pm u'(x) + \frac{h^2}{2}u''(x) \pm \frac{h^3}{6}u'''(\eta_\pm).$$

It is then easy to show that

$$[u(x+h) - u(x-h)]/2h = u'(x) + \frac{h^2}{6}u'''(\eta),$$

for some $\eta \in [x-h, x+h]$. In that case, our discrete equation would become

$$-[p(x_j) - hp'(x_j)/2]u_{j-1} + [2p(x_j) + h^2 q(x_j)]u_j - [p(x_j) + hp'(x_j)/2]u_{j+1} = h^2 f(x_j).$$

There are two drawbacks to this approach. First, the matrix we get will not be symmetric, and thus will involve more work to solve. To see this, let

$$a_j = 2p(x_j) + h^2 q(x_j), \qquad b_j = p(x_j) - hp'(x_j)/2, \qquad c_j = p(x_j) + hp'(x_j)/2.$$

Then, we may write this system of linear equations in matrix form as:

$$\begin{pmatrix} a_1 & -c_1 & 0 & 0 & \cdots & 0 \\ -b_2 & a_2 & -c_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -b_{N-2} & a_{N-2} & -c_{N-2} \\ 0 & \cdots & 0 & 0 & -b_{N-1} & a_{N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f(x_1) \\ f(x_2) \\ \cdots \\ f(x_{N-2}) \\ f(x_{N-1}) \end{pmatrix} + \begin{pmatrix} b_1 g_a \\ 0 \\ \cdots \\ 0 \\ c_{N-1} g_b \end{pmatrix}.$$

When p is not constant, we do not expect that $b_{j+1} = c_j$. Another drawback to this approach is that we would also have to compute p'.

Instead, we consider another way to approximate the derivatives in the equation. Using Taylor series expansions, we get either

$$v(x \pm \epsilon) = v(x) \pm \epsilon v'(x) + \frac{\epsilon^2}{2}v''(x) + O(\epsilon^3) \qquad \text{or}$$
$$v(x \pm \epsilon) = v(x) \pm \epsilon v'(x) + \frac{\epsilon^2}{2}v''(x) \pm \frac{\epsilon^3}{6}v'''(x) + O(\epsilon^4),$$

depending on where we decide to end the expansion. Hence,

$$[v(x+\epsilon) - v(x-\epsilon)]/(2\epsilon) = v'(x) + O(\epsilon^2) \qquad \text{or}$$
$$[v(x+\epsilon) - v(x-\epsilon)]/(2\epsilon) = v'(x) + \frac{\epsilon^2}{6}v'''(x) + O(\epsilon^3).$$

First, choosing $v = pu'$, $x = x_j$ and $\epsilon = h/2$, we get that

$$(pu')'(x_j) = [(pu')(x_j + h/2) - (pu')(x_j - h/2)]/h + O(h^2).$$

Then, choosing $v = u$, $\epsilon = h/2$, and $x = x_j + h/2$ and $x = x_j - h/2$, we get that

$$u'(x_j + h/2) = [u(x_j + h) - u(x_j)]/h - \frac{h^2}{24}u'''(x_j + h/2) + O(h^3),$$
$$u'(x_j - h/2) = [u(x_j) - u(x_j - h)]/h - \frac{h^2}{24}u'''(x_j - h/2) + O(h^3).$$

Inserting these quantities, we obtain:

$$(pu')'(x_j) = \frac{1}{h}\left\{[p(x_j + h/2)\frac{u(x_j + h) - u(x_j)}{h} - [p(x_j - h/2)\frac{u(x_j) - u(x_j - h)}{h}\right\}$$
$$+ \frac{h}{24}\left\{p(x_j + h/2)u'''(x_j + h/2) - p(x_j - h/2)u'''(x_j - h/2)\right\} + O(h^2).$$

Although the middle terms seems to only $O(h)$, we get by applying the Mean Value Theorem, that

$$p(x_j + h/2)u'''(x_j + h/2) - p(x_j - h/2)u'''(x_j - h/2) = O(h),$$

and so

$$(pu')'(x_j) = \frac{1}{h}\left\{[p(x_j + h/2)\frac{u(x_j + h) - u(x_j)}{h} - [p(x_j - h/2)\frac{u(x_j) - u(x_j - h)}{h}\right\} + O(h^2).$$

Denoting $p(x_j \pm h/2)$ by $p_{j\pm 1/2}$, and omitting the $O(h^2)$ terms, we are led to the approximation:

$$L_h u_j = \frac{1}{h^2}\left(-p_{j-1/2}u_{j-1} + [p_{j-1/2} + p_{j+1/2} + h^2 q(x_j)]u_j - p_{j+1/2}u_{j+1}\right) \tag{10.1}$$

and thus to the difference equation

$$-p_{j-1/2}u_{j-1} + [p_{j-1/2} + p_{j+1/2} + h^2 q(x_j)]u_j - p_{j+1/2}u_{j+1} = h^2 f(x_j).$$

We remark that although we have not kept careful track of the error terms, it is possible to show that the local truncation error $\tau_j = L_h u(x_j) - Lu(x_j)$ satisfies

$$|\tau_j| \leq K h^2 [\max_{[a,b]} |u^{(4)}(x)| + \max_{[a,b]} |(pu^{(3)})'(x)|],$$

where K is a constant that does not depend on u or h.

Now setting $a_j = p_{j-1/2} + p_{j+1/2} + h^2 q(x_j)$, we obtain the linear system

$$\begin{pmatrix} a_1 & -p_{3/2} & 0 & 0 & \cdots & 0 \\ -p_{3/2} & a_2 & -p_{5/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -p & a_{N-2} & -p_{N-3/2} \\ 0 & \cdots & 0 & 0 & -p_{N-3/2} & a_{N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f(x_1) \\ f(x_2) \\ \cdots \\ f(x_{N-2}) \\ f(x_{N-1}) \end{pmatrix} + \begin{pmatrix} p_{1/2}\ g_a \\ 0 \\ \cdots \\ 0 \\ p_{N-\frac{1}{2}}\ g_b \end{pmatrix}$$

Two natural questions that we now address are (i) whether this linear system always has a unique solution and (ii) how good an approximation does this produce to the true solution (i.e., does the approximation converge as $h \to 0$, and if so, how fast)? To answer these questions, we first establish a fundamental property of this discrete system, i.e., that it satisfies a discrete maximum principle.