10.3. **Finite element method.** We begin by considering the model problem:

$$Lu \equiv -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \quad a < x < b, \qquad u(a) = g_a, \quad u(b) = g_b.$$

where we assume that $p(x) \geq p_* > 0$, $q(x) \geq 0$.

Unlike the finite difference method, the finite element method is not based on the differential equation, but on a variational formulation of the boundary value problem. To describe this, we first consider a simple formula derived by integration by parts. Let

$$(u, v) = \int_a^b u(x)v(x)\, dx.$$

Then,

$$(Lu, v) = \int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right] v\, dx = -pu'v|_a^b + \int_a^b (p(x)u'v' + quv]\, dx.$$

To make use of this formula, we note that if u is a solution of the boundary value problem and is sufficiently smooth, then $Lu = f$. However, we have no information about $u'(a)$ or $u'(b)$, both of which appear on the right hand side of the integration by parts formula. However for all v (sufficiently smooth) satisfying $v(a) = v(b) = 0$, we have

$$(f, v) = (Lu, v) = \int_a^b (p(x)u'v' + quv)\, dx \equiv a(u, v).$$

Hence, the solution, u, of the boundary value problem satisfies the variational equation

$$a(u, v) = (f, v)$$

for all (sufficiently smooth) functions v satisfying $v(a) = v(b) = 0$.

We can also reverse this process. If u satisfies $a(u, v) = (f, v)$ for all (sufficiently smooth) functions v satisfying $v(a) = v(b) = 0$, then integrating by parts, we see that $(Lu, v) = (f, v)$. From this, we can conclude that if $u \in C^2(a, b)$, $p \in C^1(a, b)$, $q \in C^0(a, b)$, and $f \in C^0(a, b)$, then $Lu = f$ for $a < x < b$. To see this, note that then $w(x) = Lu(x) - f(x) \in C^0(a, b)$. However, if $\int_a^b w(x)v(x)\, dx = 0$ for all $v \in C^0(a, b)$ with $v(a) = v(b) = 0$, then $w(x) = 0$. To see this, assume that there is a point x_0 such that $w(x_0) \neq 0$ (say $w(x_0) > 0$). Then there is a number ϵ such that $w(x) > 0$ for all $x_0 - \epsilon < x < x_0 + \epsilon$. Now choose $v(x)$ satisfying

$$v(x) = \begin{cases} 0, & x \leq x_0 - \epsilon \\ [x - x_0 + \epsilon]/\epsilon, & x_0 - \epsilon \leq x \leq x_0 \\ [x_0 + \epsilon - x]/\epsilon, & x_0 \leq x \leq x_0 + \epsilon \\ 0, & x_0 + \epsilon \leq x \end{cases}.$$

Then, since both $w(x)$ and $v(x)$ are ≥ 0 on the interval $[x_0 - \epsilon, x_0 + \epsilon]$ and strictly positive on the interval $[x_0 - \epsilon/2, x_0 + \epsilon/2]$, we have

$$\int_a^b w(x)v(x)\, dx = \int_{x_0-\epsilon}^{x_0+\epsilon} w(x)v(x)\, dx \geq \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} w(x)v(x)\, dx > 0,$$

which contradicts our assumption that this integral equals zero. Hence, we conclude that $w(x) = (Lu - f)(x) = 0$. To get a solution of the boundary value problem, we must also require that u satisfies the boundary conditions $u(a) = g_a$, $u(b) = g_b$.

To make all this more precise, we need to specify what types of functions we allow as solutions. For the differential equation, we assume that that data p, q, f satisfy $p \in C^1(a, b)$, $q \in C^0(a, b)$, and $f \in C^0(a, b)$. We then usually require that $u \in C^2(a, b)$, where by $C^r(a, b)$ we mean a function that has continuous derivatives up to order r on the interval (a, b). When $r = 0$, we mean the function is continuous. Under these assumptions, all terms in the differential equation are defined at every point, and to be a solution we require that u satisfies the differential equation at every point.

In the variational equation, the highest derivative that appears on either u or v is the first derivative, so we can make sense of this equation even for functions that only have first derivatives. In addition, since these functions only appear inside an integral sign, we can make sense of the variational equation even if u' and v' are not continuous. In particular, we can allow jumps in these derivatives. The right condition is that u and v must belong to the space

$$V \equiv H^1[a, b] = \{v : \int_a^b [v^2 + (v')^2]\, dx < \infty\}.$$

A simple example of a function in this space that does not have a continuous derivative is given by

$$v(x) = (x - a)/(c - a), \quad a < x \le c, \qquad v(x) = (b - x)/(b - c), \quad c < x < b.$$

Note that this function is continuous, but

$$v'(x) = 1/(c - a), \quad a < x < c, \qquad v'(x) = -1/(b - c), \quad c < x < b.$$

Since $v'(x)$ has different limits from the right and left as $x \to c$, it is not defined at $x = c$.

We also want to define subspaces of V that include boundary conditions. The two spaces we need are

$$V^0 = \{v \in V : v(a) = 0,\ v(b) = 0\}, \qquad V^g = \{v \in V : v(a) = g_a,\ v(b) = g_b\}.$$

Using these spaces, we can then define a variational problem corresponding to the boundary value problem we started with, i.e.,

Problem P: Find $u \in V^g$ such that $a(u, v) = (f, v)$ for all $v \in V^0$.

The finite element method is based on this variational formulation of the boundary value problem. The basic idea is to choose a finite dimensional subspace V_h of V and subspaces V_h^0 and V_h^g of V_h satisfying the boundary conditions indicated by the superscript and define an approximate variational problem as follows:

Problem P_h: Find $u_h \in V_h^g$ such that $a(u_h, v) = (f, v)$ for all $v \in V_h^0$.

The spaces V_h used in the finite element method are made up of piecewise polynomials, which we describe in the following way.

10.3.1. *Piecewise Polynomial Approximation.* Consider a partition $\mathcal{P}$ of an interval $[a, b]$ by points $x_0, \ldots, x_n$, i.e., $a = x_0 < x_1 < \ldots < x_n = b$.

Definition: We say $Q(x)$ is a C^r piecewise polynomial of degree $\leq k$ with respect to the partition $\mathcal{P}$ if $Q \in C^r[a, b]$ and Q has the form $Q(x) = q_j(x)$ for $x \in (x_{j-1}, x_j)$, $j = 1, \ldots n$, where $q_j(x)$ is a polynomial of degree $\leq k$ for each value of j.

Note that since $Q(x) \in C^r$ and the q_j are polynomials, its first r derivatives are continuous and its $r+1$st derivative is defined everywhere except possibly at the points x_j. For use in the application we are considering, we will use C^0 (i.e., continuous) piecewise polynomials.

Examples:

The simplest example is when $k = 1$, the space of continuous piecewise linear functions. On any subinterval, a linear function is uniquely determined by knowing its value at two distinct points (these values are called the degrees of freedom of the function). For example, we may write any linear function $L(x)$ in the form:

$$L(x) = \frac{x-b}{a-b} L(a) + \frac{x-a}{b-a} L(b),$$

and so $L(x)$ is uniquely determined by specifying the values $L(a)$ and $L(b)$.

If we consider the space of discontinuous piecewise linear functions, we would have two degrees of freedom on each subinterval, and since there are n subintervals, there would be $2n$ degrees of freedom. However, since Q is continuous, we must have $q_j(x_j) = q_{j+1}(x_j)$, $j = 1, \ldots, n-1$. Hence, there are only $n+1$ degrees of freedom. Note that by choosing the degrees of freedom to be the values of the function at the mesh points, we insure the continuity of Q, i.e., we have $q_j(x_j) = q_{j+1}(x_j)$. The dimension of this space $= n+1$, the number of degrees of freedom. Thus, we see that continuous, piecewise linear functions are uniquely determined by their values at the $n+1$ mesh points $x_0, \ldots, x_n$.

$k = 2$: We want to determine the dimension and the degrees of freedom for the space of continuous, piecewise quadratics. The space of discontinuous piecewise quadratics would have 3 degrees of freedom per subinterval, and since there are n subintervals, the overall dimension is $3n$. However, as in the case of linears, the requirement of continuity imposes one constraint at each interior mesh point, so the dimension of continuous, piecewise quadratics is $3n - (n-1) = 2n+1$. To ensure continuity, we choose $n+1$ of these degrees of freedom to be the values of the function Q at the mesh points. To uniquely determine Q on each subinterval, we need to specify one additional degree of freedom per subinterval. A simple way to do this is to specify $Q(x_{j-1/2})$, $j = 1, \ldots, n$.

$k = 3$: In a similar way, we can construct the space of continuous, piecewise, cubic polynomials, whose dimension will be $3n+1 = 4n - (n-1)$. The degrees of freedom are the values of Q at the mesh points plus its values at any two interior points on each subinterval.

In order to use these spaces in the finite element method, we need to construct a basis for them. In the case of continuous, piecewise linear functions, we want to find functions $\phi_i(x)$,

$i = 0, 1, \ldots, n$, such that any piecewise linear function $Q_1(x)$ can be written as $Q_1(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$, where the α_i are constants.

As we shall see when we consider the discrete set of equations produced by the finite element method, we want to choose a basis that is both easy to construct and also has the property that any basis function is only non-zero on a small set of consecutive subintervals. This can be achieved for continuous, piecewise linear functions by the choice $\{\phi_i\}_{i=0}^{n}$, where

$$\begin{aligned} \phi_i(x) &= 0, \quad x \notin [x_{i-1}, x_{i+1}], \\ &= (x - x_{i-1})/(x_i - x_{i-1}), \quad x \in [x_{i-1}, x_i], \\ &= (x_{i+1} - x)/(x_{i+1} - x_i), \quad x \in [x_i, x_{i+1}]. \end{aligned}$$

The basis function $\phi_i(x)$ is called a hat function. Note that $\phi_i(x_j) = 0$ for $i \neq j$ and $= 1$ for $i = j$. Hence,

$$Q_1(x_j) = \sum_{i=0}^{n} \alpha_i \phi_i(x_j) = \alpha_j.$$

Thus, $Q_1(x) = \sum_{i=0}^{n} Q_1(x_i)\phi_i(x)$, so that $Q_1(x)$ is uniquely determined by its degrees of freedom $Q_1(x_i)$, i.e., its values at the mesh points.

When the points x_j are equally spaced, we get a simplification. Let

$$\begin{aligned} \phi(x) &= 0, \quad x \geq 1, \text{ and } x \leq -1, \\ &= 1 - x, \quad 0 \leq x \leq 1, \\ &= 1 + x, \quad -1 \leq x \leq 0. \end{aligned}$$

Then $\phi_i(x) = \phi([x - x_i]/h)$, where $h = x_{i+1} - x_i$.