13. Finite element methods for parabolic problems

We consider the parabolic problem: Find $u = u(x,t)$ satisfying

$$u_t - \frac{\partial}{\partial x}(p\frac{\partial u}{\partial x}) + qu = f, \quad a < x < b, \quad 0 < t \le T,$$

$$u(a,t) = 0, \quad u(b,t) = 0, \quad 0 < t \le T, \qquad u(x,0) = \psi(x), \quad a < x < b.$$

Let $V^0 = \{v \in H^1(a,b) : v(a) = v(b) = 0\}$. A variational formulation of this problem is to seek u such that $u(x,0) = \psi(x)$, and for each fixed $t > 0$, $u \in V^0$ satisfies

$$(\partial u/\partial t, v) + a(u,v) = (f,v), \quad v \in V^0,$$

where $(\cdot,\cdot)$ denotes the L^2 inner product on (a,b) and now

$$a(u,v) = \int_a^b \left[p\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + quv \right] dx.$$

13.1. **Continuous time Galerkin scheme.** We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace $V_h^0 \subset V^0$ and look for an approximation u_h such that $u_h(x,0) = \psi_h(x,0)$ (where ψ_h an approximation to ψ) and for each fixed $t > 0$, $u_h \in V_h^0$ satisfies

$$(\partial u_h/\partial_t, v) + a(u_h, v) = (f,v), \quad v \in V_h^0.$$

To see what is involved in solving this problem, we write $u_h(x,t) = \sum_{j=1}^m \alpha_j(t)\phi_j(x)$, where $\{\phi_j\}_{j=1}^m$ is a basis for V_h^0. Inserting this into the variational equations, and choosing v to be each of the basis functions ϕ_i, we get

$$\sum_{j=1}^m \alpha_j'(t)(\phi_j,\phi_i) + \sum_{j=1}^m \alpha_j(t)a(\phi_j,\phi_i) = (f,\phi_i), \quad i = 1,\ldots,m.$$

Let

$$M_{ij} = (\phi_j,\phi_i), \qquad A_{ij} = a(\phi_j,\phi_i), \qquad F_i = (f,\phi_i), \qquad \alpha = (\alpha_1,\ldots,\alpha_m)^T.$$

Our equations then have the form

$$M\alpha'(t) + A\alpha = F,$$

a first order system of ordinary differential equations. If we write ψ_h in the form $\sum_{j=1}^m \beta_j\phi_j$, then we immediately get the initial condition that $\alpha_j(0) = \beta_j$. A simple example is when V_h is chosen to be the space of continuous piecewise linear functions on a uniform mesh of width h on $[a,b]$, and ψ_h is chosen to be the interpolant of ψ in this space. In that case, $\beta_j = \psi(a + jh)$.

The following error estimate is known for this semidiscrete approximation.

Theorem 36. *If V_h consists of piecewise polynomials of degree $\le r$, the initial approximation ψ_h satisfies $\|\psi - \psi_h\|_{L^2} \le Ch^{r+1}\|\psi\|_{r+1}$, and u is sufficiently smooth, then for $t \ge 0$,*

$$\|u(t) - u_h(t)\|_{L^2} \le Ch^{r+1}\left[\|\psi\|_{r+1} + \int_0^t \|u_t\|_{r+1}\, ds \right].$$

13.2. **Fully discrete schemes: Finite Differences in Time.** One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate u_t by the backward Euler approximation, we get the scheme: Find $U^n \in V_h^0$, satisfying $U^0(x) = \psi_h(x)$ and for $n = 0, 1, \ldots, N-1$ (with $T = Nk$),

$$([U^{n+1} - U^n]/k, v) + a(U^{n+1}, v) = (f^{n+1}, v) \quad v \in V_h^0.$$

Using the matrices defined previously, and defining $U^n(x) = \sum_{j=1}^m \alpha_j^n \phi_j(x)$, the discrete variational formulation above corresponds to the linear system

$$(M + kA)\alpha^{n+1} = M\alpha^n + kF^{n+1}, \qquad n = 0, 1, \ldots.$$

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find $U^n \in V_h^0$, satisfying $U^0(x) = \psi_h(x)$ and for $n = 0, 1, \ldots, N-1$,

$$([U^{n+1} - U^n]/k, v) + a([U^{n+1} + U^n]/2, v) = ([f^{n+1} + f^n]/2), \quad v \in V_h^0.$$

In this case, we get the linear system

$$(M + \tfrac{1}{2}kA)\alpha^{n+1} = (M - \tfrac{1}{2}kA)\alpha^n + k(F^{n+1} + F^n)/2, \qquad n = 0, 1, \ldots.$$

For the backward Euler method, we have the following error estimate ($t_n = nk$).

Theorem 37. *Under the assumptions of the previous theorem, we have*

$$\|u(t_n) - U^n\| \le Ch^{r+1}\Big[\|\psi\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1}\Big] + k\int_0^{t_n} \|u_{tt}(s)\|\, ds, \quad n \ge 0.$$

13.3. **Fully discrete schemes: Finite Elements in Time.** Instead of obtaining a fully discrete method by discretizing in time using finite differences, we now consider two methods for discretizing in time using finite elements. The first is the continuous Galerkin method: We let $0 = t_0 < t_1 < \cdots t_N = T$ be a partition of $[0, T]$ and let S_k be a finite element space consisting of continuous piecewise polynomials of degree $\le q$ in the time variable t. Then define $W_{h,k}$ to be the tensor product space $W_{h,k} = V_h \otimes S_k$. For example, if $q = 1$ and we consider the time slab $\Omega \times [t_{n-1}, t_n]$, we can write a function in $W_{h,k}$ in the form

$$w^{hk} = [(t - t_{n-1})/k]v_h^n(x) + [(t_n - t)/k]v_h^{n-1}(x).$$

We then define $U^{h,k} \in W_{h,k}$ such that

$$\int_0^T [(U_t^{h,k}, v_t) + a(U^{h,k}, v_t)]\, dt = \int_0^T (f, v_t)\, dt, \quad \text{for all } v \in W_{h,k}.$$

While this appears to be a global problem in time, in fact it is a marching scheme, i.e., we can compute $U^{h,k}$ on $[t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$, successively by solving

$$\int_{t_{n-1}}^{t_n} [(U_t^{h,k}, w) + a(U^{h,k}, w)]\, dt = \int_{t_{n-1}}^{t_n} (f, w)\, dt, \quad \text{for all } w \in V_h \otimes P^{q-1}([t_{n-1}, t_n]),$$

where $P^{q-1}([t_{n-1}, t_n])$ denotes the set of polynomials of degree $\leq q-1$ on the interval $[t_{n-1}, t_n]$. To see this, consider the case of $q = 1$, piecewise linear in time. If we choose

$$v = \begin{cases} v^{n-1}(x), & 0 \leq t \leq t_{n-1} \\ [(t_n - t)/k]v^{n-1}(x) + [(t - t_{n-1})/k]v^n(x), & t_{n-1} \leq t \leq t_n, \\ v^n(x), & t \geq t_n, \end{cases}$$

then $v_t = [v^n(x) - v^{n-1}(x)]/k$ for $t_{n-1} \leq t \leq t_n$ and zero elsewhere. Hence, the integral from 0 to T reduces to an integral over $[t_{n-1}, t_n]$ and by choosing $v^n(x)$ and $v^{n-1}(x)$ appropriately, we can get any function $w \in V_h \otimes P^0$.

Notice also that in the case of $q = 1$, if we write

$$U^{h,k} = [(t - t_{n-1})/k]U_h^n(x) + [(t_n - t)/k]U_h^{n-1}(x),$$

then

$$\int_{t_{n-1}}^{t_n} [(U_t^{h,k}, w) + a(U^{h,k}, w)]\, dt = (U_h^n(x) - U_h^{n-1}(x), w) + \frac{k}{2}[a(U_h^n(x), w) + a(U_h^{n-1}(x), w)],$$

so we get a type of Crank-Nicholson-Galerkin scheme, where the right hand side is averaged.

A second possibility is to use the discontinuous Galerkin approach. Let

$$w_+^n = \lim_{t \to t_n+} w(t), \qquad w_-^n = \lim_{t \to t_n-} w(t), \qquad \text{and} \qquad [w^n] = w_+^n - w_-^n.$$

We now define S_k as the set of all *discontinuous* piecewise polynomials of degree $\leq q$ on the mesh on $[0, T]$ and $W_{h,k} = V_h \otimes S_k$. Then we seek $U \in W_{h,k}$ as the solution of

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)]\, dt + \sum_{n=1}^{N} ([U^{n-1}], w_+^{n-1}) + (U_-^0, w_+^0)$$
$$= (\psi, w_+^0) + \int_0^{t_N} (f, w)\, dt, \quad \text{for all } w \in W_{h,k}.$$

Since the finite element space is discontinuous in time, we can choose w so that it is non-zero only on the subinterval $[t_{n-1}, t_n]$. We again get a time marching scheme that determines U successively on $[t_{n-1}, t_n]$ by solving

$$\int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)]\, dt + (U_+^{n-1}, w_+^{n-1}) = (U_-^{n-1}, w_+^{n-1}) + \int_{t_{n-1}}^{t_n} (f, w)\, dt, \quad \text{for all } w \in W_{h,k}.$$

On the first subinterval, we will have

$$\int_{t_0}^{t_1} [(U_t, w) + a(U, w)]\, dt + (U_+^0, w_+^0) = (\psi, w_+^0) + \int_0^{t_1} (f, w)\, dt.$$

Note that the true solution will satisfy these equations, since $u_+^{n-1} = u_-^{n-1}$.

In the continuous scheme, we have a single value for U at $t = t_n$. In the discontinuous scheme, we have two values, one from the minus side and one from the plus side. So, if we choose $q = 1$, then on the subinterval $[t_{n-1}, t_n]$, we are writing

$$U = [(t - t_{n-1}/k]U_-^n(x) + [(t_n - t)/k]U_+^{n-1}(x).$$