14. Finite Difference Methods for Elliptic Equations in 2 Dimensions

14.1. **The Dirichlet problem for Poisson's equation.** We consider the finite difference approximation of the boundary value problem:

Problem P: $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$.

For simplicity, we first consider the case when Ω is the unit square $(0,1)\times(0,1)$. To obtain a finite difference approximation, we place a mesh of width h with sides parallel to the coordinate axes on $\bar{\Omega}$ (Ω together with its boundary $\partial\Omega$ and denote the set of mesh points lying inside Ω by Ω_h and the set of mesh points lying on the $\partial\Omega$ by $\partial\Omega_h$. We then seek numbers u_{ij} as approximations to the true solution $u(ih, jh)$, where $i, j = 0, 1, \dots, N$ and $Nh = 1$. To obtain u_{ij}, we derive a system of equations that approximate the equations determining the true solution $u(ih, jh)$, i.e., the equations

$$-\Delta u(ih, jh) = f(ih, jh), \qquad (ih, jh) \in \Omega.$$

To get these approximate equations, we use Taylor series expansions, i.e, we write

$$u(x \pm h, y) = u(x,y) \pm h\frac{\partial u}{\partial x}(x,y) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\xi_\pm, y),$$

for some $x \le \xi_+ \le x+h$ and $x-h \le \xi_- \le x$. Adding these equations, we get

$$\begin{aligned} u(x+h,y) - 2u(x,y) + u(x-h,y) &= h^2\frac{\partial^2 u}{\partial x^2}(x,y) \\ &\quad + \frac{h^4}{24}\left(\frac{\partial^4 u}{\partial x^4}(\xi_+, y) + \frac{\partial^4 u}{\partial x^4}(\xi_-, y)\right) = h^2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(\xi, y), \end{aligned}$$

where $x-h \le \xi \le x+h$ and we have used the Mean Value Theorem for sums in the last step, i.e., if $g_i \ge 0$ and $\sum_{i=1}^M g_i = 1$, then there is a number c satisfying $\min x_i \le c \le \max c_i$ such that $\sum_{i=1}^M g_i f(x_i) = f(c)$. Using similar expansions in the y variable, we get

$$u(x, y+h) - 2u(x,y) + u(x, y-h) = h^2\frac{\partial^2 u}{\partial y^2}(x,y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial y^4}(x, \eta),$$

where $y-h \le \eta \le y+h$. Adding these equations and dividing by h^2, we get

$$\begin{aligned} \Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= \frac{1}{h^2}\{u(x+h,y) + u(x-h,y) \\ &\quad + u(x,y+h) + u(x,y-h) - 4u(x,y)\} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi, y) + \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x, \eta). \end{aligned}$$

Defining a finite difference operator

$$\Delta_h u(x,y) = \frac{1}{h^2}\{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)\},$$

and supposing that

$$\max_{(x,y)\in\bar{\Omega}}\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| \le M_4, \qquad \max_{(x,y)\in\bar{\Omega}}\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right| \le M_4,$$

we get

$$|\,\Delta\, u(ih,jh) - \Delta_h\, u(ih,jh)| \le \frac{M_4}{6}h^2.$$

We then use the discrete Laplace operator Δ_h to define a set of discrete equations from which we can determine u_{ij}, i.e., we consider the problem:

Problem P_h: Find a mesh function $U_h = (u_{ij})$ (i.e., U_h is only defined at the mesh points), such that

$$-\,\Delta_h\, U_h = f \text{ on } \Omega_h, \qquad U_h = g \text{ on } \partial\Omega_h.$$

This is a system of $(N+1)^2$ linear equations for the $(N+1)^2$ unknowns u_{ij}, $i, j = 0, \dots N$. We next consider the form of these equations in the special case $h = 1/4$.

Since the boundary values u_{00}, u_{10}, u_{20}, u_{30}, u_{40}, u_{01}, u_{02}, u_{03}, u_{04}, u_{41}, u_{42}, u_{43}, u_{44}, u_{14}, u_{24}, u_{34} are all given by the corresponding values of g, we need only determine the values u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}. The equations for these are:

$$\begin{aligned} 4u_{11} - u_{21} - u_{12} - u_{01} - u_{10} &= h^2 f_{11}, \\ 4u_{21} - u_{31} - u_{11} - u_{20} - u_{22} &= h^2 f_{21} \end{aligned}$$

and so on, where $f_{ij} = f(ih,jh)$. Rewriting in matrix form, we get

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = \begin{pmatrix} h^2 f_{11} + u_{10} + u_{01} \\ h^2 f_{21} + u_{20} \\ h^2 f_{31} + u_{30} + u_{41} \\ h^2 f_{12} + u_{02} \\ h^2 f_{22} \\ h^2 f_{32} + u_{42} \\ h^2 f_{13} + u_{03} + u_{14} \\ h^2 f_{23} + u_{24} \\ h^2 f_{33} + u_{34} + u_{43} \end{pmatrix}$$

As in the approximation of two-point boundary value problems, we again get a matrix that is sparse and symmetric.

To see that this linear system always has a unique solution, we can again use a discrete maximum principle (whose proof is similar to the 1-dimensional case).

As in the 1-dimensional case, we can also use the discrete maximum principle to bound the error between the true and approximate solutions. To do so, we use an analogous argument to first establish the following stability result.

Theorem 38. *Let $v(x,y)$ be a function defined on $\Omega_h \cup \partial\Omega_h$. Then*

$$\max_{\Omega_h \cup \partial\Omega_h} |v| \le \max_{\partial\Omega_h} |v| + \frac{1}{2} \max_{\Omega_h} |\,\Delta_h\, v|.$$

We then proceed as before to derive an error estimate.

Theorem 39. *Suppose u is the solution of Problem P and U_h is the solution of Problem P_h. Then*

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le \frac{1}{2} \max_{\Omega_h} |\,\Delta_h u - \Delta u|.$$

Proof. Set $v = u - U_h$, where we now consider the restriction of u to the mesh, so that we can view as a function defined on the mesh. Then $v = 0$ on $\partial\Omega_h$ and

$$\Delta_h v = \Delta_h u - \Delta_h U_h = \Delta_h u - \Delta u + \Delta u - \Delta_h U_h = \Delta_h u - \Delta u + f - f = \Delta_h u - \Delta u.$$

By Theorem 38,

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le \max_{\partial\Omega_h} |u - U_h| + \frac{1}{2} \max_{\Omega_h} |\,\Delta_h u - \Delta u| = \frac{1}{2} \max_{\Omega_h} |\,\Delta_h u - \Delta u|.$$

□

Corollary 9. *If $u \in C^4(\Omega)$, then $\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le h^2 M_4/12$.*

14.2. **Extensions to domains with curved boundaries.** We now consider the same boundary value problem, but on a more general domain Ω with a smooth boundary. For simplicity, we restrict to a convex domain. Let $E_h = \{(ih, jh), i, j \text{ integers}\}$ and set $\Omega_h = \Omega \cap E_h$. We write $\Omega_h = \Omega_h^0 + \Omega_h^*$, where

$$\Omega_h^0 = \{(x, y) \in \Omega_h : (x \pm h, y), (x, y \pm h) \in \Omega_h\}, \qquad \Omega_h^* = \Omega_h - \Omega_h^0,$$

i.e., mesh points are in Ω_h^0 if their 4 nearest neighbors are also in Ω_h. Ω_h^* then denotes the remainder of the interior mesh points. We then define $\partial\Omega_h$ to be the neighbors of points in Ω_h^* which lie on the intersection of at least one mesh line and $\partial\Omega$. For points in Ω_h^0, the operator Δ_h defined previously is well defined, but for points in Ω_h^*, we must modify the definition. Consider the case where $(x, y) \in \Omega_h^*$, $(x + h, y)$ and $(x, y + h) \in \Omega_h$, but $(x - h, y)$ and $(x, y - h)$ both lie outside of Ω. Then there will be points $(x - \alpha h, y)$ and $(x, y - \beta h)$ that lie on $\partial\Omega_h$ for some $0 < \alpha, \beta < 1$. At the point (x, y), we then define

$$\begin{aligned}\Delta_h v(x, y) = \frac{2}{h^2}\Big\{ &\frac{1}{\alpha + 1} v(x + h, y) + \frac{1}{\alpha(\alpha + 1)} v(x - \alpha h, y) + \frac{1}{\beta + 1} v(x, y + h) \\ &+ \frac{1}{\beta(\beta + 1)} v(x, y - \beta h) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) v(x, y)\Big\} \qquad \text{(Shortly-Weller formula).}\end{aligned}$$

Note that for $\alpha = \beta = 1$, we recover the previous formula. Using Taylor series expansions, one can show that for all $v \in C^3(\bar{\Omega})$,

$$|\,\Delta_h v(x, y) - \Delta v(x, y)| \le 2M_3 h/3, \qquad M_3 = \max_{\bar{\Omega}} \left[\max \left|\frac{\partial^3 v}{\partial x^3}\right|, \left|\frac{\partial^3 v}{\partial y^3}\right|\right],$$

but that the formula does not give an $O(h^2)$ approximation unless $\alpha = \beta = 1$. Using our previous analysis, we might expect that the error $|u - U_h|$ would be only $O(h)$, since the error was bounded by the maximum of the local truncation errors. However, using a different technique, (discrete Green's functions) it is possible to show that the error is still $O(h^2)$.

14.3. **Other approaches to approximation on domains with curved boundaries.** The approach using discrete Green's functions can also be used to derive error estimates for other approximation schemes. A simple scheme is to define U_h as the solution of

$$-\Delta_h U_h(P) = f(P), \quad P \in \Omega_h^0, \qquad U_h(P) = g(P'), \quad P \in \Omega_h^*,$$

where P' is one of the neighbors of P on $\partial\Omega_h$. In this case, we only use the standard 5 point difference approximation to the Laplacian. The result is:

$$|u(P) - U_h(P)| \le M_1 h + \frac{M_4 d^2}{96} h^2, \qquad M_1 = \max_{\bar{\Omega}}(\max |\partial u/\partial x|, |\partial u/\partial y|).$$

Note that the crude approximation of the boundary condition gives only an $O(h)$ error estimate.

14.4. **Other boundary conditions.** We next consider the boundary condition

$$\alpha(P)u(P) + \beta(P)\frac{\partial u}{\partial n}(P) = g(P).$$

Consider first the case of a point on a straight boundary, say $x = 1$, and $0 < y < 1$. At the boundary point $(1, y)$, an $O(h)$ approximation to $\partial u/\partial n = \partial u/\partial x$ is given by $[u(1, y) - u(1 - h, y)]/h$, so the boundary condition would be approximated by:

$$\alpha(1, y)u(1, y) + \beta(1, y)[u(1, y) - u(1 - h, y)]/h = g(1, y).$$

An $O(h^2)$ approximation to $\partial u/\partial x$ is given by the centered difference: $[u(1 + h, y) - u(1 - h, y)]/(2h)$. This introduces an new unknown at the point $1 + h, y$ outside the domain. Hence, we need an additional equation. Assuming that the solution is smooth and the partial differential equation holds on the boundary as well, we can use the 5 point difference approximation to the Laplacian applied at the boundary point, i.e., we have the equation

$$U_h(1 + h, y) + U_h(1 - h, y) + U_h(1, y + h) + U_h(1, y - h) - 4U_h(1, y) = h^2 f(1, y).$$

This equation can be used to eliminate the new unknown.

If the boundary is curved, draw the normal line through the point P and assume it intersects a mesh line at a point C where C lies between the mesh points A and B. Then we approximate $\partial u/\partial n(P)$ by $[u(P) - u(C)]/|P - C|$, where $|P - C|$ denotes the distance between P and C and $u(C)$ is defined by linear interpolation using $u(A)$ and $u(B)$, i.e.,

$$u(C) = \frac{|B - C|}{|B - A|}u(A) + \frac{|C - A|}{|B - A|}u(B).$$

Inserting this formula gives a linear relation equation involving $u(A)$. $u(B)$, and $u(P)$.

14.5. **Higher order approximations.** To get higher order approximations to $\Delta\, u(x, y)$, we need to take more points at a larger distance from (x, y). Using Taylor series expansions, we have

$$u(x \pm kh, y) = u(x,y) \pm kh\frac{\partial u}{\partial x}(x,y) + \frac{k^2h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) \pm \frac{k^3h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{k^4h^4}{24}\frac{\partial^4 u}{\partial x^4}(x,y) \pm \frac{k^5h^5}{120}\frac{\partial^5 u}{\partial x^5}(x,y) + \frac{k^6h^6}{6!}\frac{\partial^6 u}{\partial x^6}(\xi_\pm, y).$$

Hence,

$$u(x+kh,y) + u(x-kh,y) - 2u(x,y) = 2\frac{k^2h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) + 2\frac{k^4h^4}{24}\frac{\partial^4 u}{\partial x^4}(x,y) + O(h^6).$$

For $k = 1, 2$, this gives

$$u(x+h,y) + u(x-h,y) - 2u(x,y) = h^2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(x,y) + O(h^6),$$

$$u(x+2h,y) + u(x-2h,y) - 2u(x,y) = 4h^2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{16h^4}{12}\frac{\partial^4 u}{\partial x^4}(x,y) + O(h^6).$$

Taking 16 times the first equation minus the second equation, we get

$$16u(x+h,y) + 16u(x-h,y) - u(x+2h,y) - u(x-2h,y) - 30u(x,y) = 12h^2\frac{\partial^2 u}{\partial x^2}(x,y) + O(h^6).$$

Taking a similar expansion in the y variable, we get

$$\{16[u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)] - [u(x+2h,y) + u(x-2h,y) + u(x,y+2h) + u(x,y-2h)] - 60u(x,y)]\}/(12h^2) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + O(h^4).$$

Note that in the case of the unit square, this higher order approximation cannot be used at interior mesh points at a distance h from the boundary of Ω, since it would involve mesh points outside the domain. At these points, we can use the 5-point difference approximation without affecting the overall accuracy of the method. To see this, one uses that fact that

$$|u(P) - U_h(P)| \le C \max_{Q\in\Omega_h^0} |\,\Delta\, u(Q) - \Delta_h\, u(Q)| + h^2 \max_{Q\in\Omega_h^*} |\,\Delta\, u(Q) - \Delta_h\, u(Q)|.$$

In Ω_h^0, which we now interpret to mean those points $(x,y) \in \Omega_h$ such that $(x \pm 2h, y)$ and $(x, y \pm 2h) \in \Omega_h \cup \partial\Omega_h$, we have $|\,\Delta\, u(Q) - \Delta_h\, u(Q)| \le Ch^4$, while in Ω_h^*, points at a distance h from $\partial\Omega_h$, we have $|\,\Delta\, u(Q) - \Delta_h\, u(Q)| \le Ch^2$. Inserting these results, we obtain the error estimate $|u(P) - U_h(P)| \le Ch^4$.

On domains with curved boundaries, the situation is much more complicated. Use of the Shortly-Weller formula would decrease the rate of convergence. So instead, we consider finite element methods, which handle these difficulties in a more natural way.