MATH 575 ASSIGNMENT 1

1a. The 3-point difference operator

$$D_h^2 u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

satisfies

$$|D_h^2 u(x) - u''(x)| \leq \frac{h^2}{12} M_4,$$

where M_4 is the maximum of $|u''''|$ on the interval $(x-h, x+h)$. Find the best possible approximation of $u''(x)$ using the *five* values $u(x-2h)$, $u(x-h)$, $u(x)$, $u(x+h)$, and $u(x+2h)$.

1b. State and prove an error estimate for this approximation analogous to the one given above.

2. The 5-point difference approximation to the Laplacian Δ may be written in geometrical form (called a stencil) as

$$\frac{1}{h^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that a 9-point difference approximation of the form

$$\Delta_h = \frac{1}{h^2}\begin{pmatrix} \alpha & \beta & \alpha \\ \beta & \gamma & \beta \\ \alpha & \beta & \alpha \end{pmatrix}$$

approximates Δ to no better than $O(h^2)$, no matter how we choose α, β and γ.

HINT: Use Taylor series expansions and observe that terms of odd order cancel out. Try to simplify your solution sheet by omitting intermediate calculations.

The next problem asks you to write a computer code in Matlab. If you are not familiar with Matlab, some basic instructions and commands can be found at:

`http://math.rutgers.edu/~falk/math575/matlab1.html`

If you do not have access to a computer that has Matlab, please stop by my office and I will arrange for an account for you on a Math Department computer. Rather than entering your commands directly into Matlab, it is usually preferable to write your program in a file which ends in the extension ".m" If your program is stored in the file *program.m*, you can then execute it by typing *program* at the Matlab prompt. However, before doing so, make sure that your current directory is the one containing the file *program.m*.

3. Consider the two-point boundary value problem

$$-u'' = f, \qquad 0 < x < 1, \qquad u(0) = 0, \qquad u(1) = 0.$$

a) Write *Matlab* programs which approximate the solution of this boundary value problem using a finite difference scheme (based on the three point difference operator given in Problem 1a) for the two choices of f given by

(i) $f = xe^x(x+3)$, for which the solution is $u = x(1-x)e^x$.

(ii) $f = 1$ when $0 < x < c$, and $f = 0$ when $c < x < 1$, where $c = \sqrt{2}/2$. The exact solution is given by $u = -x^2/2 + (c - c^2/2)x$ for $0 < x < c$ and $c^2(1-x)/2$ for $c < x < 1$. In this case the differential equation will hold for all $0 < x < 1$ except for $x = c$ where f is not defined.

Do this for $n = 4, 8, 16, 32, 64$ subintervals. For each of these values of n, have your program compute the maximum of the absolute value of the error at the mesh points, i.e., $E_h = \max_{0 \le i \le n} |u(x_i) - u_h(x_i)|$, using a uniform mesh of width $h = 1/n$. Use format long.

b) Now suppose that the error $E_h \approx Ch^\alpha$ for some constants C and α. Then

$$\frac{E_{h_1}}{E_{h_2}} \approx \frac{Ch_1^\alpha}{Ch_2^\alpha} \qquad \text{implies} \qquad \alpha \approx \ln \frac{E_{h_1}}{E_{h_2}} / \ln \frac{h_1}{h_2}.$$

Determine α for each two successive values of h for both boundary value problems. i.e., $E_h = \max_{0 \le i \le n} |u(x_i) - u_h(x_i)|$, using a uniform mesh of width $h = 1/n$. Use format long. Hand in a table showing the values of E_h and the values of α, along with a copy of your program.

c) Comment on the relationship of the results you obtained to the error estimates derived in class. In particular, since the solution u of (ii) is a piecewise polynomial, u and all its derivatives are continuous for $0 < x < c$ and $c < x < 1$. Are u, u', and u'', also continuous at $x = c$?