MATH 575 ASSIGNMENT 5

1. Let x be the solution of $Ax = b$ and consider approximations to x given by the iteration:

$$x^{k+1} = [I - \alpha A]x^k + \alpha b,$$

where A is the $n \times n$ matrix given by

$$A = \begin{pmatrix} 1+c & c & \cdots & c \\ c & 1+c & \cdots & c \\ \vdots & & \ddots & \vdots \\ c & c & \cdots & 1+c \end{pmatrix}.$$

1a. Show that if $c \geq 0$, then for all $x \neq 0$, $x^T Ax > 0$, i.e., A is positive definite.

1b. Show that $x = (1, 1, \ldots, 1)^T$ is an eigenvector of A and find the corresponding eigenvalue.

1c. For $i = 1, \ldots, n-1$, let x^i denote the vector defined by $x^i_i = 1$, $x^i_n = -1$, and $x^i_j = 0$ if $j \neq i$ and $j \neq n$. Show that x^i is an eigenvector of A and find the corresponding eigenvalue.

1d. Use these results and the result proved in class about the convergence of this iterative method to find the values of α for which this iteration converges when $c = 1/n$.

2. Consider the solution of the linear system $Ax = b$ by the iterative method

$$x^{k+1} = (1-\alpha)x^k + \alpha P^{-1}(b - Qx^k),$$

where $A = P + Q$ and α is a positive constant.

2a. Find the iteration matrix M which satisfies

$$x - x^{k+1} = M(x - x^k),$$

where x is the exact solution of the linear system.

2b. Show that M can also be written in the form $I - \alpha P^{-1}A$, where I is the identity matrix. matrix.

2c. If $P^{-1}A$ is symmetric and positive definite, for what values of α will this iteration converge?

3. We wish to approximate the solution of the problem

$$-\Delta u = 1, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

where $\Omega = (0,1) \times (0,1)$. Write a *Matlab* program to approximate this problem using a finite difference method where Δu is approximated by the 5-point difference approximation and a mesh of size $h = 1/N$. Order the unknowns from left to right in each row, starting from the smallest value of y, and then proceed to the next row.

3a. To make sure the matrix A is being generated properly, print out and hand in the matrix A for the case $N = 4$ that is generated by your computer code.

3b. Next, solve the linear system of equations $Ax = b$ for the choice $N = 100$ using *Matlab's* built in `pcg` function, which solves by the preconditioned conjugate gradient method.

First try the command `[x0,fl0,rr0,it0,rv0] = pcg(A,b,1e-8,100);`
Read the document page to see what the various arguments stand for. Check the flag `fl0` to see if the method converges before the maximum number of iterations is reached.

Next try the preconditioned version using the incomplete Cholesky factorization with zero fill-in as the preconditioner. Since this requires that A be a sparse matrix, we first need to convert A to a sparse matrix, if you have not stored it in this form already. This is done by typing:

```
As = sparse(A);

L = ichol(As);

[x1,fl1,rr1,it1,rv1] = pcg(As,b,1e-8,100,L,L');
```

Hand in the values of `rr1` and `it1`, along with a copy of your computer program.

4a. Consider the approximation of the boundary value problem $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is the unit square. Suppose we approximate this problem using a finite difference method with Δ_h taken to be the usual 5-point difference approximation to Δu on the grid $\Omega_h = \{(ih, jh) : 0 < i, j < n$ where $h = 1/n$. Hence, there are $(n-1)^2$ unknowns. Show that the grid function

$$\phi_{m,p}(x, y) = \sin(m\pi x)\sin(p\pi y), \qquad (x, y) \in \Omega_h, \qquad 1 \le m, p \le n-1$$

is an eigenfunction of $-\Delta_h$ (i.e., show that $-\Delta_h\phi_{m,p}(x, y) = \lambda_{m,p}\phi_{m,n}(x, y)$ and that the corresponding eigenvalue $\lambda_{m,p}$ is

$$(2/h^2)[2 - \cos(\pi mh) - \cos(\pi ph)].$$

Hint: Use the trigonometric identity $\sin a + \sin b = 2\sin([a + b]/2)\cos([a - b]/2)$.

4b. Show that the largest eigenvalue is $(4/h^2)[1 + \cos(\pi h)]$.

4c. Show that the smallest eigenvalue is $(4/h^2)[1 - \cos(\pi h)]$.

4d. Using Taylor series expansions, show that

$$\cos z \ge 1 - z^2/2.$$

4e. Show that the smallest eigenvalue is $\le 2\pi^2$.

4f. Show that if $h \le \sqrt{2}/\pi$, then the largest eigenvalue is $\ge 4/h^2$.

4g. Show that the condition number κ of the matrix corresponding to $-\Delta_h$ satisfies

$$\kappa = \lambda_n/\lambda_1 \ge 2/(\pi^2 h^2) = O(h^{-2}), \qquad \text{for } h \text{ sufficiently small.}$$