MATH 575 ASSIGNMENT 6

1. Consider the approximation of the one dimensional heat equation $u_t = \sigma u_{xx}$ (for constant σ) by the generalized Crank-Nicholson method:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\sigma}{h^2}\{(1-\theta)[u_{j+1}^n - 2u_j^n + u_{j-1}^n] + \theta[u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}]\}, \qquad 0 \le \theta \le 1.$$

a) Let $\tau(x,t)$ denote the local trunctation error for this method at the point (x,t) defined by

$$\begin{aligned}\tau(x,t) = u_t(x,t) - \sigma u_{xx}(x,t) - \frac{u(x,t+k) - u(x,t)}{k} \\ + \frac{\sigma}{h^2}\{(1-\theta)[u(x+h,t) - 2u(x,t) + u(x-h,t)] \\ + \theta[u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)]\},\end{aligned}$$

Show that the local truncation error of this method is $O(k + h^2)$.

b) If u satisfies the heat equation and is sufficiently smooth (so that $u_{tt} = \sigma u_{xxt}$) and $\theta = 1/2$, show that the error in part (a) is $O(k^2) + O(h^2)$.

2. Write a Matlab program to approximate the initial boundary value problem:

$$u_t = u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = x(1-x)$$

by the explicit method

$$[u_j^{n+1} - u_j^n]/k = [u_{j+1}^n - 2u_j^n + u_{j-1}^n]/h^2$$

for the three mesh sizes (i) $k = 1/256$, $h = 1/16$, (ii) $k = 1/512$, $h = 1/16$, and (iii) $k = 1/1024$, $h = 1/16$. Plot the approximate solution and true solution on the same plot at $t = 1$. Explain your results. Note that the true solution

$$u(x,t) = \frac{8}{\pi^3} \sum_{n=1,3,5,\ldots} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin(n\pi x)$$

and the first two terms of the series give a good approximation. See the file *heatsoln.m* for some help in defining and plotting the exact solution.

3. Suppose the partial differential equation

$$u_t = \sigma u_{xx} + \beta u_x + \alpha u$$

is approximated by the difference equation

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\sigma}{h^2}[U_{j+1}^n - 2U_j^n + U_{j-1}^n] + \frac{\beta}{2h}[U_{j+1}^n - U_{j-1}^n] + \alpha U_j^n$$

and the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ and initial condition $u(x,0) = \phi(x)$ are satisfied exactly by U_j^n at the appropriate mesh points. If α, β, and σ are constants with $\sigma > 0$, $\alpha \neq 0$, $0 < h < 2\sigma/|\beta|$, and $\sigma k/h^2 \leq 1/2$, show that

a)

$$E^{n+1} \leq E^n[1 + |\alpha|k] + k\tau,$$

where

$$E^n = \max_{0 \leq j \leq J} |e_j^n|, \quad e_j^n = u(jh, nk) - U_j^n, \quad \tau = \max |\tau(x,t)|,$$

$\tau(x,t)$ denoting the local truncation error at the point (x,t).

b) By iterating this inequality and using the fact that $1 + x \leq e^x$, show that for $t = nk$

$$E^n \leq \frac{e^{|\alpha|t} - 1}{|\alpha|}\tau.$$

c) The local trucation error $\tau(x,t)$ has the form $O(h^s + k^t)$. What are s and t ? (A derivation is not required).