MATH 575 ASSIGNMENT 7

1. Suppose the partial differential equation

$$u_t = \sigma u_{xx} + \beta u_x + \alpha u$$

is approximated by the difference equation

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\sigma}{h^2}[U_{j+1}^n - 2U_j^n + U_{j-1}^n] + \frac{\beta}{2h}[U_{j+1}^n - U_{j-1}^n] + \alpha U_j^n.$$

1a. Find the amplification factor for this difference scheme.

1b. Use the result of (1a) to show that if α, β, and σ are constants with $\sigma > 0$ and $0 < h < 2\sigma/|\beta|$, then the scheme is stable for $\sigma k/h^2 \leq 1/2$. Hint: Use the fact that $|G| = [G\bar{G}]^{1/2}$ and first show that

$$G\bar{G} = C_{-1}^2 + C_0^2 + C_1^2 + 2C_0[C_{-1} + C_1]\cos(ph) + 2C_{-1}C_1\cos(2ph).$$

Next use the fact that C_{-1}, C_0, and C_1 are all ≥ 0 and $\cos\theta \leq 1$ to show that $G\bar{G} \leq [C_{-1} + C_0 + C_1]^2$.

2. Consider the approximation of $u_t = \sigma u_{xx}$ given for $0 \leq \theta \leq 1$ by

$$[U_j^{n+1} - U_j^n]/k = \frac{\sigma}{h^2}\{(1-\theta)[U_{j+1}^n - 2U_j^n + U_{j-1}^n] + \theta[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}]\}.$$

Show that for $0 \leq \theta \leq 1/2$, the scheme is stable when $\sigma k/h^2 \leq 1/[2(1-2\theta)]$ and for $1/2 \leq \theta \leq 1$, the scheme is unconditionally stable. HINT: Use the identity $\cos(x) = 1 - 2\sin^2(x/2)$ to write the amplification factor

$$\lambda = \left[1 - \frac{4\sigma k}{h^2}(1-\theta)\sin^2(ph/2)\right] \Big/ \left[1 + \frac{4\sigma k}{h^2}\theta\sin^2(ph/2)\right]$$

and then show that under the conditions given above, $|\lambda| \leq 1$.

3. The simplest three level explicit scheme for the solution of the heat equation $u_t = u_{xx}$ is given by

$$[U_j^{n+1} - U_j^{n-1}]/(2k) = [U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2.$$

Show that the scheme is unstable for all values of the constant $r = k/h^2 > 0$.

HINT: Write the equation as a two level system:

$$U_j^{n+1} = 2k[U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2 + V_j^n,$$
$$V_j^{n+1} = U_j^n,$$

and show that the eigenvalues of the amplification matrix G are given by

$$\lambda = -4r\sin^2(ph/2) \pm \sqrt{16r^2\sin^4(ph/2) + 1}.$$