MATH 575 ASSIGNMENT 8

1a. Starting from the divergence theorem

$$\int_\Omega \operatorname{div} \boldsymbol{\phi}\, dx = \int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{n}\, ds,$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to Ω, derive the integration by parts formula

$$\int_\Omega [\mathbf{grad} w \cdot \boldsymbol{v} + w \operatorname{div} \boldsymbol{v}]\, dx = \int_{\partial\Omega} w\, \boldsymbol{v} \cdot \boldsymbol{n}\, ds.$$

Hint: Set $\boldsymbol{\phi} = w\boldsymbol{v}$.

1b. Let $H(\text{div};\Omega) = \{\boldsymbol{v} : \boldsymbol{v} \in L^2(\Omega), \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\}$. Consider the variational problem: Given $\boldsymbol{f} \in [L^2(\Omega)]^2$, find $\boldsymbol{u} \in H(\text{div};\Omega)$ satisfying:

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \boldsymbol{v} \in H(\text{div};\Omega).$$

What partial differential equation and boundary condition are satisfied by $\boldsymbol{u}$? Hint: Use the integration by parts formula of part (a) for an appropriate choice of w.

1c. Find a variational formulation that produces the same partial differential equation as in (1b), but with the boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Hint: You will need to modify the space $H(\text{div}, \Omega)$.

1d. Let $\boldsymbol{V}_h$ be a finite dimensional subspace of $H(\text{div};\Omega)$ and consider the Galerkin approximation scheme: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ satisfying:

$$\int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{v}\, dx + \int_\Omega \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \quad \boldsymbol{v} \in \boldsymbol{V}_h.$$

If $\{\boldsymbol{\phi}_j\}_{j=1}^m$ is a basis for $\boldsymbol{V}_h$, we can write $\boldsymbol{u}_h = \sum_{j=1}^m \alpha_j \boldsymbol{\phi}_j$. The approximation scheme given above is then equivalent to a linear system $A\alpha = b$. Give formulas for the entries A_{ij} of the matrix A and b_i of the vector b in terms of the basis functions $\{\boldsymbol{\phi}_j\}_{j=1}^m$ and the function $\boldsymbol{f}$.

1e. Show that the problem of (1d) has a unique solution. State any facts you are using to obtain your result.

1f. Suppose Ω is the unit square $(0,1) \times (0,1)$. Let Γ_N denote the bottom boundary of Ω, i.e., $\Gamma_N = \{(x,y) : y = 0, 0 \le x \le 1\}$ and Γ_D denote the remaining part of the boundary of Ω.

Find a variational formulation that produces the same partial differential equation as in (1b), but with the boundary conditions $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_D and $\operatorname{div} \boldsymbol{u} = g$ on Γ_N. Hint: You need to find a bilinear form $a(\boldsymbol{u}, \boldsymbol{v})$, a linear functional $F(\boldsymbol{v})$, and a space $\boldsymbol{V} \subset H(\text{div}, \Omega)$ incorporating appropriate boundary conditions, such that the variational formulation will have the form: Find $\boldsymbol{u} \in \boldsymbol{V}$ such that $a(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v})$, $\boldsymbol{v} \in \boldsymbol{V}$. Hint: Write $\partial\Omega = \Gamma_D + \Gamma_N$ and combine the ideas in parts (b) and (c).