1. Finite Difference Methods for Elliptic Equations

1.1. **The Dirichlet problem for Poisson's equation.** We consider the finite difference approximation of the boundary value problem:

Problem P: $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$.

For simplicity, we first consider the case when Ω is the unit square $(0,1)\times(0,1)$. To obtain a finite difference approximation, we place a mesh of width h with sides parallel to the coordinate axes on $\bar\Omega$ (Ω together with its boundary $\partial\Omega$) and denote the set of mesh points lying inside Ω by Ω_h and the set of mesh points lying on the $\partial\Omega$ by $\partial\Omega_h$. We then seek numbers u_{ij} as approximations to the true solution $u(ih, jh)$, where $i, j = 0, 1, \ldots, N$ and $Nh = 1$. To obtain u_{ij}, we derive a system of equations that approximate the equations determining the true solution $u(ih, jh)$, i.e., the equations

$$-\Delta u(ih, jh) = f(ih, jh), \qquad (ih, jh) \in \Omega.$$

To get these approximate equations, we use Taylor series expansions, i.e, we write

$$u(x \pm h, y) = u(x,y) \pm h\frac{\partial u}{\partial x}(x,y) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y) \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\xi_\pm, y),$$

for some $x \le \xi_+ \le x+h$ and $x-h \le \xi_- \le x$. Adding these equations, we get

$$\begin{aligned} u(x+h,y) - 2u(x,y) + u(x-h,y) &= h^2\frac{\partial^2 u}{\partial x^2}(x,y) \\ &+ \frac{h^4}{24}\left(\frac{\partial^4 u}{\partial x^4}(\xi_+, y) + \frac{\partial^4 u}{\partial x^4}(\xi_-, y)\right) = h^2\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(\xi, y), \end{aligned}$$

where $x - h \le \xi \le x+h$ and we have used the Mean Value Theorem for sums in the last step, i.e., if $g_i \ge 0$ and $\sum_{i=1}^M g_i = 1$, then there is a number c satisfying $\min x_i \le c \le \max c_i$ such that $\sum_{i=1}^M g_i f(x_i) = f(c)$. Using similar expansions in the y variable, we get

$$u(x, y+h) - 2u(x,y) + u(x, y-h) = h^2\frac{\partial^2 u}{\partial y^2}(x,y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial y^4}(x, \eta),$$

where $y - h \le \eta \le y + h$. Adding these equations and dividing by h^2, we get

$$\begin{aligned} \Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) &= \frac{1}{h^2}\{u(x+h,y) + u(x-h,y) \\ &+ u(x,y+h) + u(x,y-h) - 4u(x,y)\} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi, y) + \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x, \eta). \end{aligned}$$

Defining a finite difference operator

$$\Delta_h u(x,y) = \frac{1}{h^2}\{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)\},$$

and supposing that

$$\max_{(x,y)\in\bar\Omega}\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| \le M_4, \qquad \max_{(x,y)\in\bar\Omega}\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right| \le M_4,$$

we get

$$|\Delta u(ih, jh) - \Delta_h u(ih, jh)| \leq \frac{M_4}{6} h^2.$$

We then use the discrete Laplace operator Δ_h to define a set of discrete equations from which we can determine u_{ij}, i.e., we consider the problem:

Problem P_h: Find a mesh function $U_h = (u_{ij})$ (i.e., U_h is only defined at the mesh points), such that

$$-\Delta_h U_h = f \text{ on } \Omega_h, \qquad U_h = g \text{ on } \partial\Omega_h.$$

This is a system of $(N+1)^2$ linear equations for the $(N+1)^2$ unknowns u_{ij}, $i, j = 0, \ldots N$. We next consider the form of these equations in the special case $h = 1/4$.

Since the boundary values u_{00}, u_{10}, u_{20}, u_{30}, u_{40}, u_{01}, u_{02}, u_{03}, u_{04}, u_{41}, u_{42}, u_{43}, u_{44}, u_{14}, u_{24}, u_{34} are all given by the corresponding values of g, we need only determine the values u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}. The equations for these are:

$$4u_{11} - u_{21} - u_{12} - u_{01} - u_{10} = h^2 f_{11},$$
$$4u_{21} - u_{31} - u_{11} - u_{20} - u_{22} = h^2 f_{21}$$

and so on, where $f_{ij} = f(ih, jh)$. Rewriting in matrix form, we get

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = \begin{pmatrix} h^2 f_{11} + u_{10} + u_{01} \\ h^2 f_{21} + u_{20} \\ h^2 f_{31} + u_{30} + u_{41} \\ h^2 f_{12} + u_{02} \\ h^2 f_{22} \\ h^2 f_{32} + u_{42} \\ h^2 f_{13} + u_{03} + u_{14} \\ h^2 f_{23} + u_{24} \\ h^2 f_{33} + u_{34} + u_{43} \end{pmatrix}$$

Note several properties of this linear system: (1) The right hand side is known; (2) the symmetry properties of the matrix depend on the ordering of the elements; (3) this is an example of a sparse matrix – many zero elements. If we decreased the mesh size further, we would introduce more zeroes.

Questions: (1) Does this type of linear system have a unique solution? (2) How does the error between the true and approximate solutions depend on the mesh size h? (3) What is an efficient way to solve the linear system when the number of equations becomes large?

To see that this linear system always has a unique solution, we first establish a property of the discrete Laplace operator Δ_h, known as a discrete maximum principle.

Theorem 1. *(i) If v is a function defined on $\Omega_h \cup \partial\Omega_h$ and satisfies $\Delta_h v(x, y) \geq 0$ for all $(x, y) \in \Omega_h$, then $\max_{\Omega_h} v \leq \max_{\partial\Omega_h} v$.*
(ii) Alternatively, if v satisfies $\Delta_h v(x, y) \leq 0$ for all $(x, y) \in \Omega_h$, then $\min_{\Omega_h} v \geq \min_{\partial\Omega_h} v$.

Proof. The proof is by contradiction. Let $(x_0, y_0) \in \Omega_h$ at which v has a maximum, i.e., $v(x_0, y_0) = M$, where $M \geq v(x, y)$ for all $(x, y) \in \Omega_h$ and $M > v(x, y)$ for $(x, y) \in \partial\Omega_h$. By assumption, $\Delta_h v(x_0, y_0) \geq 0$. Hence,

$$M = v(x_0, y_0) \leq \frac{1}{4}\{v(x_0 + h, y_0) + v(x_0 - h, y_0) + v(x_0, y_0 + h) + v(x_0, y_0 - h)\}.$$

But $M \geq v(x, y)$ then implies that $v(x_0 \pm h, y_0) = M$ and $v(x_0, y_0 \pm h) = M$. Repeating this argument, we eventually conclude that $v(x, y) = M$ for all $(x, y) \in \Omega_h \cup \partial\Omega_h$. This contradicts our initial assumption, so (i) follows. To establish (ii), we let $w(x, y) = -v(x, y)$. Then $\Delta_h w(x, y) = -\Delta_h v(x, y) \geq 0$, so by (i), $\max_{\Omega_h}[-v(x, y)] \leq \max_{\partial\Omega_h}[-v(x, y)]$. But $\max[-v(x, y)] = -\min v(x, y)$, so $-\min_{\Omega_h} v(x, y) \leq -\min_{\partial\Omega_h}[v(x, y)]$. Then (ii) follows by multiplying by (-1), which reverses the sign of the inequality. □

We note that we can extend this result to non-square domains.

Theorem 2. *The linear system of equations corresponding to the difference equations*

$$-\Delta_h U_h(x, y) = f(x, y), \quad (x, y) \in \Omega_h, \qquad U_h(x, y) = g(x, y), \quad (x, y) \in \partial\Omega_h$$

has a unique solution.

Proof. We use the fact that a square linear system $Az = b$ will have a unique solution if and only if the only solution of the homogeneous system $Az = 0$ is $z = 0$. Hence, we need to show that the only solution to Problem P_h when f and g are zero is $U_h = 0$. But by Theorem 1, since $\Delta_h U$ is both ≥ 0 and ≤ 0, both the maximum and minimum of $U(x, y)$ occur on $\partial\Omega_h$. Hence, $0 \leq U(x, y) \leq 0$ for all $(x, y) \in \Omega_h \cup \partial\Omega_h$ and so $U \equiv 0$. □