3. A posteriori error estimates and an adaptive finite element method

Having computed an approximation to the solution of a partial differential equation using a finite element method defined on an initial mesh, we consider the best way to obtain a more accurate solution. The simplest approach is simply to uniformly refine all triangles in the triangulation $\mathcal{T}_h$. This can be done by subdividing each triangle into four triangles by connecting the midpoints of the edges of each triangle. The problem with this approach is that in regions where the true solution is smooth, we expect a fairly course mesh to already produce an accurate approximation. Hence, we can reduce the work involved by seeking to refine the mesh only in regions where the solution is not so smooth. A finite element method in which we successively adapt the mesh is called an *adaptive finite element method*. It is based on having local estimates for the error, so that we can refine the mesh where the error is largest. Such estimates are based on the ideas of *a posteriori error analysis*, which computes an approximation to the error using the just computed approximate solution. To study these ideas, we first introduce a theoretical tool needed for the analysis.

3.1. A posteriori error estimates.

In the error estimates derived previously, we obtained *a priori* bounds for the error which did not depend on the computed approximate solution, but only on the unknown solution u. We now seek error bounds that we can compute directly from the computed solution u_h. The ultimate aim of these bounds, as explained above, is to use them to determine a strategy to refine the mesh to compute better approximations.

Assume that Ω is a convex polygon in the plane and consider again the model problem

$$Lu \equiv -\operatorname{div}(p\nabla u) + qu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

which we write in weak form: Find $u \in V \equiv \mathring{H}^1(\Omega)$ such that

$$a(u,v) = F(v), \quad v \in \mathring{H}^1(\Omega),$$

where

$$a(u,v) = \int_\Omega [p\nabla u \cdot \nabla v + quv]\, dx, \qquad F(v) = \int_\Omega fv\, dx.$$

Let $u_h \in V_h$ satisfy:

$$a(u_h, v) = F(v), \quad v \in V_h.$$

Theorem 6. *Let h_T denote the diameter of the triangle T and h_e the length of the edge e. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_{L^2(\Omega)} \le C \left(\sum_{T\in\mathcal{T}_h} \eta_T^2 \right)^{1/2},$$

where

$$\eta_T^2 = h_T^4 \|R_T(u_h)\|^2_{L^2(T)} + \frac{1}{2}\sum_{e\in\partial T} h_e^3 \|R_e(u_h)\|^2_{L^2(e)}, \qquad R_T(u_h) = (f - Lu_h)|_T,$$

$$R_e(u_h) = -p\nabla u_h|_{T'} \cdot n_{T'} - p\nabla u_h|_T \cdot n_T = -p\left(\nabla u_h|_T \cdot n_T - \nabla u_h|_{T'} \cdot n_T\right),$$

and e is the common edge shared by the triangles T and T'. (Note that $n_{T'} = -n_T$).

Proof. We use the duality argument from the proof of the L^2 estimates, i.e., we let $w \in \mathring{H}^1(\Omega)$ be the solution of

$$a(v, w) = (u - u_h, v), \quad \text{for all } v \in \mathring{H}^1(\Omega).$$

Writing the integral over Ω as the sum of the integrals over $T \in \mathcal{T}_h$, using Galerkin orthogonality, and integrating by parts, we get

$$\begin{aligned}\|u - u_h\|^2_{L^2(\Omega)} &= a(u - u_h, w) = a(u - u_h, w - w_I) = (f, w - w_I) - a(u_h, w - w_I)\\ &= \sum_{T\in\mathcal{T}_h} [(f, w - w_I)_T - a(u_h, w - w_I)_T]\\ &= \sum_{T\in\mathcal{T}_h} \left[(f - Lu_h, w - w_I)_T - \int_{\partial T} p\nabla u_h \cdot n(w - w_I)\, ds\right].\end{aligned}$$

We now write the integrals over ∂T as the sum of the integrals over the three edges of T. Note that except for the edges lying on $\partial\Omega$, each edge will appear exactly twice when we sum over all the triangles. Since w and w_h are zero on $\partial\Omega$, the boundary edge integrals will be zero. For each interior edge e of T, let T' denote the other triangle in $\mathcal{T}_h$ that also has e as an edge. From the definitions of R_T and R_e, we can rewrite the above in the form

$$\|u - u_h\|^2_{L^2(\Omega)} = \sum_{T\in\mathcal{T}_h} \left[(R_T(u_h), w - w_I)_T + \frac{1}{2}\sum_{e\in\partial T}\int_e R_e(u_h)(w - w_I)\, ds\right],$$

where the factor 1/2 occurs because each term involving $R_e(u_h)$ occurs twice in the sum. Now

$$(R_T(u_h), w - w_I)_T \le \|R_T(u_h)\|_{L^2(T)}\|w - w_I\|_{L^2(T)} \le Ch_T^2\|R_T(u_h)\|_{L^2(T)}\|w\|_{H^2(T)},$$

and

$$\int_e R_e(u_h)(w - w_I)\, ds \le \|R_e(u_h)\|_{L^2(e)}\|w - w_I\|_{L^2(e)} \le Ch_e^{3/2}\|R_e(u_h)\|_{L^2(e)}\|w\|_{H^2(T)}.$$

This last estimate is not something we have seen before. To get such an estimate, we can start from the integration by parts formula

$$\int_T \operatorname{div} \boldsymbol{v} z\, dx + \int_T \boldsymbol{v} \cdot \nabla z\, dx = \int_{\partial T} z\boldsymbol{v} \cdot \boldsymbol{n}\, ds.$$

To simplify, but still see how the powers of h enter, consider the special triangle with vertices $(0, 0)$, $(h, 0)$, and $(0, h)$. Let e_1 denote the edge joining $(0, 0)$ and $(h, 0)$ and $\boldsymbol{v}$ be the function $(x/h, y/h - 1)$. Note that $\operatorname{div} \boldsymbol{v} = 2/h$ and $|\boldsymbol{v}_i| \le 1$ on T. On the edge e_1, $\boldsymbol{n} = (0, -1)$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 1$. On the edge e_3 joining $(0, h)$ and $(0, 0)$, $\boldsymbol{n} = (-1, 0)$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. On the edge e_2 joining $(h, 0)$ and $(0, h)$, $\boldsymbol{n} = (1/\sqrt{2}, 1/\sqrt{2})$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Choosing $z = (w - w_I)^2$, we get

$$\begin{aligned}\int_{\partial T} z\boldsymbol{v} \cdot \boldsymbol{n}\, ds = \int_{e_1} (w - w_I)^2\, ds &\le (2/h)\int_T (w - w_I)^2\, dx + 2\int_T |w - w_I||\nabla(w - w_I)|\, dx\\ &\le (2/h)\|w - w_I\|^2_{0,T} + 2\|w - w_I\|_{0,T}\|\nabla(w - w_I)\|_{0,T}.\end{aligned}$$

Since

$$\|w - w_I\|_{0,T} \le Ch_T^2\|w\|_{2,T}, \qquad \|\nabla(w - w_I)\|_{0,T} \le Ch_T\|w\|_{2,T},$$

we get

$$\|w - w_I\|_{L^2(e)} \le C h_T^{3/2} \|w\|_{2,T}.$$

More generally, we would get on shape regular meshes, that on a general edge e,

$$\|w - w_I\|_{L^2(e)} \le C h_e^{3/2} \|w\|_{2,T}.$$

Combining these results, we get

$$\|u - u_h\|_{L^2(\Omega)}^2 \le C \sum_{T \in \mathcal{T}_h} \left[h_T^2 \|R_T(u_h)\|_{L^2(T)} \|w\|_{H^2(T)} + \frac{1}{2} \sum_{e \in \partial T} h_e^{3/2} \|R_e(u_h)\|_{L^2(e)} \|w\|_{H^2(T)} \right].$$

Since $\sum_i a_i b_i \le \left(\sum_i a_i^2\right)^{1/2} \left(\sum_i b_i^2\right)^{1/2}$, we get

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|R_T(u_h)\|_{L^2(T)} \|w\|_{H^2(T)}$$
$$\le \left(\sum_{T \in \mathcal{T}_h} h_T^4 \|R_T(u_h)\|_{L^2(T)}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \|w\|_{H^2(T)}^2 \right)^{1/2} \le \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2} \|w\|_{H^2(\Omega)},$$

and

$$\sum_{T \in \mathcal{T}_h} \frac{1}{2} \sum_{e \in \partial T} h_e^{3/2} \|R_e(u_h)\|_{L^2(e)} \|w\|_{H^2(T)}$$
$$\le \sum_{T \in \mathcal{T}_h} \left(\sum_{e \in \partial T} \frac{1}{2} h_e^3 \|R_e(u_h)\|_{L^2(e)}^2 \right)^{1/2} \left(\sum_{e \in \partial T} \frac{1}{2} \|w\|_{H^2(T)}^2 \right)^{1/2}$$
$$\le \left(\sum_{T \in \mathcal{T}_h} \frac{1}{2} \sum_{e \in \partial T} h_e^3 \|R_e(u_h)\|_{L^2(e)}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} \frac{1}{2} \sum_{e \in \partial T} \|w\|_{H^2(T)}^2 \right)^{1/2}$$
$$\le \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2} \left(\frac{3}{2} \right)^{1/2} \|w\|_{H^2(\Omega)}.$$

Using the fact that $\|w\|_{H^2(\Omega)} \le C \|u - u_h\|_{L^2(\Omega)}$ and combining these results, we get

$$\|u - u_h\|_{L^2(\Omega)} \le C \left(\sum_T \eta_T^2 \right)^{1/2}.$$

□

3.2. **Error indicators.** We now show how to use the a posteriori estimate given in Theorem 6 to develop an adaptive finite element method. We first associate to each triangle an *error indicator*

$$\eta_T^2 = h_T^4 \|R_T(u_h)\|_{L^2(T)}^2 + \frac{1}{2} \sum_{e \in \partial T} h_e^3 \|R_e(u_h)\|_{L^2(e)}^2.$$

Our a posteriori error estimate is expressed in terms of the local error indicators as:

$$\|u - u_h\|_{L_2(\Omega)} \leq c \left(\sum_T \eta_T^2 \right)^{1/2} \equiv c\eta(\mathcal{T}_h).$$

The adaptive strategy then consists of a loop with the steps SOLVE-ESTIMATE-MARK-REFINE, defined as follows: For $k \geq 0$,

- SOLVE: Computes u_k on the current mesh $\mathcal{T}_k$
- ESTIMATE: For each triangle $T \in \mathcal{T}_k$, compute the error indicator η_T.
  If $(\sum_T \eta_T^2)^{1/2}) \leq tol$, then quit.
- MARK: Select a subset of triangles $\mathcal{M}_k \subset \mathcal{T}_k$ using Dörfler marking, i.e.,
  $\eta(\mathcal{M}_k) \geq \theta\, \eta(\mathcal{T}_k)$ for $0 < \theta < 1$ (bulk chasing) with $\eta(\mathcal{M}_k)$ minimal.
- REFINE: Refines the marked elements $\mathcal{M}_k$ and outputs a conforming mesh $\mathcal{T}_{k+1}$

In two dimensions there are a number of refinement strategies, such as bisecting each triangle using newest vertex bisection illustrated in the figure below.

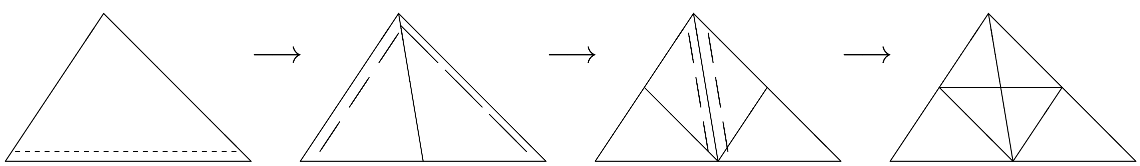

FIGURE 1. Refinement of triangles in two dimensions by newest-vertex bisection. Dashed lines indicate the refinement edges, which are the sides opposite the most recently created nodes.

To obtain a uniform refinement, one divides each triangle into four by connecting the midpoints of the edges. To get a conforming triangulation, neighboring triangles will also need to be refined by connecting the new vertex to the vertex opposite the subdivided edge.

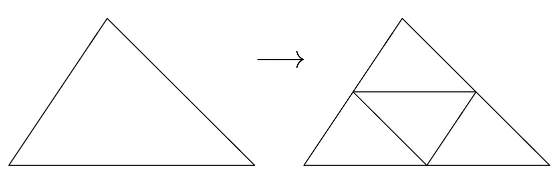

FIGURE 2. Uniform mesh refinement.