4. Approximation of Elliptic Variational Inequalities

We have previously considered approximation of variational equalities, i.e., problems of the form: Find $u \in V$ such that $a(u, v) = (f, v)$ for all $v \in V$. When $a(u, v) = a(v, u)$, this problem can also be formulated as a minimization problem, i.e., defining $J(v) = \frac{1}{2}a(v, v) - (f, v)$, we consider the problem: Find $u \in V$ such that $J(u) \leq J(v)$ for all $v \in V$.

To get a variational inequality, we let K be a closed convex subset of V (i.e., if $u, v \in K$, then for $0 \leq t \leq 1$, $(1-t)u + tv \in K$) and consider the problem: Find $u \in K$ such that $a(u, v-u) \geq (f, v-u)$ for all $v \in K$. When $a(u, v) = a(v, u)$, this problem is also equivalent to a minimization problem: Find $u \in K$ such that $J(u) \leq J(v)$ for all $v \in K$.

To see how the variational inequality is obtained from the minimization problem, let u be the solution of the minimization problem and $v \in K$. Since $u \in K$ and K is a convex set $(1-t)u + tv \in K$ for all $0 \leq t \leq 1$. Hence $J(u) \leq J((1-t)u + tv)$. Using the definition of J, rewriting $(1-t)u + tv$ as $u + t(v-u)$, and expanding the terms, we get for all $v \in K$,

$$\frac{1}{2}a(u,u) - (f,u) = J(u) \leq J((1-t)u+tv) = \frac{1}{2}a((1-t)u+tv, (1-t)u+tv) - (f, (1-t)u+tv)$$
$$\leq \frac{1}{2}a(u,u) + ta(u, v-u) + \frac{1}{2}t^2 a(v-u, v-u) - (f,u) - t(f, v-u).$$

Canceling common terms, we get

$$0 \leq ta(u, v-u) - t(f, v-u) + \frac{1}{2}t^2 a(v-u, v-u).$$

Since this inequality holds for any $0 < t < 1$, we can divide by t to get

$$0 \leq a(u, v-u) - (f, v-u) + \frac{1}{2}ta(v-u, v-u).$$

Letting $t \to 0$, we see that u must satisfy

$$a(u, v-u) \geq (f, v-u), \quad v \in K.$$

The canonical example of such a problem is the "obstacle" problem, in which $V = \mathring{H}^1(\Omega)$,

$$K = \{v \in V : v \geq \psi \text{ a.e. in } \Omega\}, \qquad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad (f, v) = \int_\Omega f\, v \, dx,$$

where $\psi(x)$ is a given function which we will assume $\in H^2(\Omega)$.

In the case of the variational equality, the variational equation says that $-\Delta u = f$ in Ω. In the case of the variational inequality describing the obstacle problem, this is no longer the case. Instead, we have that for any x, either $-\Delta u = f$ or $u = \psi$.

To get an approximation scheme, we let $V_h \subset V$ be finite dimensional and construct a closed convex subset K_h of V_h such that the following two conditions are satisfied: (i) Writing $v_h \in V_h$ as $v_h = \sum_{j=1}^M \beta_j \phi_j$, where $\{\phi_j\}$ are a basis for V_h, the set K_h should reduce to a finite number of constraints on the β_j; (ii) K_h should be a good approximation to K in a sense to be clearer in the error estimates.

In the case of the obstacle problem, a simple choice for the approximation scheme is to let V_h be the space of continuous piecewise linear functions with respect to a triangulation $\mathcal{T}_h$ of Ω (which we assume to be a convex polygon). We then define $K_h = \{v_h \in V_h : v_h \geq \psi_I \text{ for all } x \in \Omega\}$, where ψ_I is the interpolant of ψ in V_h. A key point here is that since v_h and ψ_I are piecewise linear, $v_h \in K_h$ is equivalent to requiring that $v_h(\boldsymbol{a}_i) \geq \psi_I(\boldsymbol{a}_i)$ for all vertices $\boldsymbol{a}_i$ of $\mathcal{T}_h$, so condition (i) above is satisfied. To see this, recall that on a triangle T, we may write $u_h(\boldsymbol{x}) = \sum_{i=1}^3 u_h(\boldsymbol{a}_i)\lambda_i(\boldsymbol{x})$, where λ_i is the barycentric coordinate associated to the point $\boldsymbol{a}_i$, i.e., λ_i is a linear function on T that is equal to one at $\boldsymbol{a}_i$ and equal to zero at the other two vertices of T. If $u_h(\boldsymbol{a}_i) \geq \psi_I(\boldsymbol{a}_i) = \psi(\boldsymbol{a}_i)$, then since the $\lambda_i \geq 0$, we get for all $\boldsymbol{x} \in T$, $u_h(\boldsymbol{x}) \geq \sum_{i=1}^3 \psi_I(\boldsymbol{a}_i)\lambda_i(\boldsymbol{x}) = \psi_I(\boldsymbol{x})$.

The difficulty for the error analysis is that although $V_h \subset V$, K_h is not a subset of K. The reason for this is that $\psi_I(x)$ could be $< \psi(x)$, so requiring that $v_h \geq \psi_I$ does not guarantee that $v_h \geq \psi$. In the case of variational equalities, we proved that

$$\|u - u_h\|_1 \leq \frac{M}{\alpha}\|u - v_h\|_1, \quad v \in V_h,$$

and so for C^0 piecewise linear functions, we get $\|u - u_h\|_1 \leq Ch\|u\|_2$. In the case of variational equalities, we have by integration by parts (for u sufficiently smooth) that $-\Delta u = f$ in Ω. The situation is more complicated for variational inequalities. In the case of the obstacle problem, it is known that

$$-\Delta u - f \geq 0, \qquad u \geq \psi, \qquad (u - \psi)(\Delta u + f) = 0.$$

Using these facts we can prove the following error estimate.

Lemma 9. *Let u and u_h denote the true and approximate solutions, respectively. Then if $f \in L^2(\Omega)$, $u \in H^2(\Omega)$, and $\psi \in H^2(\Omega)$,*

$$\|u - u_h\|_1 \leq Ch[\|u\|_2 + \|f\|_0 + \|\psi\|_2].$$

where $\|f\|_0 := \|f\|_{L^2(\Omega)}$, $\|u\|_1 := \|u\|_{H^1(\Omega)}$, and $\|u\|_2 := \|u\|_{H^2(\Omega)}$.

Proof. We first show that

$$a(u - u_h, u - u_h) \leq a(u - u_h, u - v_h) + (\Delta u + f, u - v_h) + (\Delta u + f, \psi_I - \psi).$$

To get this result, we write

$$\begin{aligned}
a(u - u_h, u - u_h) &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\
&\leq a(u - u_h, u - v_h) - (\Delta u, v_h - u_h) + (f, u_h - v_h) \\
&\qquad \text{since } a(u, v) = (-\Delta u, v), \qquad a(u_h, u_h - v_h) \leq (f, u_h - v_h) \\
&\leq a(u - u_h, u - v_h) - (\Delta u + f, v_h - \psi_I + \psi_I - u_h) \\
&\leq a(u - u_h, u - v_h) - (\Delta u + f, v_h - \psi_I) \\
&\qquad \text{since } -(\Delta u + f) \geq 0, \qquad \psi_I - u_h \leq 0 \\
&\leq a(u - u_h, u - v_h) - (\Delta u + f, v_h - u + u - \psi + \psi - \psi_I) \\
&\leq a(u - u_h, u - v_h) - (\Delta u + f, v_h - u + \psi - \psi_I) \\
&\qquad \text{since } -(\Delta u + f, u - \psi) = 0.
\end{aligned}$$

Hence, using the arithmetic-geometric mean inequality $ab \le \alpha a^2/2 + b^2/(2\alpha)$, the Schwarz inequality $(f, g) \le \|f\|_0 \|g\|_0$, and the properties of the bilinear form $a(u, v)$, we get

$$\begin{aligned}\alpha\|u-u_h\|_1^2 \le a(u-u_h, u-u_h) &\le M\|u-u_h\|_1\|u-v_h\|_1 \\ &\quad + \|\Delta u + f\|_0[\|v_h - u\|_0 + \|\psi - \psi_I\|_0] \\ &\le \frac{\alpha}{2}\|u-u_h\|_1^2 + \frac{M^2}{2\alpha}\|u-v_h\|_1^2 + \|\Delta u + f\|_0[\|v_h-u\|_0 + \|\psi-\psi_I\|_0].\end{aligned}$$

Choosing $v_h = u_I \in K_h$, we know

$$\|u-u_I\|_1 \le Ch\|u\|_2, \qquad \|u-u_I\|_0 \le Ch^2\|u\|_2, \qquad \|\psi-\psi_I\|_0 \le Ch^2\|\psi\|_2.$$

Inserting these results, and again using the arithmetic-geometric mean inequality, we get

$$\begin{aligned}\frac{\alpha}{2}\|u-u_h\|_1^2 &\le Ch^2\|u\|_2^2 + Ch^2[\|\Delta u\|_0 + \|f\|_0][\|u\|_2 + \|\psi\|_2] \\ &\le Ch^2[\|u\|_2^2 + \|\Delta u\|_0^2 + \|f\|_0^2 + \|\psi\|_2^2]. \\ &\le Ch^2[\|u\|_2 + \|f\|_0 + \|\psi\|_2]^2.\end{aligned}$$

In the above, C denotes a generic constant, which in not necessarily the same in any two places. However, it is independent of h, u, f, and ψ. We also use the fact that $\|\Delta u\|_0 \le C\|u\|_2$. Hence,

$$\|u-u_h\|_1 \le Ch[\|u\|_2 + \|f\|_0 + \|\psi\|_2].$$

□

To solve the approximate problem, we can use the minimization formulation. Writing $v \in K_h = \sum_{j=1}^M \beta_j\phi_j(x)$, where the ϕ_j are the C^0 piecewise linear basis functions, i.e., the piecewise linear functions that are one at vertex $\boldsymbol{a}_j$ and zero at all the other vertices, we get

$$J(v) = \frac{1}{2}a(v,v) - (f,v) = \frac{1}{2}\sum_{i=1}^M\sum_{j=1}^M \beta_i\beta_j a(\phi_i,\phi_j) - \sum_{i=1}^M \beta_i(f,\phi_i) = \beta^T A\beta - \beta^T F,$$

where A is the matrix with entries $A_{ij} = a(\phi_i, \phi_j)$, F is the column vector with entries $F_i = (f, \phi_i)$ and β is the column vector with entries β_i. Hence J is a quadratic function of the β_i. The constraint set K_h consists of functions $v \in V_h$ such that $v \ge \psi_I$. The key fact is that since v and ψ_I are linear functions on each triangle, $v - \psi_I \ge 0$ on a triangle T if and only if $v - \psi_I \ge 0$ at the vertices of T. But at a vertex $\boldsymbol{a}_k$, $v(\boldsymbol{a}_k) = \sum_{j=1}^M \beta_j\phi_j(\boldsymbol{a}_k) = \beta_k$. Hence, the constraint set K_h is equivalent to the finite set of conditions $\beta_j \ge \psi_I(\boldsymbol{a}_j)$ for $j = 1, \dots M$. This is a finite number of linear constraints on the β_j. Hence the finite dimensional problem becomes the minimization of a quadratic form subject to linear constraints, which can be solved by quadratic programming methods.