6. Iterative methods for variational inequalities

We consider the iterative solution of the approximate variational inequality: Find $u_h \in K_h$ such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \text{for all } v_h \in K_h,$$

where

$$K_h = \{v_h \in V_h : v_h \geq \psi_I, \quad x \in \Omega\}$$

and V_h is the space of continuous, piecewise linear functions on a mesh τ_h of Ω. If we write $u_h = \sum_{j=1}^M \alpha_j \phi_j$ and $v_h = \sum_{i=1}^M \beta_i \phi_i$, then the variational inequality becomes

$$\sum_{i,j} \alpha_j(\beta_i - \alpha_i) a(\phi_j, \phi_i) \geq \sum_i (\beta_i - \alpha_i)(f, \phi_i),$$

for all $\alpha, \beta \in \kappa$, where

$$\kappa = \{\beta \in R^M : \beta_i \geq \psi_I(x_i)\}.$$

Defining $A_{ij} = a(\phi_j, \phi_i)$ and $b_i = (f, \phi_i)$, we can rewrite this in the form

$$(\beta - \alpha)^T (A\alpha - b) \geq 0,$$

and then in the form

$$(\beta - \alpha)^T (\alpha - [\alpha - \rho(A\alpha - b)]) \geq 0,$$

where ρ is a positive constant. If we define $\gamma = \alpha - \rho(A\alpha - b)$, then we have that $\alpha = P_\kappa \gamma$, where $(P_\kappa \gamma)_i = \gamma_i$ if $\gamma_i \geq \psi_I(x_i)$, and $(P_\kappa \gamma)_i = \psi_I(x_i)$ if $\gamma_i < \psi_I(x_i)$. To see that this gives the solution of the problem, let I^+ denote the set of i where $\gamma_i \geq \psi_I(x_i)$ and I^- denote the set of i where $\gamma_i < \psi_I(x_i)$. Then for the choice $\alpha = P_\kappa \gamma$, we have

$$\begin{aligned}
(\beta - \alpha)^T (\alpha - \gamma) &= \sum_i (\beta - \alpha)_i (\alpha - \gamma)_i \\
&= \sum_{i \in I^+} (\beta - \alpha)_i (\alpha - \gamma)_i + \sum_{i \in I^-} (\beta - \alpha)_i (\alpha - \gamma)_i \\
&= \sum_{i \in I^+} (\beta - \gamma)_i (\gamma - \gamma)_i + \sum_{i \in I^-} (\beta_i - \psi_I(x_i))(\psi_I(x_i) - \gamma)_i \geq 0,
\end{aligned}$$

since the first sum is zero and the second sum is the product of nonnegative terms.

This form motivates the iterative method: Find $\alpha^{k+1} \in \kappa$ such that

$$(\beta - \alpha^{k+1})^T (\alpha^{k+1} - [\alpha^k - \rho(A\alpha^k - b)]) \geq 0, \qquad \beta \in \kappa.$$

The solution in this case is given by $\alpha^{k+1} = P_\kappa[\alpha^k - \rho(A\alpha^k - b)]$.

To analyze this method, we note that

$$(P_\kappa y)_i - (P_\kappa z)_i = \begin{cases} y_i - z_i, & \text{if } y_i \geq \psi_I(x_i), z_i \geq \psi_I(x_i), \\ y_i - \psi(x_i), & \text{if } y_i \geq \psi_I(x_i), z_i < \psi_I(x_i), \\ \psi(x_i) - z_i, & \text{if } y_i < \psi_I(x_i), z_i \geq \psi_I(x_i), \\ 0, & \text{if } y_i < \psi_I(x_i), z_i < \psi_I(x_i). \end{cases}$$

Hence, $|(P_\kappa y)_i - (P_\kappa z)_i| \le |y_i - z_i|$ and so

$$\|P_\kappa y - P_\kappa z\|^2 = \sum_{i=1}^{N} |(P_\kappa y)_i - (P_\kappa z)_i|^2 \le \sum_{i=1}^{N} |y_i - z_i|^2 = \|y - z\|^2.$$

Then we get

$$\begin{aligned}\|\alpha - \alpha^{k+1}\| &= \|P_\kappa[\alpha - \rho(A\alpha - b)] - P_\kappa[\alpha^k - \rho(A\alpha^k - b)]\| \\ &\le \|\alpha - \rho(A\alpha - b) - [\alpha^k - \rho(A\alpha^k - b)]\| = \|[I - \rho A](\alpha - x^k)\|,\end{aligned}$$

so we are exactly in the case of the corresponding analysis of variational equalities.

Since the convergence of this method will be very slow, we need to adapt the faster methods for the solution of the linear systems arising from the discretization of partial differential equations to the case of variational inequalities.