9. Finite difference methods for the transport equation

We next consider the approximation of the initial value problem for the transport equation in one space dimension, i.e., Find $u(x,t)$ satisfying

$$Lu \equiv \frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = f, \qquad u(x,0) = \phi(x).$$

When $f \equiv 0$ and α is a constant, it is easy to check that the solution of this problem is given by $u(x,t) = \phi(x - \alpha t)$. Note that the solution at the point (x,t) only depends on the initial data at the point $x - \alpha t$. This point (or for more general problems, the set of points) on the initial line that determine the solution at the point (x,t) is called the domain of dependence of the point (x,t). Consider the case when $\alpha > 0$.

The following condition, called the CFL (Courant-Friedrich-Lewy) condition, is necessary for convergence of an approximation scheme.

Theorem 10. *A necessary condition for convergence is that the numerical domain of dependence contains the true domain of dependence.*

Thus, we see that choosing forward difference approximations for both u_t and u_x will not lead to a convergent approximation scheme. In that case, we would obtain in the case $f \equiv 0$

$$[U_j^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_j^n]/h = 0.$$

Setting $\lambda = \alpha k/h$, this is equivalent to:

$$U_j^{n+1} = (1+\lambda)U_j^n - \lambda U_{j+1}^n.$$

Note that U_j^{n+1} depends on data to the right of x and directly below it, while the exact solution depends on initial data to the left of x. Hence, the CFL condition is not satisfied and the scheme does not converge as $h, k \to 0$.

If we consider instead the scheme:

$$[U_j^{n+1} - U_j^n]/k + \alpha[U_j^n - U_{j-1}^n]/h = f_j^n, \quad \text{i.e.,}$$
$$U_j^{n+1} = (1-\lambda)U_j^n + \lambda U_{j-1}^n + k f_j^n,$$

then the solution at the point $(x,t) = (x, nk)$ depends on the initial data in the interval $[x - nh, x]$. The CFL condition will be satisfied if we choose $nh \geq \alpha nk$, i.e., $\lambda = \alpha k/h \leq 1$.

For this simple problem, convergence is easy to prove. Let v_j^n be a mesh function and

$$L_{h,k} v_j^n = [v_j^{n+1} - v_j^n]/k + \alpha[v_j^n - v_{j-1}^n]/h.$$

Then

$$v_j^{n+1} = (1-\lambda)v_j^n + \lambda v_{j-1}^n + k L_{h,k} v_j^n.$$

For $0 \leq \lambda \leq 1$, we easily obtain the following stability result.

Lemma 10. *For* $0 \leq \lambda = ak/h \leq 1$, *and* $nk \leq T$,

$$\max_j |v_j^n| \leq \max_j |v_j^0| + T \max_{j,n} |L_{h,k} v_j^n|.$$

Proof. Let $V^n = \max_j |v_j^n|$. Then for $0 \le n \le N$, with $Nk = T$,

$$|v_j^{n+1}| \le (1-\lambda)|v_j^n| + \lambda|v_{j-1}^n| + k|L_{h,k}v_j^n| \\ \le (1-\lambda)V^n + \lambda V^n + k\max_j |L_{h,k}v_j^n| \le V^n + k\max_{j,n}|L_{h,k}v_j^n|.$$

Hence,

$$|v_j^{n+1}| \le V^n + k\max_{j,n}|L_{h,k}v_j^n|.$$

Since this equation holds for all values of j, we get

$$V^{n+1} = \max_j |v_j^{n+1}| \le V^n + k\max_{j,n}|L_{h,k}v_j^n|.$$

Iterating this equation, we obtain

$$V^n \le V^0 + nk\max_{j,n}|L_{h,k}v_j^n| \le T\max_{j,n}|L_{h,k}v_j^n|.$$

□

To obtain an error estimate, we apply this stability result to $v_j^n = u(jh, nk) - U_j^n$. An easy calculation shows that

$$|L_{h,k}v_j^n| = |L_{h,k}u(jh,nk) - L_{h,k}U_j^n| = |L_{h,k}u(jh,nk) - f_j^n| \\ = |L_{h,k}u(jh,nk) - Lu(jh,nk)| = O(k) + O(h).$$

Since $v_j^0 = 0$, we get for $0 \le n \le N$,

$$\max_{j,n}|u(jh,nk) - U_j^n| \le T[O(k) + O(h)].$$

There are other schemes one might consider, and we shall examine these later. Even though all of the following schemes satisfy the CFL condition, not all of them are convergent. The CFL condition is necessary for convergence, but not sufficient. The following are all explicit schemes.

$$\frac{U_j^{n+1} - U_j^n}{k} + \alpha\frac{U_{j+1}^n - U_{j-1}^n}{2h} = f_j^n,$$

$$\frac{U_j^{n+1} - (1/2)U_{j+1}^n - (1/2)U_{j-1}^n}{k} + \alpha\frac{U_{j+1}^n - U_{j-1}^n}{2h} = f_j^n, \quad \text{Lax -Friedrichs.}$$

$$\frac{U_j^{n+1} - U_j^n}{k} + \alpha\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \alpha^2\frac{k}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = f_j^n, \quad \text{Lax Wendroff,}$$

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} + \alpha\frac{U_{j+1}^n - U_{j-1}^n}{2h} = f_j^n, \quad \text{leapfrog,}$$

An example of an implicit scheme is given by the Crank-Nicolson method.

$$\frac{U_j^{n+1} - U_j^n}{k} + \alpha\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1} + U_{j+1}^n - U_{j-1}^n}{4h} = \frac{1}{2}[f_j^{n+1} + f_j^n].$$

The derivation of all but one of the above schemes is fairly straightforward. Below we give a derivation of the Lax-Wendroff scheme (for constant α), when $f = 0$.

A Taylor series expansion gives

$$u(x,t+k) = u(x,t)+ku_t(x,t)+\frac{k^2}{2}u_{tt}(x,t)+O(k^3) = u(x,t)-k\alpha u_x(x,t)+\frac{k^2}{2}u_{tt}(x,t)+O(k^3).$$

Differentiating the equation, we get

$$u_{tt} = -[\alpha u_x]_t = -\alpha[u_t]_x = -\alpha[-\alpha u_x]_x = \alpha^2 u_{xx}.$$

Inserting this into the Taylor series, we have

$$u(x,t+k) = u(x,t) - k\alpha u_x(x,t) + \frac{\alpha^2 k^2}{2}u_{xx}(x,t) + O(k^3).$$

If we then use the difference approximations

$$u_x(x,t) = \frac{u(x+h,t)-u(x-h,t)}{2h} + O(h^2),$$

$$u_{xx}(x,t) = \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} + O(h^2),$$

we get

$$u(x,t+k) = u(x,t) - \frac{k\alpha}{2h}[u(x+h,t)-u(x-h,t)] + \frac{\alpha^2 k^2}{2h^2}[u(x+h,t)-2u(x,t)+u(x-h,t)] + O(h^2) + O(k^3).$$

Approximating $u(x,t)$ at the point $x = jh$, $t = nk$ by U_j^n, dividing by k, and dropping the $O(h^2)$ and $O(k^3)$ terms, we obtain the Lax-Wendroff scheme.

10. Finite difference methods for the wave equation

We next consider the approximation of the initial value problem for the wave equation in one space dimension, i.e., Find $u(x,t)$ satisfying

$$Lu \equiv \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f, \quad a < x < b, \quad t > 0,$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

We might also give boundary conditions, e.g., $u(a,t) = u(b,t) = 0$.

A closed form solution (d'Alembert solution) can be given for this problem when $f \equiv 0$, i.e.,

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds.$$

Note that the solution at the point (x^*,t^*) depends on the initial data only in the interval $[x^*-ct^*, x^*+ct^*]$, (called the domain of dependence).

For the heat equation, this property is not true, i.e., if we consider the pure initial value problem:

$$\frac{\partial u}{\partial t} = \sigma\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \phi(x), \quad -\infty < x < \infty,$$

then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(\xi-x)^2/(4t)}}{\sqrt{4\pi t}} \phi(\xi)\, d\xi.$$

Hence $u(x,t)$ depends on the initial data $\phi(x)$ everywhere in the interval $(-\infty, \infty)$.

A simple finite difference approximation of the wave equation is given by:

$$[U_j^{n+1} - 2U_j^n + U_j^{n-1}]/k^2 = c^2[U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2 + f_j^n,$$

a three-level explicit scheme. If we set $\lambda = (ck/h)$, we may rewrite this scheme in the form

$$U_j^{n+1} = 2[1-\lambda^2]U_j^n + \lambda^2[U_{j+1}^n + U_{j-1}^n] - U_j^{n-1} + k^2 f_j^n.$$

To proceed, we need initial values at two levels to get started. We take $U_j^0 = \phi_j$. To get a value for U_j^1, we can use a Taylor series approximation. The simplest is to approximate $u_t(x,0)$ by $[u(x,k) - u(x,0)]/k$. Then

$$u(x,k) \approx u(x,0) + ku_t(x,0) = \phi(x) + k\psi(x).$$

More accurate approximations can be obtained if ϕ is sufficiently differentiable and the wave equation is satisfied at $t = 0$. Then

$$u_{tt}(x,0) = c^2 u_{xx}(x,0) = c^2\phi''(x),$$

so

$$u(x,k) = u(x,0) + ku_t(x,0) + \frac{k^2}{2}u_{tt}(x,0) + O(k^3) \approx \phi(x) + k\psi(x) + \frac{k^2}{2}c^2\phi''(x).$$

Next, we look at the numerical domain of dependence of the difference equation. Since U_j^{n+1} depends on the values U_{j-1}^n, U_j^n, U_{j+1}^n, and U_j^{n-1}, it is easy to see that the value of the approximate solution at the point (x^*, t^*) will depend on the initial data only in the interval $[x^* - (h/k)t^*, x^* + (h/k)t^*]$. Hence, the CFL condition will be satisfied, i.e., the numerical domain of dependence contains the true domain of dependence of the wave equation if

$$h/k \geq c, \qquad \text{i.e.,} \qquad \lambda = ck/h \leq 1.$$

Hence, we can conclude that in general, the solution of the difference equation cannot converge to the exact solution of the wave equation as $h \to 0$ and $k \to 0$ for constant $\lambda = ck/h > 1$.

10.1. **First order symmetric hyperbolic systems.** We note that one can reduce the 1-dimensional wave equation to a first order system by introducing new variables. Let $w_1 = cu_x$, $w_2 = u_t$, and write $w = (w_1, w_2)^T$. Then we have $(w_1)_t = c(w_2)_x$, $(w_2)_t = u_{tt} = c^2u_{xx} + f = c(w_1)_x + f$. Hence, we have the system

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

with initial conditions

$$w_1(x,0) = c\phi'(x), \qquad w_2(x,0) = \psi(x).$$

More generally, we can consider a system of n first order differential equations for the n functions $w_1, \cdots, w_n$ that are the components of the vector $\boldsymbol{w}$. This will have the form

$$\frac{\partial}{\partial t}\boldsymbol{w} + A\frac{\partial}{\partial x}\boldsymbol{w} = F,$$

where A is an $n \times n$ matrix. In the simplest case of a symmetric hyperbolic system, A has n real eigenvalues $\lambda_1, \cdots, \lambda_n$ and a complete set of eigenvectors. Some of the numerical methods developed for the transport problem can then be applied to approximate this system.