11. Stability of difference schemes for pure initial value problems with periodic initial data

In this section, we develop algebraic conditions that allow us to determine for which values of the mesh sizes h and k, a given difference scheme for an initial value problem is stable. The problems we consider are of the form:

$$L\boldsymbol{u} \equiv \boldsymbol{u}_t - A\boldsymbol{u} = \boldsymbol{f}, \qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),$$

where $\boldsymbol{u}$ is a vector with m components and A is an $m \times m$ matrix of differential operators in x.

11.1. **Two-level explicit schemes.** Examples:

Heat equation $\qquad Lu = u_t - \sigma u_{xx}, \qquad A = \sigma\partial^2/\partial x^2.$

Transport equation: $\qquad Lu = u_t + \alpha u_x, \qquad A = -\alpha\partial/\partial x.$

The wave equation $u_{tt} - c^2 u_{xx}$ does not fit this framework in that form, but does fit if we rewrite it as a first order system by defining $w = cu_x$, $v = u_t$, and setting $\boldsymbol{u} = (v, w)^T$. Then

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} 0 & c\partial/\partial x \\ c\partial/\partial x & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = 0,$$

so

$$\boldsymbol{u}_t - A\boldsymbol{u} = 0, \qquad A = \begin{pmatrix} 0 & c\partial/\partial x \\ c\partial/\partial x & 0 \end{pmatrix}.$$

We consider the problem with periodic initial data and look for periodic solutions (period 2π). We illustrate the theory by first considering *explicit* schemes of the form:

$$L_{h,k}\boldsymbol{U}_j^n = \boldsymbol{f}_j^n,$$

where

$$L_{h,k}\boldsymbol{U}_j^n = k^{-1}[\boldsymbol{U}_j^{n+1} - \sum_{q=-Q}^{Q} C_q \boldsymbol{U}_{j+q}^n].$$

In the above Q is a fixed positive number and $C_q = C_q(h,k)$ are matrices that are independent of j and n.

Example: explicit scheme for the heat equation

$$L_{h,k}U_j^n = k^{-1}\left\{U_j^{n+1} - \left(1 - \frac{2\sigma k}{h^2}\right)U_j^n - \frac{\sigma k}{h^2}[U_{j+1}^n + U_{j-1}^n]\right\}.$$

Then

$$m = 1, \qquad C_{-1} = \sigma k/h^2 = C_1, \qquad C_0 = 1 - 2\sigma k/h^2.$$

Since we are considering a pure initial value problem, we can define $L_{h,k}\boldsymbol{U}^n$ at x for all x, not just the mesh points $x = jh$, i.e.,

$$L_{h,k}\boldsymbol{U}^n(x) = k^{-1}[\boldsymbol{U}^{n+1}(x) - \sum_{q=-Q}^{Q} C_q \boldsymbol{U}^n(x+qh)].$$

Hence, we can define the numerical solution for all values of x, even though in practice we only compute it at the mesh points, i.e.,

$$L_{h,k}\boldsymbol{U}^n(x) = \boldsymbol{f}^n(x), \qquad \boldsymbol{U}(x,0) = \boldsymbol{u}_0(x).$$

Note the the approximate solution is still only defined at the discrete time levels $t = nk$.

To analyze such problems, we shall use the L^2-norm on vector-valued functions of x, i.e.,

$$\|\boldsymbol{U}\|^2 = \int_0^{2\pi} |\boldsymbol{U}|^2 \, dx,$$

where $|\cdot|$ denotes the Euclidean vector norm and we are assuming that $\boldsymbol{U}(x+2\pi) = \boldsymbol{U}(x)$. For fixed $T > 0$, we shall use the term *discrete function* to mean a function $\boldsymbol{U}^n(x)$, $0 \le n \le N = T/k$, i.e., the sequence of functions, $\boldsymbol{U}^0, \boldsymbol{U}^1, \cdots, \boldsymbol{U}^N$. On such functions, we use the norm $\max_{0 \le n \le N} \|\boldsymbol{U}^n\|$.

We then say that the numerical method determined by the difference operator $L_{h,k}$ is *stable* in this norm if there exists a constant C independent of h and k such that

$$\max_{0 \le n \le N} \|\boldsymbol{V}^n\| \le C \left[\max_{0 \le n \le N-1} \|L_{h,k}\boldsymbol{V}^n\| + \|\boldsymbol{V}^0\| \right]$$

for all discrete functions $\boldsymbol{V}^n(x)$, $0 \le n \le N$, with $Nk = T$.

If a scheme is stable, then using the triangle inequality, we obtain the following error estimate.

$$\max_{0 \le n \le N} \|\boldsymbol{u}^n - \boldsymbol{U}^n\| \le C \left[\max_{0 \le n \le N-1} \|L_{h,k}\boldsymbol{u}^n - (L\boldsymbol{u})^n\| + \|\boldsymbol{u}^0 - \boldsymbol{U}^0\| \right].$$

If we choose $\boldsymbol{U}^0 = \boldsymbol{u}^0$, then the scheme is convergent if it is consistent, i.e., if

$$\max_{0 \le n \le N-1} \|L_{h,k}\boldsymbol{u}^n - (L\boldsymbol{u})^n\| \to 0 \quad \text{as} \quad h, k \to 0.$$

This combination is the often cited theorem that stability + consistency implies convergence.

Since checking that a scheme is consistent is fairly straightforward for finite difference schemes, (using Taylor series expansions), we concentrate on the issue of stability. Let us introduce a formally weaker notion of stability, again with $T > 0$ fixed and $h, k \to 0$.

Condition (A): There exists a constant $K > 0$ independent of h and k such that

$$\max_{0 \le n \le N} \|\boldsymbol{V}^n\| \le K \|\boldsymbol{V}^0\|$$

for all discrete functions $\boldsymbol{V}^n$ such that $L_{h,k}\boldsymbol{V}^n = 0$. Clearly, if a scheme is stable, then it satisfies Condition (A). In fact, one can also show that if a scheme satisfies Condition (A), then it is stable. Assuming this is the case, we wish to develop an algebraic criterion for checking Condition (A).

For any vector-valued function $\boldsymbol{v}(x)$ that is 2π periodic, let $\boldsymbol{v}(x) = \sum_{p=-\infty}^{\infty} \hat{\boldsymbol{v}}(p)e^{ipx}$ denote its Fourier series. Then

$$\|\boldsymbol{v}\|^2 = 2\pi \sum_{p=-\infty}^{\infty} |\hat{\boldsymbol{v}}(p)|^2, \qquad \hat{\boldsymbol{v}}(p) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{v}(x)e^{-ipx}\,dx.$$

Now $$kL_{h,k}\boldsymbol{V}^n(x) = \boldsymbol{V}^{n+1}(x) - \sum_{q=-Q}^{Q} C_q \boldsymbol{V}^n(x+qh)$$

$$= \sum_p \hat{\boldsymbol{V}}^{n+1}(p)e^{ipx} - \sum_p \sum_{q=-Q}^{Q} C_q \hat{\boldsymbol{V}}^n(p)e^{ip(x+qh)} = \sum_p [\hat{\boldsymbol{V}}^{n+1}(p) - G(p,h,k)\hat{\boldsymbol{V}}^n(p)]e^{ipx},$$

where

$$G(p,h,k) = \sum_{q=-Q}^{Q} C_q e^{ipqh}.$$

In particular, if $L_{h,k}\boldsymbol{V}^n = 0$, then

$$\hat{\boldsymbol{V}}^{n+1}(p) = G(p,h,k)\hat{\boldsymbol{V}}^n(p).$$

Iterating this equation, we obtain

$$\hat{\boldsymbol{V}}^n(p) = G^n(p,h,k)\hat{\boldsymbol{V}}^0(p).$$

The matrix $G(p,h,k)$ is the *amplification matrix* for the pth mode.

Let $\|G\|$ denote the 2-norm (spectral norm) of G given by $\max_i |\lambda_i(G^*G)|^{1/2}$, where $\lambda_i(G^*G)|$ are the eigenvalues of G^*G. Note that if G is symmetric, then $\|G\| = \max_i |\lambda_i(G)|$ $= \rho(G)$, the spectral radius of G. Then we have the following result.

Lemma 11. *Condition (A) holds if and only if there exists a constant K independent of h, k, and p such that*

$$\textit{Condition (B)} \qquad \max_{0\le n\le N-1} \|G^n(p,h,k)\| \le K, \qquad \forall p,h,k.$$

Proof. We only prove that Condition (B) implies Condition (A). If $\boldsymbol{V}^n$ is a discrete function with $L_{h,k}\boldsymbol{V}^n = 0$, then for $0 \le n \le N$, with $Nk = T$,

$$\|\boldsymbol{V}^n\|^2 = 2\pi\sum_p |\hat{\boldsymbol{V}}^n(p)|^2 = 2\pi \sum_p |G^n(p,h,k)\hat{\boldsymbol{V}}^0(p)|^2$$
$$\le 2\pi \sum_p \|G^n(p,h,k)\|^2|\hat{\boldsymbol{V}}^0(p)|^2 \le 2\pi K^2 \sum_p |\hat{\boldsymbol{V}}^0(p)|^2 = K^2\|\boldsymbol{V}^0\|^2.$$

□

Next, we give an important necessary condition for stability (von Neumann).

Theorem 11. *Suppose the explicit 2-level scheme is stable. Then there exists a constant* M *independent of* k *and* h *such that*

$$\max_p \rho[G(p,h,k)] \leq 1 + Mk.$$

Corollary 2. *If* G *is a normal matrix, i.e.,* $GG^* = G^*G$*, then the von Neumann condition is both necessary and sufficient for stability.*

Proof. (Of sufficiency). If $G(p,h,k)$ is normal, then

$$\|G^n\| = \|G\|^n = [\rho(G)]^n.$$

Hence if $\rho(G) \leq 1 + Mk$, then

$$\|G^n\| \leq (1+Mk)^n \leq e^{Mkn} \leq e^{MT} = K, \quad 0 \leq n \leq N,$$

so the method is stable. □