11.2. **Stability of difference schemes – examples.** In this section we present some examples to illustrate the theory.

Example 1: explicit scheme for the heat equation. In this case, we saw that $C_{-1} = C_1 = \sigma k/h^2$ and $C_0 = 1 - 2\sigma k/h^2$. Hence the amplification matrix

$$G(p,h,k) = \sum_{q=-1}^{1} e^{ipqh} C_q = e^{-iph}\sigma k/h^2 + (1 - 2\sigma k/h^2) + e^{iph}\sigma k/h^2$$
$$= 1 - 2\sigma k/h^2 + 2\sigma k/h^2 \cos(ph).$$

For stability, we want $|G| \le 1 + Mk$. Now $\cos(2\theta) = 1 - 2\sin^2\theta$, so

$$G = 1 - 2\sigma k/h^2 + 2(\sigma k/h^2)[1 - 2\sin^2(ph/2)] = 1 - 4(\sigma k/h^2)\sin^2(ph/2).$$

Then $-1 - Mk \le G \le 1 + Mk$ if

$$-2 - Mk \le -4(\sigma k/h^2)\sin^2(ph/2) \le Mk.$$

The right inequality is always true, since $\sigma, k > 0$. For the left inequality, we need

$$(\sigma k/h^2)\sin^2(ph/2) \le (1/2) + Mk/4, \quad \forall p.$$

Since $\sin^2(ph/2)$ can be arbitrarily close to 1, the stability condition becomes:

$$\sigma k/h^2 \le (1/2) + Mk/4.$$

If we let $h, k \to 0$ in such a way that $\sigma k/h^2$ remains constant, then we obtain the stability restriction $\sigma k/h^2 \le 1/2$.

Example 2: transport equation $u_t + \alpha u_x = 0$. If we consider the scheme:

$$[U_j^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_j^n]/h = 0, \quad \text{i.e.,}$$
$$k^{-1}[U_j^{n+1} - (1 + \alpha k/h)U_j^n + (\alpha k/h)U_{j+1}^n] = 0,$$

then $C_0 = 1 + \alpha k/h$ and $C_1 = -\alpha k/h$. Hence,

$$G(p,h,k) = 1 + \alpha k/h - (\alpha k/h)e^{iph}.$$

For $p = \pi/h$,

$$G(p,h,k) = 1 + 2\alpha k/h > 1 + Mk,$$

no matter how $k, h \to 0$. Hence, the scheme is unstable. Recall that the CFL condition is also violated in this case.

Example 3: If, instead, we consider the scheme:

$$[U_j^{n+1} - U_j^n]/k + \alpha[U_j^n - U_{j-1}^n]/h = 0, \quad \text{i.e.,}$$
$$k^{-1}[U_j^{n+1} - (1 - \alpha k/h)U_j^n - \alpha k/h U_{j+1}^n] = 0,$$

then $C_0 = 1 - \alpha k/h$ and $C_{-1} = \alpha k/h$. Hence,

$$G(p,h,k) = 1 - \alpha k/h + (\alpha k/h)e^{-iph}.$$

For $\lambda = \alpha k/h$ satisfying $0 \le \lambda \le 1$,

$$|G| \le |1 - \alpha k/h + (\alpha k/h)e^{-iph}| \le |1 - \lambda| + |\lambda e^{-iph}| \le 1 - \lambda + \lambda \le 1.$$

Hence, the scheme is stable for $0 < \alpha k/h \leq 1$. For $\lambda > 1$ and $ph \approx \pi$,

$$|G| \approx |1 - 2\alpha k/h| = 2\lambda - 1 > 1,$$

so the method is unstable.

Example 4:

$$[U_j^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_{j-1}^n]/(2h) = 0, \quad \text{i.e.,}$$

$$k^{-1}[U_j^{n+1} - U_j^n + \alpha k/(2h)(U_{j+1}^n - U_{j-1}^n)] = 0,$$

then $C_0 = 1$, $C_{-1} = \alpha k/(2h)$, $C_1 = -\alpha k/(2h)$. Hence,

$$G(p, h, k) = 1 + \alpha k/(2h)e^{-iph} - \alpha k/(2h)e^{iph} = 1 - i(ak/h)\sin(ph).$$

Then

$$|G| = [1 + (\alpha^2 k^2/h^2)\sin^2(ph)]^{1/2} \approx [1 + \alpha^2 k^2/h^2]^{1/2}$$

for $ph \approx= \pi/2$. If $k = O(h)$, then the method is unstable, while if $k = ch^2$, then

$$|G| = [1 + \alpha^2 ck]^{1/2} \leq [1 + \alpha^2 ck + \alpha^4 c^2 k^2/4]^{1/2} = 1 + \alpha^2 ck/2.$$

Hence, the method is stable in this case. However, this is a bad scheme, since it requires a very small time step.

11.3. **Three-level explicit schemes.** A scheme that was mentioned earlier was the approximation of the wave equation $u_{tt} = c^2 u_{xx}$ by the method

$$[U_j^{n+1} - 2U_j^n + U_j^{n1}]/k^2 = c^2[U_{j+1}^n - 2U_j^n + U_{j-1}^n]/h^2.$$

If we set $\lambda = ck/h$ and introduce a new variable $V_j^n = U_j^{n-1}$. then we convert this scheme to a two level scheme for the vector (U_j^n, V_j^n), i.e., we have

$$U_j^{n+1} = (2 - 2\lambda^2)U_j^n + \lambda^2(U_{j+1}^n + U_{j-1}^n) - V_j^n, \qquad V_j^{n+1} = U_j^n.$$

In matrix form, this becomes:

$$\begin{pmatrix} U_j^{n+1} \\ V_j^{n+1} \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{j-1}^n \\ V_{j-1}^n \end{pmatrix} + \begin{pmatrix} 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_j^n \\ V_j^n \end{pmatrix} + \begin{pmatrix} \lambda^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{j+1}^n \\ V_{j+1}^n \end{pmatrix}.$$

Hence, the amplification matrix for this method is

$$G(p, h, k) = \begin{pmatrix} \lambda^2[e^{-iph} + e^{iph}] + 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2\lambda^2\cos(ph) + 2 - 2\lambda^2 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 - 4\lambda^2\sin^2(ph/2) & -1 \\ 1 & 0 \end{pmatrix}.$$

We next show that the von Neumann condition is satisfied if as $h, k \to 0$, $\lambda = ck/h \leq 1$. Let $\beta = 2 - 4\lambda^2\sin^2(ph/2)$. Then the eigenvalues of the matrix G are the roots of

$$\det \begin{pmatrix} \beta - x & -1 \\ 1 & -x \end{pmatrix} = x^2 - \beta x + 1 = 0, \quad \text{i.e.,} \quad x = (\beta \pm \sqrt{\beta^2 - 4})/2.$$

Now for $0 \le \lambda \le 1$, $-2 \le \beta \le 2$ and so $\beta^2 - 4 \le 0$. For $|\beta| < 2$, the roots are complex conjugates and so $|x|^2 = (\beta^2 + 4 - \beta^2)/4 = 1$. If $|\beta| = 2$, then $|x| = 1$. Hence, $\rho(G) \le 1$ and so the von Neumann condition is satisfied. However,

$$GG^* = \begin{pmatrix} \beta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \beta^2+1 & \beta \\ \beta & 1 \end{pmatrix},$$
$$G^*G = \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta^2+1 & -\beta \\ -\beta & 1 \end{pmatrix},$$

and so $GG^* \ne G^*G$, i.e., G is not a normal matrix. Hence, the von Neumann condition does not imply stability. To investigate stability for this problem, one can verify directly that for $0 < \lambda < 1$,

$$\|G^n(p,h,k)\| \le K, \quad \forall p,h,k, \quad 0 \le n \le N, \quad \lambda = ak/h.$$

For $\lambda = 1$, $\|G^n\| \to \infty$ and the method is not stable.

11.4. **Stability of two-level implicit schemes.** In the homogeneous case, $(f = 0)$, a constant coefficient two-level implicit scheme may be written in the form

$$\sum_{q=-Q}^{Q} B_q \boldsymbol{U}^{n+1}(x+qh) = \sum_{q=-Q}^{Q} C_q \boldsymbol{U}^n(x+qh).$$

Again writing $\boldsymbol{U}^n$ in terms of its Fourier series, i.e.,

$$\boldsymbol{U}^n(x) = \sum_{p=-\infty}^{\infty} \hat{\boldsymbol{U}}^n(p) e^{ipx},$$

we have

$$\sum_p \sum_{q=-Q}^{Q} B_q e^{ipqh} \hat{\boldsymbol{U}}^{n+1}(p) e^{ipx} = \sum_p \sum_{q=-Q}^{Q} C_q e^{ipqh} \hat{\boldsymbol{U}}^n(p) e^{ipx}.$$

Hence,

$$H_1(p,h,k)\hat{\boldsymbol{U}}^{n+1}(p) = H_0(p,h,k)\hat{\boldsymbol{U}}^n(p), \qquad \text{where}$$
$$H_1(p,h,k) = \sum_{q=-Q}^{Q} B_q e^{ipqh}, \qquad H_0(p,h,k) = \sum_{q=-Q}^{Q} C_q e^{ipqh}.$$

Setting $G(p,h,k) = H_1^{-1} H_0$, we get

$$\hat{\boldsymbol{U}}^{n+1}(p) = G(p,h,k)\hat{\boldsymbol{U}}^n(p).$$

The previous theory carries over directly to this case: the difference scheme is stable if and only if there exists a constant K independent of h, k, and p such that

$$\max_{0 \le n \le N-1} \|G^n(p,h,k)\| \le K, \qquad \forall p,h,k.$$

The von Neumann condition is again necessary for stability and is also sufficient if G is a normal matrix.

Example: implicit scheme for the heat equation

$$[U_j^{n+1} - U_j^n]/k = \sigma[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}]h^2,$$

which we rewrite in the form

$$k^{-1}[-(\sigma k/h^2)U_{j+1}^{n+1} + (1 + 2\sigma k/h^2)U_j^{n+1} - (\sigma k/h^2)U_{j-1}^{n+1}] = k^{-1}U_j^n.$$

Then

$$B_{-1} = B_1 = -\sigma k/h^2, \qquad B_0 = 1 + 2\sigma k/h^2, \qquad C_0 = 1.$$

Hence,

$$\begin{aligned} H_1 &= -(\sigma k/h^2)(e^{iph} + e^{-iph}) + 1 + 2\sigma k/h^2 \\ &= 1 + 2\sigma k/h^2 - 2(\sigma k/h^2)\cos(ph) = 1 + (4\sigma k/h^2)\sin^2(ph/2). \end{aligned}$$

Since $H_0 = 1$, we get that

$$0 \leq G = 1/[1 + (4\sigma k/h^2)\sin^2(ph/2)] \leq 1,$$

and so the implicit method is unconditionally stable.