12. Finite element methods for parabolic problems

We consider the parabolic problem:

$$u_t - \operatorname{div}(p\nabla u) + qu = f, \quad (x,t) \in \Omega \times (0,T],$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = g(x), \quad x \in \Omega.$$

A variational formulation of this problem is to seek $u(t) \in \mathring{H}^1(\Omega)$ such that $u(0) = g$,

$$(\partial u/\partial t, v) + a(u,v) = (f,v), \quad v \in \mathring{H}^1(\Omega),$$

where as in the elliptic case, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product and

$$a(u,v) = \int_\Omega [p\nabla u \cdot \nabla v + quv]\,dx.$$

12.1. **Continuous time Galerkin scheme.** We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace $V_h \subset \mathring{H}^1(\Omega)$ and look for an approximation $u_h(t) \in V_h$, $t \in [0,T]$, satisfying: $u_h(0) = g_h$ (g_h an approximation to g) and

$$(\partial u_h/\partial t, v) + a(u_h, v) = (f,v), \quad v \in V_h.$$

To see what is involved in solving this problem, we write $u_h(t) = \sum_{j=1}^m \alpha_j(t)\phi_j(x)$. Inserting this into the variational equations, and choosing v to be each of the basis functions ϕ_i, we get

$$\sum_{j=1}^m \alpha_j'(t)(\phi_j,\phi_i) + \sum_{j=1}^m \alpha_j(t)a(\phi_j,\phi_i) = (f,\phi_i), \quad i = 1,\dots,m.$$

Let

$$M_{ij} = (\phi_j,\phi_i), \qquad A_{ij} = a(\phi_j,\phi_i), \qquad F_i = (f,\phi_i), \qquad \alpha = (\alpha_1,\dots,\alpha_m)^T.$$

Our equations then have the form

$$M\alpha'(t) + A\alpha = F,$$

a first order system of ordinary differential equations.

One can obtain a simple error estimate for this approximation scheme by comparing the approximate solution to the elliptic projection $w_h(t) \in V_h$, satisfying

$$a(u(t) - w_h(t), v_h) = 0, \qquad v_h \in V_h.$$

We showed previously that if V_h consists of piecewise polynomials of degree $\le r$, and u is sufficiently smooth, then

$$\|u(t) - w_h(t)\| + h\|u(t) - w_h(t)\|_1 \le Ch^{r+1}|u(t)|_{r+1}.$$

Theorem 12. *If V_h consists of piecewise polynomials of degree $\le r$ and u is sufficiently smooth, then for $t \ge 0$,*

$$\|u(t) - u_h(t)\| \le \|g - g_h\| + Ch^{r+1}\left[\|g\|_{r+1} + \int_0^t \|u_t\|_{r+1}\,ds\right].$$

Proof. We estimate the error by writing $u - u_h = (u - w_h) + (w_h - u_h)$. From the above, we have

$$\|u(t) - w_h(t)\| \le Ch^{r+1}\|u(t)\|_{r+1} \le Ch^{r+1}\|u(0) + \int_0^t u_t(s)\,ds\|_{r+1} \le Ch^{r+1}\left[\|g\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1}\,ds\right].$$

It thus remains to estimate $\|u_h - w_h\|$. Using the continuous and discrete variational formulations and the definition of $w_h(t)$, we get

$$(\partial[u_h - w_h]/\partial t, v) + a(u_h - w_h, v) = (\partial[u_h - u]/\partial t, v) + a(u_h - u, v) + (\partial[u - w_h]/\partial t, v) + a(u - w_h, v) = (\partial[u - w_h]/\partial t, v), \quad v \in V_h.$$

Choosing $v = u_h - w_h$, and observing that

$$\|u_h - w_h\|\frac{d}{dt}\|u_h - w_h\| = \frac{1}{2}\frac{d}{dt}\|u_h - w_h\|^2 = ([u_h - w_h]_t, u_h - w_h),$$

we get

$$\|u_h - w_h\|\frac{d}{dt}\|u_h - w_h\| + \|u_h - w_h\|_E^2 = ([u - w_h]_t, u_h - w_h) \le \|[u - w_h]_t\|\|u_h - w_h\|.$$

Hence,

$$\frac{d}{dt}\|u_h - w_h\| \le \|[u - w_h]_t\|.$$

Integrating this equation between 0 and t, we get

$$\|u_h(t) - w_h(t)\| \le \|u_h(0) - w_h(0)\| + \int_0^t \|[u - w_h]_t(s)\|\,ds \le \|u_h(0) - u(0)\| + \|u(0) - w_h(0)\| + \int_0^t \|[u - w_h]_t(s)\|\,ds. \le \|g - g_h\| + Ch^{r+1}\left[\|g\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1}\,ds\right].$$

Using the triangle inequality, and combining estimates, we then obtain

$$\|u(t) - u_h(t)\| \le \|u(t) - w_h(t)\| + \|u_h(t) - w_h(t)\| \le \|g - g_h\| + Ch^{r+1}\left[\|g\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1}\,ds\right].$$

□

12.2. **Fully discrete schemes.** One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate u_t by the backward Euler approximation, we get the scheme: Find $U^n \in V_h$, satisfying $U^0 = g_h$ and for $n \ge 0$

$$([U^{n+1} - U^n]/k, v) + a(U^{n+1}, v) = (f^{n+1}, v) \quad v \in V_h.$$

Using the matrices defined previously, and defining $U^n(x) = \sum_{j=1}^m \alpha^n \phi_j(x)$, the discrete variational formulation above corresponds to the linear system

$$(M + kA)\alpha^{n+1} = M\alpha^n + kF^{n+1}, \qquad n = 0, 1, \ldots.$$

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find $U^n \in V_h$, satisfying $U^0 = g_h$ and for $n \geq 0$

$$([U^{n+1} - U^n]/k, v) + a([U^{n+1} + U^n]/2, v) = ([f^{n+1} + f^n]/2), \quad v \in V_h.$$

In this case, we get the linear system

$$(M + \tfrac{1}{2}kA)\alpha^{n+1} = (M - \tfrac{1}{2}kA)\alpha^n + k(F^{n+1} + F^n)/2, \qquad n = 0, 1, \ldots.$$

For the backward Euler method, we have the following error estimate ($t_n = nk$).

Theorem 13.

$$\|u(t_n) - U^n\| \leq \|g - g_h\| + Ch^{r+1}\left[\|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1}\right] + k\int_0^{t_n} \|u_{tt}(s)\|\, ds, \quad n \geq 0.$$

Proof. As before, we write $u(t_n) - U^n = (u(t_n) - W^n) + (W^n - U^n)$, where $W^n = w_h(t_n) \in V_h$ (the elliptic projection) satisfies

$$a(u(t) - w_h(t), v_h) = 0, \qquad v_h \in V_h.$$

From our previous result, we have

$$\|u(t_n) - W^n\| \leq Ch^{r+1}\left[\|g\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1}\, ds\right].$$

To estimate $U^n - W^n$, we again use our continuous and discrete variational formulations, but this time obtaining

$$\begin{aligned}([(U-W)^{n+1} - (U-W)^n]/k, v) + a((U-W)^{n+1}, v) &= ([(U-u)^{n+1} - (U-u)^n]/k, v) \\ + a((U-u)^{n+1}, v) &+ ([(u-W)^{n+1} - (u-W)^n]/k, v) + a((u-W)^{n+1}, v) \\ = (u_t^{n+1} - [u^{n+1} - u^n]/k, v) &+ ([(u-W)^{n+1} - (u-W)^n]/k, v) \equiv (\rho^n, v) \quad v \in V_h.\end{aligned}$$

Choosing $v = (U - W)^{n+1}$, we get

$$\begin{aligned}\|(U-W)^{n+1}\|^2 + k\|(U-W)^{n+1}\|_E^2 &= ((U-W)^n, (U-W)^{n+1}) + k(\rho^n, (U-W)^{n+1}) \\ &\leq [\|(U-W)^n\| + k\|\rho^n\|]\|(U-W)^{n+1}\|.\end{aligned}$$

Hence,

$$\|(U-W)^{n+1}\| \leq \|(U-W)^n\| + k\|\rho^n\|.$$

Iterating this equation, we get

$$\|(U-W)^n\| \leq \|(U-W)^0\| + k\sum_{j=0}^{n-1} \|\rho^j\|.$$

By Taylor series,

$$u(x,t) = u(x,t+k) - ku_t(x,t+k) + \int_t^{t+k} (s-t)u_{tt}(s)\,ds.$$

Hence, for $t_j = jk$,

$$u_t^{j+1} - [u^{j+1} - u^j]/k = k^{-1} \int_{t_j}^{t_{j+1}} (s-t_j)u_{tt}(s)\,ds.$$

So

$$\begin{aligned}
\|\rho^j\| &= \left\| k^{-1} \int_{t_j}^{t_{j+1}} [(s-t_j)u_{tt}(s) + (u-w_h)_t(s)]\,ds \right\| \\
&\le k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + \|(u-w_h)_t(s)\|]\,ds \\
&\le k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}]\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
k\sum_{j=0}^{n-1} \|\rho^j\| &\le \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}]\,ds \\
&\le k\int_{t_0}^{t_n} \|u_{tt}(s)\|\,ds + Ch^{r+1}\int_{t_0}^{t_n} \|u_t(s)\|_{r+1}]\,ds.
\end{aligned}$$

The theorem follows by combining all these results. □