1.2. **Stability and error estimates.** We now use the discrete maximum principle to bound the error between u and U_h. We first establish the following stability result.

Theorem 3. *Let $v(x, y)$ be a function defined on $\Omega_h \cup \partial\Omega_h$. Then*

$$\max_{\Omega_h \cup \partial\Omega_h} |v| \le \max_{\partial\Omega_h} |v| + \frac{1}{2} \max_{\Omega_h} |\Delta_h v|.$$

Proof. Let $\phi(x, y) = x^2/2$. Then $0 \le \phi(x, y) \le 1/2$ for all $(x, y) \in \Omega_h \cup \partial\Omega_h$. Furthermore,

$$\Delta_h \phi(x, y) = \frac{1}{2h^2}\{(x+h)^2 + (x-h)^2 + x^2 + x^2 - 4x^2\} = 1.$$

Define functions $v_+(x, y)$ and $v_-(x, y)$ by $v_\pm(x, y) = \pm v(x, y) + M\phi(x, y)$, where $M = \max_{\Omega_h} |\Delta_h v|$. Then for all $(x, y) \in \Omega_h$,

$$\Delta_h v_\pm(x, y) = \pm \Delta_h v(x, y) + M \ge 0.$$

Hence, by Theorem 1, for all $(x, y) \in \Omega_h$,

$$v_\pm(x, y) \le \max_{\partial\Omega_h} v_\pm(x, y) = \max_{\partial\Omega_h}[\pm v(x, y) + M\phi(x, y)] \le \max_{\partial\Omega_h}[\pm v(x, y] + M/2.$$

Since $M\phi \ge 0$,

$$\pm v(x, y) = v_\pm(x, y) - M\phi(x, y) \le v_\pm(x, y).$$

Hence, for all $(x, y) \in \Omega_h$,

$$\pm v(x, y) \le \max_{\partial\Omega_h}[\pm v(x, y)] + M/2 \le \max_{\partial\Omega_h} |v| + M/2,$$

and so

$$\max_{\Omega_h} |v| \le \max_{\partial\Omega_h} |v| + \max_{\Omega_h} |\Delta_h v|.$$

Since $\max_{\partial\Omega_h} |v|$ is also bounded by the right hand side of the above, (ii) follows immediately. □

Theorem 4. *Suppose u and U_h are the solutions of Problems P and P_h, respectively. Then*

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le \frac{1}{2} \max_{\Omega_h} |\Delta_h u - \Delta u|.$$

Proof. Set $v = u - U_h$, where we now consider the restriction of u to the mesh, so that we can view v as a function defined on the mesh. Then $v = 0$ on $\partial\Omega_h$ and

$$\Delta_h v = \Delta_h u - \Delta_h U_h = \Delta_h u - \Delta u + \Delta u - \Delta_h U_h = \Delta_h u - \Delta u + f - f = \Delta_h u - \Delta u.$$

By Theorem 3, applied to $v = u - U_h$,

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le \max_{\partial\Omega_h} |u - U_h| + \frac{1}{2} \max_{\Omega_h} |\Delta_h u - \Delta u| = \frac{1}{2} \max_{\Omega_h} |\Delta_h u - \Delta u|.$$

□

Corollary 1. *If $u \in C^4(\Omega)$, then $\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le h^2 M_4/12$.*

Remarks: The quantity $\Delta u - \Delta_h u$ is called the *consistency error* in the approximation of Δu by $\Delta_h u$. The statement that $\max|\Delta u - \Delta_h u| \to 0$ as $h \to 0$ says that the approximation is *consistent*.

Since Problem P_h has a unique solution, there will be a constant C_h depending on h such that for any mesh function v, we have an estimate of the form:

$$\max_{\Omega_h \cup \partial\Omega_h} |v| \le C_h(\max_{\partial\Omega_h} |v| + \max_{\Omega_h} |\Delta_h v|).$$

If there exists a constant C (the *stability* constant) such that $C_h \le C$ for all $0 < h \le h_0$, (i.e, the estimate holds with a constant independent of h), we say the approximation scheme is *stable*.

Our previous argument showed the error $\max|u - U_h|$ is bounded by the stability constant times the maximum of the consistency error. So we have stability + consistency implies convergence, i.e., $\lim_{h\to 0} |u - U_h| = 0$. Since we imposed boundary conditions exactly, there was no consistency error due to approximation of the boundary conditions. In other problems (e.g., domains with curved boundaries or Neumann boundary conditions), we will also need to account for this.

The statement *stability + consistency implies convergence* holds in much more generality and is a fundamental notion underlying many areas of numerical analysis. We shall frequently see various versions of this as we consider the approximation of other partial differential equations and other discretization schemes.

1.3. **Extensions to domains with curved boundaries.** We now consider the same boundary value problem, but on a more general domain Ω with a smooth boundary. For simplicity, we restrict to a convex domain. Let $E_h = \{(ih, jh), i, j \text{ integers}\}$ and set $\Omega_h = \Omega \cap E_h$. We write $\Omega_h = \Omega_h^0 + \Omega_h^*$, where

$$\Omega_h^0 = \{(x, y) \in \Omega_h : (x \pm h, y), (x, y \pm h) \in \Omega_h\}, \qquad \Omega_h^* = \Omega_h - \Omega_h^0,$$

i.e., mesh points are in Ω_h^0 if their 4 nearest neighbors are also in Ω_h. Ω_h^* then denotes the remainder of the interior mesh points. We then define $\partial\Omega_h$ to be the neighbors of points in Ω_h^* which lie on the intersection of at least one mesh line and $\partial\Omega$. For points in Ω_h^0, the operator Δ_h defined previously is well defined, but for points in Ω_h^*, we must modify the definition. Consider the case where $(x, y) \in \Omega_h^*$, $(x + h, y)$ and $(x, y + h) \in \Omega_h$, but $(x - h, y)$ and $(x, y - h)$ both lie outside of Ω (see Figure below). Then there will be points $(x - \alpha h, y)$ and $(x, y - \beta h)$ that lie on $\partial\Omega_h$ for some $0 < \alpha, \beta < 1$. At the point (x, y), we then define

$$\begin{aligned}\Delta_h v(x, y) = \frac{2}{h^2}\Big\{ &\frac{1}{\alpha + 1} v(x + h, y) + \frac{1}{\alpha(\alpha + 1)} v(x - \alpha h, y) + \frac{1}{\beta + 1} v(x, y + h) \\ &+ \frac{1}{\beta(\beta + 1)} v(x, y - \beta h) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) v(x, y)\Big\} \qquad \text{(Shortly-Weller formula)}.\end{aligned}$$

Note that for $\alpha = \beta = 1$, we recover the previous formula. Using Taylor series expansions, one can show that for all $v \in C^3(\bar{\Omega})$,

$$|\Delta_h v(x,y) - \Delta v(x,y)| \leq 2M_3 h/3, \qquad M_3 = \max_{\bar{\Omega}} \left[\max \left| \frac{\partial^3 v}{\partial x^3} \right|, \left| \frac{\partial^3 v}{\partial y^3} \right| \right],$$

but that the formula does not give an $O(h^2)$ approximation unless $\alpha = \beta = 1$. Using our previous analysis, we might expect that the error $|u - U_h|$ would be only $O(h)$, since the error was bounded by the maximum of the local truncation errors. However, using a more precise analysis and the fact that the difference operator with larger local truncation error only lives on a strip of width ch near the boundary of Ω, it is possible to show that the error $|u - U_h|$ is still $O(h^2)$. More precisely, we have the following result.

Theorem 5. *Suppose u and U_h are the solutions of Problems P and P_h, respectively. If $u \in C^4(\Omega)$, then*

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \leq \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3,$$

where M_3 and M_4 are the bounds on appropriate third and fourth derivatives of u given previously and d is the diameter of the smallest circumscribed circle containing Ω.

Approximation Δ_h near $\partial\Omega$