13. Approximation of Parabolic Variational Inequalities

We previously considered the finite element approximation of the parabolic problem:

$$u_t - \operatorname{div}(p\nabla u) + qu = f, \quad (x,t) \in \Omega \times (0,T],$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = g(x), \quad x \in \Omega.$$

based on the variational formulation: Find $u(t) \in \mathring{H}^1(\Omega)$ such that $u(0) = g$,

$$(u_t, v) + a(u,v) = (f,v), \quad v \in \mathring{H}^1(\Omega),$$

where as in the elliptic case, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product and

$$a(u,v) = \int_\Omega [p\nabla u \cdot \nabla v + quv]\, dx.$$

A corresponding variational inequality (the parabolic obstacle problem) is to define

$$K = \{v \in \mathring{H}^1(\Omega) : v \geq \psi \text{ in } \Omega\}$$

and seek $u(t) \in K$ such that $u(0) = g$ and

$$(u_t, v-u) + a(u, v-u) \geq (f, v-u), \quad v \in K.$$

One simple approximation scheme is to approximate u_t by the backward Euler approximation and approximate $u(t_n)$ at each time level $t_n = nk$ by U^n, where $U^n \in V_h$, the space of continuous piecewise linear functions. As in the elliptic case, we approximate the convex set K by the convex set

$$K_h = \{v \in V_h : v \geq \psi_I \text{ in } \Omega\}.$$

Then we get the approximate problem: Find $U^0, U^1, \ldots, U^n \in K_h$, satisfying $U^0 = g_h$ and for $n \geq 0$,

$$([U^{n+1} - U^n]/k, v - U^{n+1}) + a(U^{n+1}, v - U^{n+1}) \geq (f^{n+1}, v - U^{n+1}) \quad v \in K_h.$$

If we define a bilinear form

$$b(u,v) = (u,v) + ka(u,v),$$

then at each time level $t_{n+1} = (n+1)k$, we are solving the elliptic variational inequality: Find $U^{n+1} \in K_h$ satisfying

$$b(U^{n+1}, v - U^{n+1}) \geq (U^n + kf^{n+1}, v - U^{n+1}), \quad v \in K_h.$$

The following error estimate can be obtained for this approximation scheme.

$$\max_n \|u^n - U^n\|_0 + \left(\sum_{n=1}^N k\|u^n - U^n\|_1^2\right)^{1/2} \leq C(k^{1/2} + h).$$

A complete proof can be found in C. Johnson, *A convergence estimate for an approximation of a parabolic variational inequality*, SIAM J. Numer. Anal., Vol 13., No. 4, September 1976.