14. Space-Time finite element methods for parabolic problems

We consider the parabolic problem:

$$u_t - \operatorname{div}(p\nabla u) + qu = f, \quad (x,t) \in \Omega \times (0,T],$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \qquad u(x,0) = g(x), \quad x \in \Omega.$$

The continuous time Galerkin method is based on the variational formulation of this problem given by: Seek $u(t) \in \mathring{H}^1(\Omega)$ such that $u(0) = g$,

$$(u_t, v) + a(u,v) = (f,v), \quad v \in \mathring{H}^1(\Omega).$$

For the Galerkin method, we choose a finite dimensional subspace $V_h \subset \mathring{H}^1(\Omega)$ (e.g., piecewise polynomials of degree $\leq r$) and look for an approximation $u_h(t) \in V_h$, $t \in [0,T]$, satisfying: $u_h(0) = g_h$ (g_h an approximation to g) and

$$((u_h)_t, v) + a(u_h, v) = (f,v), \quad v \in V_h.$$

Instead of obtaining a fully discrete method by discretizing in time using finite differences, we now consider two methods for discretizing in time using finite elements. The first is the continuous Galerkin method: We let $0 = t_0 < t_1 < \cdots t_N = T$ be a partition of $[0,T]$ and let S_k be a finite element space consisting of continuous piecewise polynomials of degree $\leq q$ in the time variable t. Then define $W_{h,k}$ to be the tensor product space $W_{h,k} = V_h \otimes S_k$. For example, if $q = 1$ and we consider the time slab $\Omega \times [t_{n-1}, t_n]$, we can write a function in $W_{h,k}$ in the form

$$w^{hk} = \frac{t - t_{n-1}}{k} v_h^n(x) + \frac{t_n - t}{k} v_h^{n-1}(x),$$

where v_h^{n-1} and $v_h^n \in V_h$.

Define $g_h \in V_h$ by

$$a(g_h, v) = a(g,v), \quad v \in V_h$$

and $U^{h,k} \in W_{h,k}$ such that

$$\int_0^T [(U_t^{h,k}, v_t) + a(U^{h,k}, v_t)]\, dt = \int_0^T (f, v_t)\, dt, \quad \text{for all } v \in W_{h,k}.$$

While this appears to be a global problem in time, in fact it is a marching scheme, i.e., we can compute $U^{h,k}$ on $[t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$, successively by solving

$$\int_{t_{n-1}}^{t_n} [(U_t^{h,k}, w) + a(U^{h,k}, w)]\, dt = \int_{t_{n-1}}^{t_n} (f, w)\, dt, \quad \text{for all } w \in V_h \otimes P^{q-1}([t_{n-1}, t_n]),$$

where $P^{q-1}([t_{n-1}, t_n])$ denotes the set of polynomials of degree $\leq q-1$ on the interval $[t_{n-1}, t_n]$. To see this, consider the case of $q = 1$, piecewise linear in time. If we choose

$$v = \begin{cases} v^{n-1}(x), & 0 \leq t \leq t_{n-1} \\ \frac{t_n - t}{k} v^{n-1}(x) + \frac{t - t_{n-1}}{k} v^n(x), & t_{n-1} \leq t \leq t_n, \\ v^n(x), & t \geq t_n, \end{cases}$$

then $v_t = [v^n(x) - v^{n-1}(x)]/k$ for $t_{n-1} \le t \le t_n$ and zero elsewhere. Hence, the integral from 0 to T reduces to an integral over $[t_{n-1}, t_n]$ and by choosing $v^n(x)$ and $v^{n-1}(x)$ appropriately, we can get any function $w \in V_h \otimes P^0$.

Notice also that in the case of $q = 1$, if we write

$$U^{h,k} = \frac{t - t_{n-1}}{k} U_h^n(x) + \frac{t_n - t}{k} U_h^{n-1}(x),$$

then

$$\int_{t_{n-1}}^{t_n} [(U_t^{h,k}, w) + a(U^{h,k}, w)]\, dt = (U_h^n(x) - U_h^{n-1}(x), w) + \frac{k}{2}[a(U_h^n(x), w) + a(U_h^{n-1}(x), w)],$$

so we get a type of Crank-Nicholson-Galerkin scheme, where the right hand side is averaged.

A second possibility is to use the discontinuous Galerkin approach. Let

$$w_+^n = \lim_{t \to t_n +} w(t), \qquad w_-^n = \lim_{t \to t_n -} w(t), \qquad \text{and} \qquad [w^n] = w_+^n - w_-^n.$$

We now define S_k as the set of all *discontinuous* piecewise polynomials of degree $\le q$ on the mesh on $[0, T]$ and $W_{h,k} = V_h \otimes S_k$. Then we seek $U \in W_{h,k}$ as the solution of

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)]\, dt + \sum_{n=1}^{N} ([U^{n-1}], w_+^{n-1}) + (U_-^0, w_+^0)$$
$$= (g, w_+^0) + \int_0^{t_N} (f, w)\, dt, \quad \text{for all } w \in W_{h,k}.$$

Since the finite element space is discontinuous in time, we can choose w so that it is non-zero only on the subinterval $[t_{n-1}, t_n]$. We again get a time marching scheme that determines U successively on $[t_{n-1}, t_n]$ by solving

$$\int_{t_{n-1}}^{t_n} [(U_t, w) + a(U, w)]\, dt + (U_+^{n-1}, w_+^{n-1}) = (U_-^{n-1}, w_+^{n-1}) + \int_{t_{n-1}}^{t_n} (f, w)\, dt, \quad \text{for all } w \in W_{h,k}.$$

On the first subinterval, we will have

$$\int_{t_0}^{t_1} [(U_t, w) + a(U, w)]\, dt + (U_+^0, w_+^0) = (g, w_+^0) + \int_{t_0}^{t_1} (f, w)\, dt, \quad \text{for all } w \in W_{h,k}.$$

Note that the true solution will satisfy these equations, since $u_+^{n-1} = u_-^{n-1}$.

In the continuous scheme, we have a single value for U at $t = t_n$. In the discontinuous scheme, we have two values, one from the minus side and one from the plus side. So, if we choose $q = 1$, then on the subinterval $[t_{n-1}, t_n]$, we are writing

$$U = \frac{t - t_{n-1}}{k} U_-^n(x) + \frac{t_n - t}{k} U_+^{n-1}(x).$$