15. A finite element method for the transport equation

We consider the approximation of the initial value problem:

$$\boldsymbol{\beta}\cdot\nabla u+\sigma u=f,\quad \text{in }\Omega,\qquad u=g\quad\text{on }\Gamma_{in}(\Omega),$$

where $\boldsymbol{\beta}$ is a unit vector, $\sigma\ge c>0$ and

$$\Gamma_{in}(\Omega)=\{x\in\partial\Omega:\boldsymbol{\beta}\cdot\boldsymbol{n}<0\},$$

where $\boldsymbol{n}$ is the unit outward normal to Ω. We shall assume that Ω is a polygon and $\mathcal{T}_h$ is a triangulation of Ω. Then it is possible to order the triangles $\{T_1,T_2,\cdots\}$ such that for each k, the domain of dependence of T_k consists of some subset of $\Gamma_{in}(\Omega)$ and $\{T_1,\cdots,T_{k-1}\}$. With such an ordering, one can develop an finite element method in an explicit fashion, triangle by triangle.

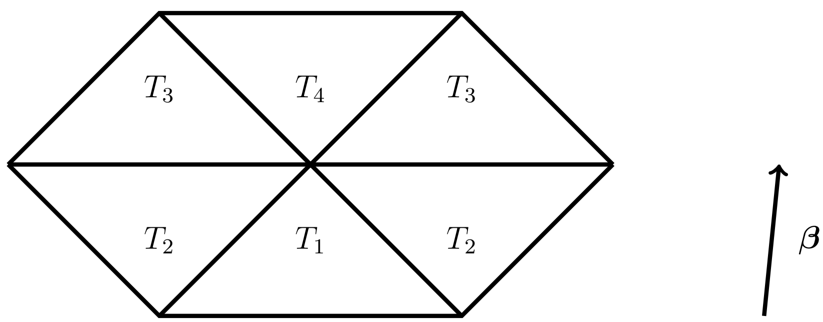

Figure 3. Ordering of the triangles

Letting $P_n(T)$ denote the polynomials of degree $\le n$ on T, the Discontinuous Galerkin method for this problem has the form:

For $T\in\mathcal{T}_h$, find $u_h\in P_n(T)$ such that $u_h^-=g_h$ on $\Gamma_{in}(\Omega)$ and satisfies

$$(\boldsymbol{\beta}\cdot\nabla u_h+\sigma u_h,v)_T-\int_{\Gamma_{in}(T)}(u_h^+-u_h^-)v\,\boldsymbol{\beta}\cdot n\,ds=(f,v)_T,\quad v\in P_n(T),$$

where for $x\in\Gamma_{in}(T)$, $u_h^\pm(x)=\lim_{\epsilon\to0}u_h(x\pm\epsilon\boldsymbol{\beta})$. If we solve these equations using the ordering discussed above, then u_h^- is known at the time it is needed to compute the solution on triangle T. On each triangle, we need to solve a simple square linear system of equations. Note that the solution produced is a piecewise polynomial, but one that is discontinuous across triangle edges. The key to the analysis of this method is the following identity.

Lemma 12. *Assume that $\boldsymbol{\beta}$ is a constant vector. Then*

$$(\boldsymbol{\beta}\cdot\nabla u,u)_T-\int_{\Gamma_{in}(T)}(u^+-u^-)u^+\,\boldsymbol{\beta}\cdot n\,ds$$
$$=\frac12\int_{\Gamma_{out}(T)}(u^-)^2|\beta\cdot\boldsymbol{n}|\,ds+\frac12\int_{\Gamma_{in}(T)}(u^+-u^-)^2|\beta\cdot\boldsymbol{n}|\,ds-\frac12\int_{\Gamma_{in}(T)}(u^-)^2|\beta\cdot\boldsymbol{n}|\,ds.$$

Proof. Now since $\beta \cdot n \geq 0$ on $\Gamma_{out}(T)$ and $\beta \cdot n < 0$ on $\Gamma_{in}(T)$,

$$(\boldsymbol{\beta} \cdot \nabla u, u)_T = \tfrac{1}{2}(\boldsymbol{\beta} \cdot \nabla [u^2])_T$$
$$= \int_{\partial T} \tfrac{1}{2} u^2 \beta \cdot n \, ds = \int_{\Gamma_{out}(T)} \tfrac{1}{2}(u^-)^2 |\beta \cdot n| \, ds - \int_{\Gamma_{in}(T)} \tfrac{1}{2}(u^+)^2 |\beta \cdot n| \, ds,$$

$$- \int_{\Gamma_{in}(T)} (u^+ - u^-) u^+ \, \boldsymbol{\beta} \cdot n \, ds = \int_{\Gamma_{in}(T)} \tfrac{1}{2}(u^+)^2 |\boldsymbol{\beta} \cdot n| \, ds$$
$$+ \int_{\Gamma_{in}(T)} \tfrac{1}{2}(u^+ - u^-)^2 |\boldsymbol{\beta} \cdot n| \, ds - \int_{\Gamma_{in}(T)} \tfrac{1}{2}(u^-)^2 |\boldsymbol{\beta} \cdot n| \, ds.$$

Adding these two equations gives the desired identity. □

One important implication of this identity is that it easily follows that the linear system on each triangle has a unique solution for $\sigma > 0$. To see this, we need only show that if $f = 0$ and $u_h^- = 0$, then $u_h = 0$. Choosing $v = u_h$ and using the above identity, we get

$$\frac{1}{2} \int_{\Gamma_{out}(T)} (u^-)^2 |\beta \cdot \boldsymbol{n}| \, ds + \frac{1}{2} \int_{\Gamma_{in}(T)} (u^+ - u^-)^2 |\beta \cdot \boldsymbol{n}| \, ds + \sigma \|u_h\|_T^2 = 0,$$

and so $u_h = 0$.

In analyzing this problem, it is helpful to think of u_h as evolving in layers S_i, defined by

$$S_0 = \emptyset, \qquad S_i = \{T \in \mathcal{T}_h : \Gamma_{in}(T) \subset \Gamma_{in}(\Omega - \cup_{j<i} S_j\}, \quad j = 1, 2, \cdots.$$

Within a layer, u_h can be developed in parallel. We can also define a sequence of fronts F_i, to which u_h has advanced after it has been computed in $\Omega_i = \cup_{j \leq i} S_j$.

In the case when $f = 0$, we also have a very simple stability analysis that can be expressed in the above terms.

Theorem 14.

$$\frac{1}{2} |u_h^-|_{F_i}^2 + \sigma \|u_h\|_{\Omega_i}^2 \leq \frac{1}{2} |u_h^-|_{\Gamma_{in}(\Omega)}^2.$$

Proof. Applying our identity with $f = 0$, we obtain

$$\frac{1}{2} \int_{\Gamma_{out}(T)} (u^-)^2 |\beta \cdot \boldsymbol{n}| \, ds + \frac{1}{2} \int_{\Gamma_{in}(T)} (u^+ - u^-)^2 |\beta \cdot \boldsymbol{n}| \, ds + \sigma \|u_h\|_T^2 = \frac{1}{2} \int_{\Gamma_{in}(T)} (u^-)^2 |\beta \cdot \boldsymbol{n}| \, ds.$$

Summing over all the triangles in the layer S_i and omitting the positive jump terms, we get

$$\frac{1}{2} |u_h^-|_{F_i}^2 + \sigma \sum_{T \in S_i} \|u_h\|_T^2 \leq \frac{1}{2} |u_h^-|_{F_{i-1}}^2.$$

The theorem follows by iterating this inequality. □

Also using this key identity, we are able to show that $\|u - u_h\|_{L^2(\Omega)} \leq C h^{n+1/2} \|u\|_{n+1}$. Note that this is not an optimal order error estimate, since the best approximation by polynomials of degree $\leq n$, would be $O(h^{n+1})$.