## 16. The finite volume method for elliptic problems

We consider the approximation of the problem

$$-\operatorname{div}(a\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Finite volume methods are based on an approximation of the balance equation

$$-\int_{\partial b} a\nabla u \cdot n\, ds = \int_b f\, dx,$$

valid for any subdomain $b \subset \Omega$, where $n$ denotes the unit outward normal to the boundary of $b$. Note this equation can be obtained by integrating the partial differential equation over the subdoiman $b$ and applying the divergence theorem. One approach to the finite volume method, and the one we will discuss, uses a finite element partition of $\Omega$, where the approximate solution space consists of piecewise linear functions, a collection of vertex-centered control volumes, and a test space consisting of piecewise constant functions over the control volumes.

More precisely, we begin with a family of triangulations $\{\mathcal{T}_h\}$ of the domain $\Omega$ and let

$$X_h = \{v \in H_0^1(\Omega) : v|_T \in P_1, \quad \forall T \in \mathcal{T}_h\}.$$

To construct the control volumes, we let $z_T$ denote the barycenter of $T$ and connect $z_T$ to the midpoints of the edges of $T$ with line segments. This partitions each triangle $T$ into three quadrilaterals. We use the notation $K_{z,T}$ to denote the quadrilateral in $T$ which shares the vertex $z$ of the triangle $T$. To each vertex $z$ of the triangulation, we associate a control volume $b_z$ consisting of the union of the quadrilaterals $K_{z,T}$, where the union is taken over all triangles $T$ containing the vertex $z$.

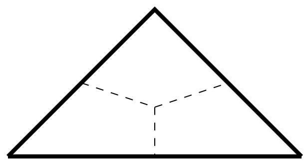

Figure 4. Partition of a triangle into 3 quadrilaterals

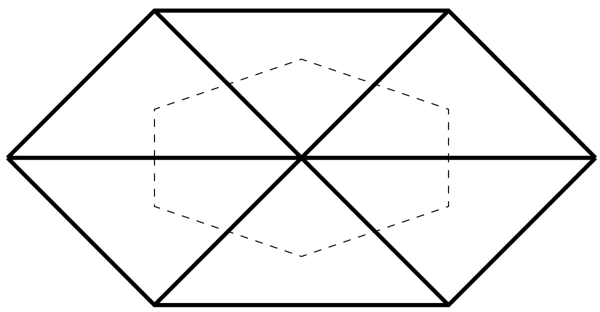

Figure 5. The control volume associated to a vertex

The finite volume method is then to find $u_h \in X_h$ such that

$$-\int_{\partial b_z} a \nabla u_h \cdot n \, ds = \int_{b_z} f \, dx, \quad \text{for all } z \in Z_h^0,$$

where $Z_h^0$ denotes the set of interior vertices of the mesh $\mathcal{T}_h$.

There is a large literature on the use of finite volume methods to approximate partial differential equations. An error analysis for the method presented here can be found in the paper "Error estimates for a finite volume element method for elliptic pdes in nonconvex polygonal domains," by P. Chatzipantelidis and R. D. Lazarov, SIAM J. Numer. Anal., (42), no 5., 2005, pp. 1932-1958.