17. Qualitative properties of finite difference schemes

17.1. **Dissipation.** Consider the approximation of the linear transport equation $u_t+\alpha u_x=0$ by the finite difference schemes:

$$[U_j^{n+1}-U_j^{n-1}](2k)+\alpha[U_{j+1}^n-U_{j-1}^n]/(2h)=0, \quad \text{leapfrog},$$
$$[U_j^{n+1}-U_j^n]/k+\alpha[U_{j+1}^n-U_{j-1}^n]/(2h)-\alpha^2k[U_{j+1}^n-2U_j^n+U_{j-1}^n]/(2h^2)=0, \text{ Lax Wendroff.}$$

Introducing $\lambda=\alpha k/h$, we may rewrite these in the form:

$$U_j^{n+1}=U_j^{n-1}-\lambda[U_{j+1}^n-U_{j-1}^n], \quad \text{leapfrog},$$
$$U_j^{n+1}=U_j^n-(\lambda/2)[U_{j+1}^n-U_{j-1}^n]+(\lambda^2/2)[U_{j+1}^n-2U_j^n+U_{j-1}^n]. \quad \text{Lax Wendroff.}$$

Suppose that $U_j^n=(-1)^{j+n}\epsilon$, $n=0,1$, i.e., a mesh function with a high frequency oscillation. Then for the leapfrog scheme,

$$U_j^{n+1}=(-1)^{j+n-1}\epsilon-\lambda[(-1)^{j+n+1}\epsilon-(-1)^{j+n-1}\epsilon]=(-1)^{j+n-1}\epsilon=(-1)^{j+n+1}\epsilon.$$

So the leapfrog scheme propagates the disturbance without damping it. If we consider Lax-Wendroff, then we get

$$\begin{aligned}U_j^{n+1}=(-1)^{j+n}\epsilon+(\lambda^2/2)\epsilon[(-1)^{j+n+1}-2(-1)^{j+n}+(-1)^{j+n-1}]\\=(-1)^{j+n}\epsilon[1-2\lambda^2]=[1-2\lambda^2]U_j^n.\end{aligned}$$

If $|\lambda|<1$, then $|1-2\lambda^2|<1$, so the oscillation decreases in amplitude at each iteration The decreasing of high frequency oscillations is called *dissipation*.

Another way to understand dissipation of a numerical approximation scheme is to look at the scheme as a approximation of a modified equation. Consider the approximation of the transport equation $Lu\equiv u_t+\alpha u_x=0$ by the Lax-Friedrichs scheme:

$$L_{h,k}U_j^n\equiv\frac{1}{k}\{U_j^{n+1}-(1/2)[U_{j-1}^n+U_{j+1}^n]\}-\frac{\alpha}{2h}[U_{j+1}^n-U_{j-1}^n]=0,$$

If we look at the local truncation error of the method, we get by Taylor series expansions that

$$\begin{aligned}L_{h,k}u(x,t)-Lu(x,t)&=\frac{1}{k}\{u(x,t+k)-\frac{1}{2}[u(x-h,t)+u(x+h,t)]\\&\quad+\frac{\alpha}{2h}[u(x+h,t)-u(x-h,t)]-u_t(x,t)-\alpha u_x(x,t)\\&=\frac{1}{k}\{(u+ku_t+\frac{k^2}{2}u_{tt}+\cdots)-(u+\frac{h^2}{2}u_{xx}+\cdots)\}\\&\quad+\frac{\alpha}{2h}[2hu_x+\frac{1}{3}h^3u_{xxx}+\cdots]-u_t-\alpha u_x\\&=\frac{k}{2}u_{tt}-\frac{h^2}{2k}u_{xx}+O(k^2)+O(h^3/k)+O(h^2).\end{aligned}$$

Hence, if $k=O(h)$, then the local truncation error is $O(k)=O(h)$. However, if we view the Lax-Friedrichs scheme as an approximation of

$$u_t+\alpha u_x+\frac{k}{2}u_{tt}-\frac{h^2}{2k}u_{xx}=0,$$

with $k = O(h)$, then the local truncation error is $O(k^2) = O(h^2)$. The above equation is called the modified equation. If u is a sufficiently smooth solution of the modified equation, then by differentiating the equation, we obtain

$$u_{tt} = -\alpha u_{tx} - \frac{k}{2}u_{ttt} - \frac{h^2}{2k}u_{xxt} = -\alpha[-\alpha u_{xx}] + O(k) = \alpha^2 u_{xx} + O(k).$$

Inserting this result, we see that the Lax-Friedrichs scheme is a second order approximation to the equation

$$u_t + \alpha u_x = \frac{h^2}{2k}\left[1 - \frac{k^2}{h^2}\alpha^2\right] u_{xx}.$$

Setting $\lambda = \alpha k/h$, we can rewrite this in the form

$$u_t + \alpha u_x = h\frac{\alpha(1-\lambda^2)}{2\lambda}u_{xx}.$$

Thus, we have effectively added a small diffusion term to the right hand side that tends to smooth out the solution as t increases.

17.2. **Dispersion.** We again consider the transport equation $u_t + \alpha u_x = 0$ with initial condition $u(x, 0) = u_0(x)$. Recalling that the Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx,$$

we can write

$$\hat{u}(p, t + k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx} u(x, t + k)\, dx.$$

But for α a constant, $u(x, t) = u_0(x - \alpha t)$ and so

$$\hat{u}(p, t + k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx} u_0(x - \alpha[t + k])\, dx.$$

Setting $y = x - \alpha k$, we get

$$\hat{u}(p, t + k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ip(y+\alpha k)} u_0(y - \alpha t)\, dy = \frac{1}{\sqrt{2\pi}}e^{-ip\alpha k}\int_{-\infty}^{\infty} e^{-ipy} u(y, t)\, dy.$$

Hence $\hat{u}(p, t + k) = e^{-ip\alpha k}\hat{u}(p, t)$.

If we approximate this problem by a two level finite difference scheme, then we showed that

$$\hat{U}^{n+1}(p) = G(p, h, k)\hat{U}^n(p).$$

So we expect that $G(p, h, k)$ should be a good approximation to $e^{-ip\alpha k}$. Write

$$G(p, h, k) = |G|e^{i\theta} = |G|e^{-ip\omega k},$$

i.e., $\theta = -p\omega k$. In general, $\omega = \omega(p)$ will depend on p and is called the phase speed, i.e., the speed at which waves of frequency p are propagated by the finite difference scheme. To propagate at the correct scheme, we would need $\omega = \alpha$ for all p. Since in general, ω depends on p, ω will only be an approximation to α. *Dispersion* is the phenomena of waves of different frequencies traveling at different speeds. Define $\alpha - \omega(p)$ to be the phase error.

Example: Lax-Wendroff scheme:

$$L_{h,k}U_j^n \equiv [U_j^{n+1} - U_j^n]/k + \alpha[U_{j+1}^n - U_{j-1}^n]/(2h) - \alpha^2 k[U_{j+1}^n - 2U_j^n + U_{j-1}^n]/(2h^2) = 0.$$

The amplification factor is:

$$g = 1 - 2\lambda^2 \sin^2(ph/2) - i\lambda \sin(ph), \qquad \lambda = \alpha k/h.$$

If we write $g = |g|e^{i\theta} = |g|(\cos\theta + i\sin\theta)$, then

$$\tan\theta = \frac{\operatorname{Imag} g}{\operatorname{Real} g} = \frac{-\lambda\sin(ph)}{1 - 2\lambda^2\sin^2(ph/2)}. \qquad \theta = -p\omega k.$$

Using Taylor series expansions, one can show that

$$\omega = \alpha[1 + O(p^2h^2)].$$

To see this, we use

$$\frac{1}{1-x} = 1 + x + O(x^2), \quad \sin x = x - x^3/6 + O(x^5),$$
$$\sin^2 x = x^2 - x^4/3 + O(x^6), \quad \tan^{-1} x = x + O(x^3)$$

to get

$$\frac{-\lambda\sin(ph)}{1 - 2\lambda^2\sin^2(ph/2)} = -\lambda(ph)\Big[1 + O(p^2h^2)\Big]\Big[1 + O(p^2h^2)\Big] = -\lambda(ph)[1 + O(p^2h^2)].$$

Then

$$-p\omega k = \theta = \tan^{-1}[\tan\theta] = \tan^{-1}\Big(-\lambda(ph)[1 + O(p^2h^2)\Big)$$
$$= [-\lambda(ph)[1 + O(p^2h^2)] + O(p^3h^3) = -\lambda(ph)[1 + O(p^2h^2)].$$

Hence,

$$\omega = \frac{\lambda(ph)[1 + O(p^2h^2)]}{pk} = \frac{\alpha kp[1 + O(p^2h^2)]}{pk} = \alpha[1 + O(p^2h^2)].$$