1.4. **Other approaches to approximation on domains with curved boundaries.** A simple scheme for domains with curved boundaries is to define U_h as the solution of

$$-\Delta_h U_h(x,y) = f(x,y), \quad (x,y) \in \Omega_h^0, \qquad U_h(x,y) = g(x',y'), \quad (x,y) \in \Omega_h^*,$$

where (x',y') is one of the neighbors of (x,y) on $\partial\Omega_h$. In this case, we only use the standard 5 point difference approximation to the Laplacian. The result is:

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le M_1 h + \frac{M_4 d^2}{96} h^2, \qquad M_1 = \max_{\bar{\Omega}}(\max |\partial u/\partial x|, |\partial u/\partial y|).$$

Note that the crude approximation of the boundary condition gives only an $O(h)$ error estimate.

An $O(h^2)$ approximation can be obtained by the following method, due to L. Collatz. Define $U_h(x,y)$ as the solution of

$$-\Delta_h U_h(x,y) = f(x,y), \quad (x,y) \in \Omega_h^0, \qquad U_h(x,y) = g(x,y), \quad (x,y) \in \partial\Omega_h,$$

and for $(x,y) \in \Omega_h^*$, define $U_h(x,y)$ as the linear interpolate of the value of U_h at two neighbors of (x,y), one in Ω_h^0 and one on $\partial\Omega_h$. For example, if $(x+h,y) \in \Omega_h^0$ and $(x-\alpha h, y) \in \partial\Omega_h$, define U_h at the point (x,y) by

$$U_h(x,y) = \frac{\alpha}{\alpha+1} U_h(x+h,y) + \frac{1}{\alpha+1} U_h(x-\alpha h, y).$$

The result is:

$$\max_{\Omega_h \cup \partial\Omega_h} |u - U_h| \le M_2 h^2 + \frac{M_4 d^2}{48} h^2, \qquad M_2 = \max_{\bar{\Omega}}(\max |\partial^2 u/\partial x^2|, |\partial u^2/\partial y^2|).$$

1.5. **Other boundary conditions.** We next consider the boundary condition

$$\alpha(x,y)u(x,y) + \beta(x,y)\frac{\partial u}{\partial n}(x,y) = g(x,y).$$

Consider first the case of a point on a straight boundary, say $x = 1$, and $0 < y < 1$. At the boundary point $(1,y)$, an $O(h)$ approximation to $\partial u/\partial n = \partial u/\partial x$ is given by $[u(1,y) - u(1-h,y)]/h$, so the boundary condition would be approximated by:

$$\alpha(1,y)u(1,y) + \beta(1,y)[u(1,y) - u(1-h,y)]/h = g(1,y).$$

An $O(h^2)$ approximation to $\partial u/\partial x$ is given by the centered difference: $[u(1+h,y) - u(1-h,y)]/(2h)$. This introduces a new unknown at the point $1+h, y$ outside the domain. Hence, we need an additional equation. Assuming that the solution is smooth and the partial differential equation holds on the boundary as well, we can use the 5 point difference approximation to the Laplacian applied at the boundary point, i.e., we have the equation

$$U_h(1+h,y) + U_h(1-h,y) + U_h(1,y+h) + U_h(1,y-h) - 4U_h(1,y) = h^2 f(1,y).$$

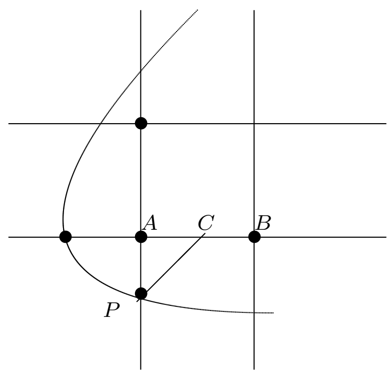
This equation can be used to eliminate the new unknown.

If the boundary is curved, draw the normal line through the point $P = (x,y)$ and assume it intersects a mesh line at a point C where C lies between the mesh points A and B. Then we approximate $\partial u/\partial n(P)$ by $[u(P) - u(C)]/|P - C|$, where $|P - C|$ denotes the distance between P and C and $u(C)$ is defined by linear interpolation using $u(A)$ and $u(B)$, i.e.,

$$u(C) = \frac{|B - C|}{|B - A|}u(A) + \frac{|C - A|}{|B - A|}u(B).$$

Inserting this formula gives a linear relation equation involving $u(A)$, $u(B)$, and $u(P)$.

Approximation of $\partial u/\partial n$



1.6. **Higher order approximations.** To get higher order approximations to $\Delta u(x, y)$, we need to take more points at a larger distance from (x, y). Using Taylor series expansions, we have

$$u(x \pm kh, y) = u(x, y) \pm kh\frac{\partial u}{\partial x}(x, y) + \frac{k^2h^2}{2}\frac{\partial^2 u}{\partial x^2}(x, y) \pm \frac{k^3h^3}{6}\frac{\partial^3 u}{\partial x^3}(x, y) + \frac{k^4h^4}{24}\frac{\partial^4 u}{\partial x^4}(x, y) \pm \frac{k^5h^5}{120}\frac{\partial^5 u}{\partial x^5}(x, y) + \frac{k^6h^6}{6!}\frac{\partial^6 u}{\partial x^6}(\xi_\pm, y).$$

Hence,

$$u(x + kh, y) + u(x - kh, y) - 2u(x, y) = 2\frac{k^2h^2}{2}\frac{\partial^2 u}{\partial x^2}(x, y) + 2\frac{k^4h^4}{24}\frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6).$$

For $k = 1, 2$, this gives

$$u(x + h, y) + u(x - h, y) - 2u(x, y) = h^2\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6),$$

$$u(x + 2h, y) + u(x - 2h, y) - 2u(x, y) = 4h^2\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{16h^4}{12}\frac{\partial^4 u}{\partial x^4}(x, y) + O(h^6).$$

Taking 16 times the first equation minus the second equation, we get

$$16u(x + h, y) + 16u(x - h, y) - u(x + 2h, y) - u(x - 2h, y) - 30u(x, y) = 12h^2\frac{\partial^2 u}{\partial x^2}(x, y) + O(h^6).$$

Taking a similar expansion in the y variable, we get

$$\begin{aligned}\{16[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)]\\ -[u(x+2h,y)+u(x-2h,y)+u(x,y+2h)+u(x,y-2h)]-60u(x,y)]\}/(12h^2)\\ =\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)+O(h^4).\end{aligned}$$

Note that in the case of the unit square, this higher order approximation cannot be used at interior mesh points at a distance h from the boundary of Ω, since it would involve mesh points outside the domain. At these points, we can use the 5-point difference approximation without affecting the overall accuracy of the method.

On domains with curved boundaries, the situation is much more complicated. Use of the Shortly-Weller formula would decrease the rate of convergence. So instead, we consider finite element methods, which handle these difficulties in a more natural way.

1.7. **More general elliptic operators.** In this section, we develop difference approximations for the operator $\operatorname{div}(a \operatorname{grad} u)$ in two dimensions, where $a = a(x,y)$. Although we could expand this operator to $a\Delta u + \nabla a \cdot \nabla u$ and then apply difference approximations, this is not the best approach, since it does not lead to a symmetric matrix. To see a better approach, consider the one dimensional analogue, $(d/dx)(adu/dx)$. The basic idea is to use the approximation $dp/dx(x_j) \approx \frac{1}{h}[p(x_{j+1/2}) - p(x_{j-1/2}]$. Choosing $p = adu/dx$, we get

$$\begin{aligned}\frac{d}{dx}\left[a\frac{du}{dx}\right](x_j) &\approx \frac{1}{h}\left\{\left[a\frac{du}{dx}\right](x_{j+1/2}) - \left[a\frac{du}{dx}\right](x_{j-1/2})\right\}\\ &\approx \frac{1}{h^2}\{a(x_{j+1/2})[u(x_{j+1})-u(x_j)] - a(x_{j-1/2})[u(x_j)-u(x_{j-1})]\}.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{div}(a \operatorname{grad} u) \approx \frac{1}{h^2}\big\{&a(x_{j+1/2},y_l)[u(x_{j+1},y_l)-u(x_j,y_l)] - a(x_{j-1/2},y_l)[u(x_j,y_l)-u(x_{j-1},y_l)]\\ &+ a(x_j,y_{l+1/2})[u(x_j,y_{l+1})-u(x_j,y_l)] - a(x_j,y_{l-1/2})[u(x_j,y_l)-u(x_j,y_{l-1})]\big\}.\end{aligned}$$