## 2. Finite Element Methods for Elliptic Equations

2.1. **Preliminaries and variational formulations.** We will study the approximation of problems of the form:

$$-\operatorname{div}(p\nabla u) + qu = f, \quad \text{in } \Omega, \tag{2.1}$$

together with one of the following boundary conditions

$$(i) \quad u = g, \qquad \text{or} \qquad (ii) \quad p\frac{\partial u}{\partial n} + \gamma u = g.$$

We shall assume that the given coefficients $p, q, \gamma$ satisfy

$$p(x) \geq p_0 > 0, \qquad q(x) \geq 0, \qquad \gamma(x) \geq 0.$$

Note that if we take $p = 1$ and $q = 0$, then we have Poisson's equation $-\Delta u = f$.

The finite element method is not based directly on the partial differential equation, but rather on a variational formulation of the problem. To discuss this, we first introduce some notation and formulas. First, we introduce the space $L^2(\Omega)$ as the set of functions $v$ for which $\int_\Omega v^2\, dx < \infty$. For integer $m \geq 0$, we then define the Sobolev spaces

$$H^m(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega)\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \ldots \alpha_n \leq m$ and

$$D^\alpha v = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We also define the norm

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2\, dx$$

Example: in two dimensions,

$$\|u\|_0^2 = \int_\Omega u^2\, dx, \qquad \|u\|_1^2 = \int_\Omega \left( u^2 + \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \right) dx$$

$$\|u\|_2^2 = \int_\Omega \left( u^2 + \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2 \right) dx$$

Define Sobolev spaces of functions satisfying boundary conditions:

$$\mathring{H}^m(\Omega) = \{v \in H^m(\Omega) : D^\alpha v = 0 \text{ on } \partial\Omega, |\alpha| \leq m - 1\}.$$

Example: $\mathring{H}^1(\Omega)$ denotes functions in $H^1(\Omega)$ that are zero on $\partial\Omega$.

Example: We will consider the space of continuous piecewise linear functions. Such functions are not in $C^1(\Omega)$, but will belong to the space $H^1(\Omega)$.

To write down a variational formulation of our boundary value problem, we need to use some identities (called Green's identities) that are just integration by parts formulas. All these identities will follow from the Divergence Theorem:

$$\int_\Omega \operatorname{div} \boldsymbol{\varphi}\, dx = \int_{\partial\Omega} \boldsymbol{\varphi} \cdot \boldsymbol{n}\, ds,$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to $\Omega$. If we set $\boldsymbol{\varphi} = v\boldsymbol{\psi}$, then

$$\operatorname{div} \boldsymbol{\varphi} = \sum_{i=1}^N \frac{\partial[v\psi_i]}{\partial x_i} = \sum_{i=1}^N \left[\psi_i \frac{\partial v}{\partial x_i} + v\frac{\partial \psi_i}{\partial x_i}\right] = \nabla v \cdot \boldsymbol{\psi} + v \operatorname{div} \boldsymbol{\psi}.$$

Hence,

$$\int_\Omega [\nabla v \cdot \boldsymbol{\psi} + v \operatorname{div} \boldsymbol{\psi}]\, dx = \int_\Omega \operatorname{div}[v\boldsymbol{\psi}]\, dx = \int_{\partial\Omega} v\boldsymbol{\psi} \cdot \boldsymbol{n}\, ds.$$

If we pick $\boldsymbol{\psi} = (0, \cdots, 0, u, 0, \cdots 0)$, with $u$ in the $i$th position, then we get the integration by parts formula:

$$\int_\Omega \left[\frac{\partial v}{\partial x_i} u + v\frac{\partial u}{\partial x_i}\right] dx = \int_{\partial\Omega} vun_i\, ds.$$

If we choose $\boldsymbol{\psi} = p\nabla u$, then

$$\int_\Omega [\nabla v \cdot p\nabla u + v \operatorname{div}(p\nabla u)]\, dx = \int_{\partial\Omega} vp\nabla u \cdot \boldsymbol{n}\, ds,$$

which we may rewrite as:

$$\int_\Omega p\nabla u \cdot \nabla v\, dx = -\int_\Omega \operatorname{div}(p\nabla u) v\, dx + \int_{\partial\Omega} pv\frac{\partial u}{\partial n}\, ds.$$

This formula is one of the classical Green's identities. Using this formula, we can now give a variational formulation of the boundary value problem. We first consider the case of boundary condition (i) with $g = 0$, i.e., $u = 0$ on $\partial\Omega$. The variational formulation of the BVP in this case is given by:

Find $u \in \mathring{H}^1(\Omega)$ such that

$$\int_\Omega [p\nabla u \cdot \nabla v + quv]\, dx = \int_\Omega fv\, dx, \quad \text{for all } v \in \mathring{H}^1(\Omega). \tag{2.2}$$

Note that in this case, we are building the boundary condition into the space in which we seek the solution (called an essential boundary condition).

The relationship of the solution of problem (2.2) to the original boundary value problem can be stated as follows.

**Lemma 1.** *If $u$ is a smooth solution of* (2.1) *with boundary condition $u = 0$ on $\partial\Omega$, then $u$ is a solution of* (2.2). *Conversely, if $u$ is a solution of* (2.2) *and $u$ is sufficiently smooth, then $u$ is a solution of* (2.1) *and satisfies $u = 0$ on $\partial\Omega$.*

*Proof.* First suppose $u$ is a smooth solution of (2.1). Then for any function $v \in \mathring{H}^1(\Omega)$,

$$\int_\Omega (-\operatorname{div} p\nabla u + qu)v\, dx = \int_\Omega fv\, dx.$$

Applying Green's formula, and observing that the boundary term vanishes since $v = 0$ on $\partial\Omega$, we find that $u$ satisfies:

$$\int_\Omega [p\nabla u \cdot \nabla v + quv]\,dx = \int_\Omega (-\operatorname{div} p\nabla u + qu)v\,dx + \int_{\partial\Omega} pv\frac{\partial u}{\partial n}\,ds = \int_\Omega fv\,dx.$$

Since $u = 0$ on $\partial\Omega$ and is smooth, $u \in \mathring{H}^1(\Omega)$ and hence is a solution of (2.2). Now suppose that $u$ is a solution of (2.2) and is smooth. Again applying Green's formula, we find that $u$ satisfies

$$\int_\Omega (-\operatorname{div} p\nabla u + qu - f)v\,dx = 0, \quad \text{for all } v \in \mathring{H}^1(\Omega).$$

Since this equation holds for all such $v$, it can be shown that $-\operatorname{div} p\nabla u + qu - f = 0$. Since $u \in \mathring{H}^1(\Omega)$, $u = 0$ on $\partial\Omega$. □

Next consider boundary condition (ii). The variational formulation of the BVP in this case is given by:

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega [p\nabla u \cdot \nabla v + quv]\,dx + \int_{\partial\Omega} \gamma uv\,ds = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds, \quad \text{for all } v \in H^1(\Omega). \tag{2.3}$$

Note that in this case, we do not build the boundary condition into the space. As we shall see, the solution of (2.3) automatically satisfies this boundary condition (called a natural boundary condition).

Also, note that in some cases, this problem will not have a solution unless the data satisfies a compatibility relation. For example, if $q = 0$ and $\gamma = 0$, then choosing $v = 1$, we must have

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0.$$

If this equation holds, we will have a solution, but it will not be unique, since if we add any constant to $u$, it will still be a solution. One way around this to to then add the extra condition that $\int_\Omega u\,dx = 0$.

The relationship of the solution of problem (2.3) to the original boundary value problem can be stated as follows.

**Lemma 2.** *If $u$ is a smooth solution of* (2.1) *with boundary condition (ii) $\partial u/\partial n + \gamma u = g$ on $\partial\Omega$, then $u$ is a solution of* (2.3). *Conversely, if $u$ is a solution of* (2.3) *and $u$ is sufficiently smooth, then $u$ satisfies* (2.1) *and boundary condition (ii).*

*Proof.* First suppose $u$ is a smooth solution of (2.1). Then for any function $v \in H^1(\Omega)$,

$$\int_\Omega (-\operatorname{div} p\nabla u + qu)v\,dx = \int_\Omega fv\,dx.$$

Applying Green's formula, we find that $u$ satisfies:

$$\begin{aligned}\int_\Omega [p\nabla u \cdot \nabla v + quv]\,dx &= \int_\Omega (-\operatorname{div} p\nabla u + qu)v\,dx + \int_{\partial\Omega} pv\frac{\partial u}{\partial n}\,ds \\ &= \int_\Omega fv\,dx + \int_{\partial\Omega}(g-\gamma u)v\,ds.\end{aligned}$$

Hence, $u$ satisfies (2.3). Now suppose $u$ satisfies (2.3) and is sufficiently smooth. Applying Green's formula, we get

$$\begin{aligned}\int_\Omega [p\nabla u \cdot \nabla v + quv]\,dx + \int_{\partial\Omega}\gamma uv\,ds &= \int_\Omega (-\operatorname{div} p\nabla u + qu)v\,dx + \int_{\partial\Omega}(p\frac{\partial u}{\partial n} + \gamma u)v\,ds \\ &= \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,ds.\end{aligned}$$

By first choosing $v \in \mathring{H}^1(\Omega)$, we find that $u$ must satisfy (2.1). Since the boundary variation holds for all $v \in H^1(\Omega)$, we can conclude that $u$ also satisfies the boundary condition (ii). $\square$