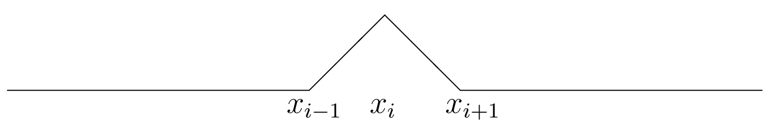
2.7. **Global bases for finite element spaces.** We have seen that if the shape functions $V(T)$ consist of polynomials and the degrees of freedom are of the form $P(\boldsymbol{b}_i)$, where the $\boldsymbol{b}_i$ are points in the triangle T, then we can find a basis function $\phi_i \in V(T)$ with the property that $\phi_i(\boldsymbol{b}_j) = 1$, if $i = j$ and $\phi_i(\boldsymbol{b}_j) = 0$, if $i \neq j$. Furthermore, we can then write $P(\boldsymbol{x}) = \sum_{i=1}^{M} P(\boldsymbol{b}_i)\phi_i(\boldsymbol{x})$.

The next step is to define basis functions for the full finite element space, V_h, not just its restriction $V(T)$ to the triangle T. This can be done in a similar way, by now finding functions $\phi_i \in V_h$ such that $\phi_i(\boldsymbol{b}_j) = 1$, if $i = j$ and $\phi_i(\boldsymbol{b}_j) = 0$, if $i \neq j$, where now the values $u(\boldsymbol{b}_j)$ denote the total set of global degrees of freedom of a function u in the finite element space.

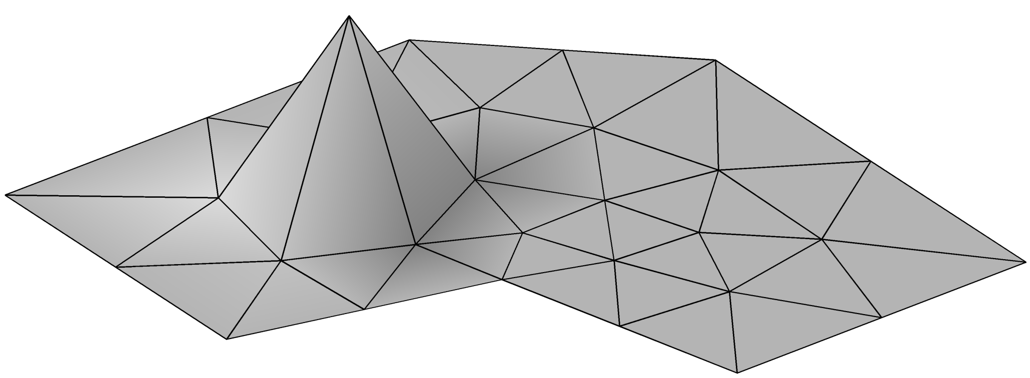
A simple example is the set of continuous piecewise linear functions on a mesh of the interval $[0, 1]$ with mesh points $0 = x_0 < x_1 < \dots x_N = 1$. Then the continuous piecewise linear function ϕ_i that satisfies $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for $j \neq i$ is given by:

$$\phi_i(x) = \begin{cases} 0 & 0 \leq x \leq x_{i-1}, \\ (x - x_{i-1})/(x_i - x_{i-1}) & x_{i-1} \leq x \leq x_i, \\ (x_{i+1} - x)/(x_{i+1} - x_i) & x_i \leq x \leq x_{i+1}, \\ 0 & x_{i+1} \leq x \leq 1. \end{cases}$$

Graph of $\phi_i(x)$



In two dimensions, the analogous continuous piecewise linear basis function is:



Note that the basis function corresponding to the degree of freedom $u(\boldsymbol{b}_i)$ is non-zero only on the triangles which have $\boldsymbol{b}_i$ as one of its vertices.

Recall, that in order to find a finite element space, consisting of piecewise polynomials, that is a subspace of $H^1(\Omega)$, the subspace must belong to $C^0(\Omega)$. In one dimension, we accomplish this by choosing $u(x_i)$ to be among the degrees of freedom for each mesh point x_i, and requiring it to be single-valued. This means that the values of the linear functions on the interval $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ will be the same at the point x_i. In two dimensions, we need to choose $u(\boldsymbol{b}_i)$ to be among the degrees of freedom for each vertex $\boldsymbol{b}_i$, since we need $(u|_T)(\boldsymbol{b}_i)$ to have the same value, no matter which triangle T with common vertex $\boldsymbol{b}_i$ we consider. For piecewise linear functions, we have seen that this set of degrees of freedom completely define the finite element space. If we consider two triangles with a common edge joining the vertices $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$, and let $\boldsymbol{x} = (1-\theta)\boldsymbol{a}_i + \theta\boldsymbol{a}_j$ be a point on that edge, then if u is a linear function, $u(\boldsymbol{x}) = (1-\theta)u(\boldsymbol{a}_i) + \theta u(\boldsymbol{a}_j)$ Hence, if $u_1 = u|_{T_1}$ and $u_2 = u|_{T_2}$ are linear functions defined on the triangles T_1 and T_2 with a common edge, and $u_1(\boldsymbol{a}_i) = u_2(\boldsymbol{a}_i)$, $u_1(\boldsymbol{a}_j) = u_2(\boldsymbol{a}_j)$, then $u_1(\boldsymbol{x}) = u_2(\boldsymbol{x})$ for all $\boldsymbol{x}$ on the edge, and so u is continuous there.

For piecewise quadratics, if we choose as global degrees of freedom the values at the vertices and the midpoints of the edges, we will again get continuity across edges, since two quadratics in one variable that agree at three points, must be identical everywhere. In general, two polynomials of degree $\le r$ in one variable that agree at $r+1$ points must be identical, so this gives a way of choosing degrees of freedom that give global continuity.

2.8. **Affine families of finite elements.** To describe a finite element on an arbitrary triangle in $\mathbb{R}^2$ in a way that will simplify computations, we use the following idea. Let $\hat{T}$ denote the reference triangle with vertices $\hat{\boldsymbol{a}}_1 = (1,0)$, $\hat{\boldsymbol{a}}_2 = (0,1)$, and $\hat{\boldsymbol{a}}_3 = (0,0)$. Given any triangle T with vertices $\boldsymbol{a}_1 = (a_{11}, a_{21})$, $\boldsymbol{a}_2 = (a_{12}, a_{22})$, $\boldsymbol{a}_3 = (a_{13}, a_{23})$, we can find a unique invertible affine mapping F_T mapping the triangle $\hat{T}$ to the triangle T given by $\boldsymbol{x} = F_T(\hat{\boldsymbol{x}}) = B_T\hat{\boldsymbol{x}} + b_T$, where B_T is an invertible 2×2 matrix and b_T a two-dimensional vector, satisfying: $F_T(\hat{\boldsymbol{a}}_i) = \boldsymbol{a}_i, \quad i = 1, \dots 3$. In fact, we can easily show that

$$B_T = \begin{pmatrix} a_{11}-a_{13} & a_{12}-a_{13} \\ a_{21}-a_{23} & a_{22}-a_{23} \end{pmatrix}, \qquad b_T = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}.$$

Hence, we have the relations:

$$x_1 = (a_{11}-a_{13})\hat{x}_1 + (a_{12}-a_{13})\hat{x}_2 + a_{13}, \qquad x_2 = (a_{21}-a_{23})\hat{x}_1 + (a_{22}-a_{23})\hat{x}_2 + a_{23}.$$

If we invert this relationship, we find that

$$\hat{x}_1 = [(a_{22}-a_{23})(x_1-a_{13}) - (a_{12}-a_{13})(x_2-a_{23})]/\det(B_T),$$
$$\hat{x}_2 = [-(a_{21}-a_{23})(x_1-a_{13}) + (a_{11}-a_{13})(x_2-a_{23})]/\det(B_T).$$

Given a set of shape functions $\hat{p} \in V(\hat{T})$ on the reference triangle $\hat{T}$, we then define the set of shape functions $V(T)$ on the triangle $T = F_T(\hat{T})$ by:

$$V(T) = \{p : p(\boldsymbol{x}) = \hat{p} \circ F_T^{-1}(\boldsymbol{x}) = \hat{p}(\hat{\boldsymbol{x}}), \quad \hat{p} \in V(\hat{T})\}.$$

With these definitions, we can translate the degrees of freedom and basis functions on the reference element to degrees of freedom and basis functions on an arbitrary triangle. For example, if we take as degrees of freedom on the reference triangle the values of a polynomial $\hat{p}$ at the points $\hat{\boldsymbol{x}}_i$, i.e., $\hat{p}(\hat{\boldsymbol{x}}_i)$, then the degrees of freedom on the triangle T will be the values $p(\boldsymbol{x})$ satisfying $p(\boldsymbol{x}) = \hat{p}(\hat{\boldsymbol{x}})$. In the case of linear functions, this means the values at the vertices of the triangle, since the map F_T is constructed to map vertices to vertices. We have already shown that the basis functions for linear functions on a triangle corresponding to vertex degrees of freedom are given by the barycentric coordinate functions $\lambda_i(\boldsymbol{x})$. By the presentation above, this means that if we denote by $\hat{\lambda}_i(\hat{\boldsymbol{x}})$, the barycentric coordinates on the reference triangle, then we have $\hat{\lambda}_i(\hat{\boldsymbol{x}}) = \lambda_i(\boldsymbol{x})$. To get an understanding about how these mappings work, consider the following example. Let T be the triangle with vertices $\boldsymbol{a}_1 = (1,0)$, $\boldsymbol{a}_2 = (0,1)$ and $\boldsymbol{a}_3 = (1,1)$. We can map $\hat{T}$ to T by the mapping $\boldsymbol{x} = F_T(\hat{\boldsymbol{x}}) = B_T\hat{\boldsymbol{x}} + \boldsymbol{b}_T$, where

$$B_T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \boldsymbol{b}_T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{i.e.,} \quad x_1 = 1 - \hat{x}_2, \qquad x_2 = 1 - \hat{x}_1.$$

Then

$$\begin{aligned}
\lambda_1(\boldsymbol{x}) &= \hat{\lambda}_1(\hat{\boldsymbol{x}}) = \hat{x}_1 = 1 - x_2, \\
\lambda_2(\boldsymbol{x}) &= \hat{\lambda}_2(\hat{\boldsymbol{x}}) = \hat{x}_2 = 1 - x_1, \\
\lambda_3(\boldsymbol{x}) &= \hat{\lambda}_3(\hat{\boldsymbol{x}}) = 1 - \hat{x}_1 - \hat{x}_2 = x_1 + x_2 - 1.
\end{aligned}$$

Thus, we can construct basis functions in terms of the barycentric coordinates on the reference triangle and then map them to obtain the basis functions on the general triangle.

We can use these ideas to calculate the integrals in the finite element method. For example, to calculate an integral of the form

$$\int_T \nabla u \cdot \nabla v \, dx\, dy = \int_T \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\, dy,$$

we transform the integral to the reference triangle by making the change of variable $\boldsymbol{x} = B\hat{\boldsymbol{x}} + \boldsymbol{b}$. Since $\hat{v}(\hat{x},\hat{y}) = v(x,y)$, we get by the chain rule that

$$\frac{\partial v}{\partial x} = \frac{\partial \hat{v}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{v}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial \hat{v}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{v}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}.$$

Now $\hat{\boldsymbol{x}} = B^{-1}(\boldsymbol{x} - \boldsymbol{b})$, so

$$\frac{\partial \hat{x}}{\partial x} = B^{-1}_{11}, \quad \frac{\partial \hat{x}}{\partial y} = B^{-1}_{12}, \quad \frac{\partial \hat{y}}{\partial x} = B^{-1}_{21}, \quad \frac{\partial \hat{y}}{\partial y} = B^{-1}_{22}.$$

Hence,

$$\begin{pmatrix} \partial v/\partial x \\ \partial v/\partial y \end{pmatrix} = (B^{-1})^T \begin{pmatrix} \partial \hat{v}/\partial \hat{x} \\ \partial \hat{v}/\partial \hat{y} \end{pmatrix}, \quad \text{i.e.,} \quad \nabla v = (B^{-1})^T \hat{\nabla}\hat{v}.$$

Since $(B^{-1})^T = (B^T)^{-1}$, we also have $\hat{\nabla}\hat{v} = B^T \nabla v$. By the change of variable formula, we get

$$\begin{aligned}
\int_T \nabla u \cdot \nabla v \, dx\, dy &= \int_{\hat{T}} (B^{-1})^T \hat{\nabla}\hat{u} \cdot (B^{-1})^T \hat{\nabla}\hat{v} |\det B| \, d\hat{x} d\hat{y} \\
&= |\det B| \int_{\hat{T}} (\hat{\nabla}\hat{u})^T B^{-1}(B^{-1})^T \hat{\nabla}\hat{v} \, d\hat{x} d\hat{y}.
\end{aligned}$$

In the case of piecewise linear functions,

$$\frac{\partial \hat{v}}{\partial \hat{x}} = \hat{v}(\hat{\boldsymbol{a}}_1) - \hat{v}(\hat{\boldsymbol{a}}_3) = v(\boldsymbol{a}_1) - v(\boldsymbol{a}_3), \qquad \frac{\partial \hat{v}}{\partial \hat{y}} = \hat{v}(\hat{\boldsymbol{a}}_2) - \hat{v}(\hat{\boldsymbol{a}}_3) = v(\boldsymbol{a}_2) - v(\boldsymbol{a}_3).$$

We then compute

$$\int_\Omega \nabla u \cdot \nabla v \, dx \, dy = \sum_T \int_T \nabla u \cdot \nabla v \, dx \, dy.$$