2.9. **Tensor product finite elements.** We define the space Q_k as the space of polynomials which are of degree $\le k$ in each of the variables $x_1, \dots x_n$ separately. The dimension of the space Q_k is $(k+1)^n$.

Example: $k = 1$:

$$Q_1(x, y) = c_0 + c_1 x + c_2 y + c_3 xy \quad (n = 2),$$
$$Q_1(x, y, z) = c_0 + c_1 x + c_2 y + c_3 z + c_4 xy + c_5 xz + c_6 yz + c_7 xyz \quad (n = 3).$$

Lemma 7. *A polynomial $q \in Q_k$ is uniquely determined by its values on the set*

$$M_k = \{\boldsymbol{x} = (i_1/k, i_2/k, \dots, i_n/k) \in \mathbb{R}^n : i_j \in \{0, 1, \dots, k\}, \quad 1 \le j \le n\}.$$

Example: $k = 1$: $n = 2$: $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$

$$q(x, y) = q(0,0)(1-x)(1-y) + q(1,0)x(1-y) + q(0,1)y(1-x) + q(1,1)xy.$$

Example: $k = 2$: $n = 2$: $(0,0)$, $(1/2,0)$, $(1,0)$, $(0,1/2)$, $(1/2,1/2)$, $(1,1/2)$, $(0,1)$, $(1/2,1)$, $(1,1)$.

2.10. **Quadrilateral elements.** These are defined by using bilinear mappings of a rectangle, i.e., we define as the reference element the unit square $\hat{K}$. Then we can define the mapping

$$\boldsymbol{F}(\hat{x}, \hat{y}) = \boldsymbol{a}_1(1-\hat{x})(1-\hat{y}) + \boldsymbol{a}_2\hat{x}(1-\hat{y}) + \boldsymbol{a}_3\hat{y}(1-\hat{x}) + \boldsymbol{a}_4\hat{x}\hat{y}.$$

that maps $\hat{K}$ to the quadrilateral K with vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$. To define the shape functions on the element K, we begin with a set $\hat{V}$ of polynomial shape functions defined on the reference element (e.g., the space $Q_1(\hat{x}, \hat{y})$). We then define a space of functions on the quadrilateral K by

$$V_F(K) = \{u : K \mapsto \mathbb{R} : \hat{u}_{K,F} \in \hat{V}\},$$

where $\hat{u}_{K,F}(\hat{x}, \hat{y}) = u \circ \boldsymbol{F}(\hat{x}, \hat{y}) = u(x, y)$. The complication here is that the elements in $V_F(K)$ need not be polynomials if the mapping $\boldsymbol{F}$ is not affine, i.e., if K is not a parallelogram. For example, consider the mapping $(x, y) = F(\hat{x}, \hat{y}) = (\hat{x}, \hat{y}(1+\hat{x}))$. Then $\hat{x} = x$ and $\hat{y} = y/(1+x)$. Hence, if we take $\hat{V}$ to be the span of $1, \hat{x}, \hat{y}, \hat{x}\hat{y}$, then $V_F(K)$ is the span of $1, x, y/(1+x), xy/(1+x)$.

2.11. C^1 **finite elements.** The Argyris element is a C^1 finite element defined on each triangle by a quintic polynomial with 21 degrees of freedom:

$$p(\boldsymbol{a}_i), (\partial p/\partial x)(\boldsymbol{a}_i), (\partial p/\partial y)(\boldsymbol{a}_i), (\partial^2 p/\partial x^2)(\boldsymbol{a}_i), (\partial^2 p/\partial x \partial y)(\boldsymbol{a}_i), (\partial^2 p/\partial y^2)(\boldsymbol{a}_i), \quad 1 \le i \le 3,$$
$$(\partial p/\partial n)(\boldsymbol{a}_{12}), (\partial p/\partial n)(\boldsymbol{a}_{13}), (\partial p/\partial n)(\boldsymbol{a}_{23}).$$

There are also other choices of degrees of freedom.

2.12. **Error estimates for piecewise polynomial interpolation – 1 dimension.** The error estimate we derived previously related the error in the finite element method to the error in the approximation of the solution by the best approximation in the finite element subspace. We now obtain bounds for this error by defining and analyzing the properties of the finite element interpolant, considering the case of piecewise linear interpolation in detail.

We begin with the Taylor series expansion with integral remainder in one dimension.

$$F(s) - F(s_0) = \int_{s_0}^{s} F'(t)\,dt = \int_{s_0}^{s} F'(t)\frac{d}{dt}(t-s)\,dt$$

$$= F'(t)(t-s)\,|_{t=s_0}^{t=s} - \int_{s_0}^{s} F''(t)(t-s)\,dt = F'(s_0)(s-s_0) - \int_{s_0}^{s} F''(t)(t-s)\,dt.$$

$$\text{Hence,}\qquad F(s) = F(s_0) + F'(s_0)(s-s_0) + \int_{s_0}^{s} F''(t)(s-t)\,dt.$$

We next give a simple derivation of the error in 1-dimension. Let $a < b$ be given points. Then the linear function interpolating u at the points a and b is given by:

$$u_I(x) = \frac{b-x}{b-a}u(a) + \frac{x-a}{b-a}u(b).$$

To estimate the error $u(x) - u_I(x)$, we expand $u(a)$ and $u(b)$ in a Taylor series about the point x, using the integral remainder formula given above.

$$u(a) = u(x)+u'(x)(a-x)+\int_x^a u''(t)(a-t)\,dt, \qquad u(b) = u(x)+u'(x)(b-x)+\int_x^b u''(t)(b-t)\,dt.$$

Then

$$u(x) - u_I(x) = u(x) - \frac{b-x}{b-a}u(a) - \frac{x-a}{b-a}u(b)$$

$$= u(x)\left[1 - \frac{b-x}{b-a} - \frac{x-a}{b-a}\right] - u'(x)\left[\frac{b-x}{b-a}(a-x) + \frac{x-a}{b-a}(b-x)\right]$$

$$- \frac{b-x}{b-a}\int_x^a u''(t)(a-t)\,dt - \frac{x-a}{b-a}\int_x^b u''(t)(b-t)\,dt = -\frac{1}{b-a}\int_a^b u''(t)G(t,x)\,dt,$$

where

$$G(t,x) = \begin{cases} (b-x)(t-a), & a < t < x \\ (x-a)(b-t), & x < t < b \end{cases}.$$

Hence,

$$|u(x) - u_I(x)|^2 = \frac{1}{(b-a)^2}\left|\int_a^b u''(t)G(t,x)\,dt\right|^2 \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b |G(t,x)|^2\,dt,$$

and so

$$\int_a^b |u(x) - u_I(x)|^2\,dx \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b\int_a^b |G(t,x)|^2\,dt\,dx$$

$$= \frac{(b-a)^4}{90}\int_a^b |u''(t)|^2\,dt.$$

Suppose now that we have a set of points $x_0, x_1, \dots, x_N$ with $x_i - x_{i-1} = h_i$, $i = 1, \dots, N$ and we denote by u_I the continuous piecewise linear interpolant of u, i.e., a continuous function that is linear on each subinterval $[x_{i-1}, x_i]$ and satisfies $u_I(x_i) = u(x_i)$, $i = 0, \dots, N$. Then, applying the above formula with $a = x_{i-1}$ and $b = x_i$, and assuming $h_i \le h$, we get

$$\int_{x_0}^{x_N} |u(x) - u_I(x)|^2\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} |u(x) - u_I(x)|^2\,dx$$

$$= \frac{1}{90} \sum_{i=1}^{N} h_i^4 \int_{x_{i-1}}^{x_i} |u''(t)|^2\,dt \le \frac{h^4}{90} \int_{x_0}^{x_N} |u''(t)|^2\,dt.$$

$$\text{Hence,} \qquad \|u - u_I\|_{L^2[x_0,x_N]} \le \frac{h^2}{\sqrt{90}} \|u''\|_{L^2[x_0,x_N]}.$$

We can derive an error estimate for the derivative in a similar way.

$$u_I'(x) = \frac{u(b) - u(a)}{b-a} = u'(x)\frac{(b-x)-(a-x)}{b-a}$$

$$+ \frac{1}{b-a}\int_x^b u''(t)(b-t)\,dt - \frac{1}{b-a}\int_x^a u''(t)(a-t)\,dt.$$

$$\text{Hence,} \qquad u'(x) - u_I'(x) = -\frac{1}{b-a}\int_a^b u''(t)H(t,x)\,dt,$$

$$\text{where} \qquad H(t,x) = \begin{cases} a-t, & a<t<x \\ b-t, & x<t<b \end{cases}.$$

$$\text{Then} \qquad |u'(x) - u_I'(x)|^2 \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b H^2(t,x)\,dt,$$

and so

$$\int_a^b |u'(x) - u_I'(x)|^2\,dx \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b H^2(t,x)\,dt\,dx = \frac{(b-a)^2}{6}\int_a^b |u''(t)|^2\,dt.$$

Translating these results to each subinterval and summing, we get

$$\|u' - u_I'\|_{L^2[x_0,x_N]} \le \frac{h}{\sqrt{6}}\|u''\|_{L^2[x_0,x_N]}.$$

If instead of continuous, piecewise linear interpolation, we define u_I to the continuous piecewise polynomial of degree $\le r$ that interpolates u at the mesh points and at $r-1$ additional points in the interior of each subinterval, then the analogous estimates are:

$$\|u - u_I\|_{L^2[x_0,x_N]} \le Ch^{r+1}\|D^{r+1}u\|_{L^2[x_0,x_N]}, \qquad \|u' - u_I'\|_{L^2[x_0,x_N]} \le Ch^{r}\|D^{r+1}u\|_{L^2[x_0,x_N]}.$$