2.13. **Error estimates in higher dimensions.** One can derive similar estimates in higher dimensions, but the proofs are more complicated.

In the case of piecewise linear interpolation, one first shows that on a triangle T,

$$\|u - u_I\|_{L^2(T)} \le C_T h_T^2 |u|_{2,T}, \qquad |u|_{2,T}^2 = \|u_{xx}\|_{L^2(T)}^2 + \|u_{xy}\|_{L^2(T)}^2 + \|u_{yy}\|_{L^2(T)}^2.$$

To get this result, define

$$F_i(s) = u([1-s]x + sx_i, [1-s]y + sy_i).$$

To simplify notation, let $\xi_i(s) = [1-s]x + sx_i$, $\eta_i(s) = [1-s]y + sy_i$. Then,

$$\begin{aligned} F_i'(s) &= u_x(\xi_i(s), \eta_i(s))(x_i - x) + u_y(\xi_i(s), \eta_i(s))(y_i - y), \\ F_i''(s) &= u_{xx}(\xi_i(s), \eta_i(s))(x_i - x)^2 + 2u_{xy}(\xi_i(s), \eta_i(s))(x_i - x)(y_i - y) \\ &\quad + u_{yy}(\xi_i(s), \eta_i(s))(y_i - y)^2. \end{aligned}$$

Using the one-dimensional Taylor series, we have

$$\begin{aligned} u(x_i, y_i) = F_i(1) &= F_i(0) + F_i'(0) + \int_0^1 F_i''(t)(1-t)\, dt \\ &= u(x,y) + u_x(x,y)(x_i - x) + u_y(x,y)(y_i - y) + \int_0^1 F_i''(t)(1-t)\, dt. \end{aligned}$$

Next recall that the linear interpolant u_I of u may be written $u_I(x,y) = \sum_{i=1}^3 \lambda_i u(x_i, y_i)$, where $\lambda_i = \lambda_i(x,y)$ are the barycentric coordinates. Then using the Taylor series given above, and the identities $\sum_{i=1}^3 \lambda_i = 1$, $\sum_{i=1}^3 \lambda_i x_i = x$, $\sum_{i=1}^3 \lambda_i y_i = y$, we get

$$\begin{aligned} u_I(x,y) = \sum_{i=1}^3 \lambda_i u(x_i, y_i) &= u(x,y) \sum_{i=1}^3 \lambda_i + u_x(x,y) \sum_{i=1}^3 (x_i - x)\lambda_i \\ &\quad + u_y(x,y) \sum_{i=1}^3 (y_i - y)\lambda_i + \sum_{i=1}^3 \lambda_i \int_0^1 F_i''(t)(1-t)\, dt \\ &= u(x,y) + \sum_{i=1}^3 \lambda_i \int_0^1 F_i''(t)(1-t)\, dt. \end{aligned}$$

Setting $R_i = \int_0^1 F_i''(t)(1-t)\, dt$, we can show

$$\left(\sum_{i=1}^3 \lambda_i R_i \right)^2 \le \sum_{i=1}^3 \int_T R_i^2.$$

Then

$$\int_T (u_I - u)^2(x,y)\, dx\, dy = \int_T \left(\sum_{i=1}^3 \lambda_i R_i \right)^2 dx\, dy \le \sum_{i=1}^3 \int_T R_i^2\, dx\, dy.$$

A typical term on the right hand side of the above looks like

$$\int_T (x - x_i)^4 \left[\int_0^1 (1-t) u_{xx}(\xi_i(t), \eta_i(t))\, dt \right]^2 dx\, dy.$$

Need some technical estimates to complete the proof, but can see that $|x - x_i| \le h$ on the triangle T gives the power of h in the estimate and that second derivatives come in.

If we assume that $h_T \le h$ for all T, then as in the one-dimensional case, we get

$$\|u - u_I\|_{L^2(\Omega)}^2 = \sum_T \int_T (u - u_I)^2 \, dx \, dy \le C \sum_T h_T^4 |u|_{2,T}^2 \le Ch^4 \sum_T |u|_{2,T}^2 = Ch^4 |u|_{2,\Omega}^2.$$

Thus, we finally obtain

$$\|u - u_I\|_{L^2(\Omega)} \le Ch^2 |u|_{2,\Omega}.$$

To get estimates for the derivatives, we usually make the assumption that the mesh is *shape regular*. Let h_T denote the diameter of T and ρ_T the diameter of the largest ball contained in T. Define the *shape constant* $\sigma_T = h_T/\rho_T$. If we consider a family of meshes $\mathcal{T}_h$, $0 < h < 1$, we say that the family is *shape regular* if for all $T \in \mathcal{T}_h$ and all $0 < h < 1$, $\sigma_T \le C$ independent of T and h. One can then show that

$$\|\nabla u - \nabla u_I\|_{L^2(T)} \le C_T h |u|_{2,T},$$

where C_T depends on the shape constant σ_T, and then for shape regular meshes, we have by summing the squares of this inequality that

$$\|\nabla u - \nabla u_I\|_{L^2(\Omega)} \le Ch |u|_{2,\Omega}.$$

If instead of continuous, piecewise linear interpolation, we define u_I to be the continuous piecewise polynomial of degree $\le r$ interpolating u at the degrees of freedom, then the analogous estimates are:

$$\|u - u_I\|_{L^2(\Omega)} \le Ch^{r+1} |u|_{r+1,\Omega}, \qquad \|\nabla u - \nabla u_I\|_{L^2(\Omega)} \le Ch^r |u|_{r+1,\Omega},$$

where

$$|u|_{r+1,\Omega}^2 = \sum_{|\alpha|=r} \|D^\alpha u\|_{L^2(\Omega)}^2.$$

2.14. **Order of convergence estimates for Ritz-Galerkin approximation schemes.** We now return to the approximation of the variational problem: Find $u \in V$ such that

$$a(u, v) = F(v), \quad \text{for all } v \in V,$$

by the standard Ritz-Galerkin approximation scheme:

Find $u_h \in V_h$ such that $\qquad a(u_h, v_h) = F(v_h), \quad$ for all $v_h \in V_h$.

We have previously established that under the conditions

$$a(v, v) \ge \alpha \|v\|_1^2, \qquad |a(u, v)| \le M \|u\|_1 \|v\|_1, \quad |F(v)| \le K \|v\|_1,$$

we have the quasi-optimal error estimate

$$\|u - u_h\|_1 \le \frac{M}{\alpha} \|u - v_h\|_1, \quad \text{for all } v_h \in V_h.$$

The interpolation error estimate derived above then shows that if we choose the space V_h to consist of continuous piecewise polynomials of degree $\leq k$ and the solution $u \in H^{r+1}(\Omega)$ with $1 \leq r \leq k$, then

$$\|u - u_h\|_1 \leq Ch^r |u|_{r+1}.$$

We observe, however, that the error $\|u - u_I\|_{L^2} \leq Ch^{r+1}|u|_{r+1}$. We next show that the same improved order of convergence in $L^2(\Omega)$ also holds for the error $u - u_h$.

Lemma 8. *Suppose Ω is a convex polygon. Then, under the hypotheses stated above,*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\|u - u_h\|_1 \leq Ch^{r+1}|u|_{r+1}.$$

Proof. The proof uses a combination of elliptic regularity and duality. For simplicity, we consider only the variational formulation corresponding to the boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

i.e., $V = \mathring{H}^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $F(v) = \int_\Omega fv \, dx$. More general problems can be done in a similar way. For such problems, the following regularity result for the solution u is known: Given $f \in L^2(\Omega)$, there exists a constant C independent of u and f, such that $\|u\|_2 \leq C\|f\|_{L^2(\Omega)}$. Note that the equation says that the combination $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 \in L^2(\Omega)$. The regularity result says that each of the second derivatives $\in L^2(\Omega)$ and satisfies the indicated bound. To establish the lemma, we introduce the "dual problem": Find $w \in V = \mathring{H}^1(\Omega)$ such that

$$a(v, w) = (u - u_h, v), \quad \text{for all } v \in V,$$

i.e., w is the solution of the boundary value problem:

$$-\Delta w = u - u_h \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

The dual problem is chosen to have the same form as the original boundary value problem, but where f is replaced by the error $u - u_h$. From the elliptic regularity result, we know that w satisfies $\|w\|_2 \leq C\|u - u_h\|_{L^2}$. Then, using Galerkin orthogonality,

$$\begin{aligned} \|u - u_h\|_{L^2}^2 &= (u - u_h, u - u_h) = a(u - u_h, w) = a(u - u_h, w - w_I) \\ &\leq M\|u - u_h\|_1\|w - w_I\|_1 \leq C\|u - u_h\|_1 h\|w\|_2 \leq Ch\|u - u_h\|_1\|u - u_h\|_{L^2}. \end{aligned}$$

Hence,

$$\|u - u_h\|_{L^2} \leq Ch\|u - u_h\|_1 \leq Ch^{r+1}|u|_{r+1}.$$

□