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Locking-free Reissner–Mindlin elements without reduced integration ☆

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Abstract

In a recent paper of Arnold et al. [D.N. Arnold, F. Brezzi, L.D. Marini, A family of discontinuous Galerkin finite elements for the Reissner–Mindlin plate, J. Sci. Comput. 22 (2005) 25–45], the ideas of discontinuous Galerkin methods were used to obtain and analyze two new families of locking free finite element methods for the approximation of the Reissner–Mindlin plate problem. By following their basic approach, but making different choices of finite element spaces, we develop and analyze other families of locking free finite elements that eliminate the need for the introduction of a reduction operator, which has been a central feature of many locking-free methods. For $k \geqslant 2$, all the methods use piecewise polynomials of degree k to approximate the transverse displacement and (possibly subsets) of piecewise polynomials of degree $k-1$ to approximate both the rotation and shear stress vectors. The approximation spaces for the rotation and the shear stress are always identical. The methods vary in the amount of interelement continuity required. In terms of smallest number of degrees of freedom, the simplest method approximates the transverse displacement with continuous, piecewise quadratics and both the rotation and shear stress with rotated linear Brezzi–Douglas–Marini elements.

Keywords: Locking-free finite elements; Reissner-Mindlin plates; Discontinuous Galerkin method

1. Introduction

In the Reissner–Mindlin model of a clamped plate, one seeks to determine the rotation vector $\boldsymbol{\theta}$ and the transverse displacement w which minimize over $\boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$ the plate energy

$$J(\boldsymbol{\theta}, w) = \frac{1}{2}\int_\Omega \mathsf{C}\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\theta})\,\mathrm{d}x + \frac{1}{2}\lambda t^{-2}\int_\Omega |\nabla w - \boldsymbol{\theta}|^2\,\mathrm{d}x - \int_\Omega g w\,\mathrm{d}x,$$

where the coefficients C and λ depend on the material properties of the plate, g is the scaled load, and t is the plate thickness. If one minimizes the energy over subspaces consisting of low order finite elements, then the resulting approximation suffers from the problem of *locking*. This problem is most easily described by noting that as t tends to 0, the solution $(\boldsymbol{\theta}, w)$ of the minimization problem approaches $(\boldsymbol{\theta}_0, w_0)$, where $\boldsymbol{\theta}_0 = \nabla w_0$. If we discretize the problem directly by seeking $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$ and $w_h \in W_h$ minimizing $J(\boldsymbol{\theta}, w)$ over $\boldsymbol{\Theta}_h \times W_h$, then as t vanishes, $(\boldsymbol{\theta}_h, w_h)$ will converge to some $(\boldsymbol{\theta}_{0,h}, w_{0,h})$ where, again, $\boldsymbol{\theta}_{0,h} = \nabla w_{0,h}$. The locking problem occurs because, for low order finite element spaces, this last condition is too restrictive to allow for good approximations of smooth functions. In particular, if continuous piecewise

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linear functions are chosen to approximate both variables, then $\boldsymbol{\theta}_{0,h} \equiv \nabla w_{0,h}$ would be continuous *and* piecewise constant, with zero boundary conditions. Only the choice $\boldsymbol{\theta}_{0,h} = 0$ can satisfy all these conditions. For t very small, the quantity $\boldsymbol{\theta}_h - \nabla w_h$, although not necessarily zero, must be very small, and hence $\boldsymbol{\theta}_h$ will be very close to zero, instead of being close to $\boldsymbol{\theta}$. We can also see the problem from the point of view of approximation: for small t, one cannot find $\boldsymbol{\theta}^I$ and w^I that are close to $\boldsymbol{\theta}$ and w, respectively, if one requires $\boldsymbol{\theta}^I - \nabla w^I$ to be of the order of t^2.

A number of approaches have been developed to avoid the locking problem. One successful idea has been to introduce an additional finite element space $\boldsymbol{\Gamma}_h$ and a reduction operator $\mathbf{P}_h : \boldsymbol{\Theta}_h \to \boldsymbol{\Gamma}_h$, and then seek approximations $\boldsymbol{\theta}_h \in \boldsymbol{\Theta}_h$ and $w_h \in W_h$ minimizing

$$J_h(\boldsymbol{\theta}, w) = \frac{1}{2}\int_\Omega \mathsf{C}\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\theta})\,\mathrm{d}x + \frac{1}{2}\lambda t^{-2}\int_\Omega |\nabla w - \mathbf{P}_h\boldsymbol{\theta}|^2\,\mathrm{d}x - \int_\Omega g w\,\mathrm{d}x.$$

A key assumption is that ∇W_h is a subset of $\boldsymbol{\Gamma}_h$, and in particular of the image of $\mathbf{P}_h$. As t tends to 0, the limiting condition will now be

$$\mathbf{P}_h\boldsymbol{\theta}_{0,h} = \nabla w_{0,h}. \tag{1.1}$$

The introduction of the operator $\mathbf{P}_h$ adds flexibility: if this operator and the finite element subspaces are chosen properly, then one can obtain good approximations which still satisfy the limiting condition (1.1). A number of locking-free individual finite elements and finite element families (e.g. [5,10,15,18,19,16,20,17]) have been obtained in this way.

In a recent paper of Arnold et al. [4], the techniques of discontinuous Galerkin (DG) methods were used to develop two families of locking-free elements. DG solutions are not required to satisfy the standard interelement continuity conditions of conforming finite element methods (that is, continuous elements in the case of the Reissner–Mindlin plate problem). Hence DG methods allow a greater flexibility, that we shall exploit.

As noted in [4], there are many variations of the DG approach. The starting point for all the methods considered in [4] is a fully discontinuous approach in which for k odd, the spaces $\boldsymbol{\Theta}_h$ and W_h are chosen to be piecewise polynomials of degree $\leqslant k$, and $\boldsymbol{\Gamma}_h$ is chosen to be piecewise polynomials of degree $\leqslant k-1$. Various degrees of interelement continuity can then be added, provided suitable bubble functions are added to $\boldsymbol{\Theta}_h$. Error estimates are obtained for two cases: first, when all finite element spaces are fully discontinuous, and, second, when $\boldsymbol{\Theta}_h$ is a continuous finite element space augmented by bubble functions, W_h is a nonconforming space (i.e., moments of order $k-1$ are continuous across interelement boundaries), and $\boldsymbol{\Gamma}_h$ is discontinuous. The second case coincides when $k=1$ with the Arnold–Falk element [6], in which $\boldsymbol{\Theta}_h$ consists of the continuous piecewise linear functions augmented by cubic bubble functions, W_h consist of the nonconforming piecewise linear functions, and $\boldsymbol{\Gamma}_h$ consists of the piecewise constants. A possible advantage of the first, fully discontinuous case, is that it allows the same degrees of freedom for the rotations and transverse displacement. This condition is considered by some engineers to simplify the implementation in the context of the commonly used conforming or nonconforming methods (and, especially, for the extension to shell problems). It might prove less important when discontinuous elements are used. Since there is still very limited experience in the practical use of discontinuous elements for plates (and for their extension to shell problems), we consider this question as yet unresolved. It might well turn out, for example, that the greater flexibility of DG methods enables the treatment of some particularly difficult shell problems, compensating for other difficulties in implementation. Much more research and experimentation are needed to fully understand the practical interest of all these possible developments, and we shall not consider this issue further here.

In this paper, the starting point for all the methods considered is to choose W_h to be piecewise polynomials of degree $\leqslant k$ (with $k \geqslant 2$), and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ to be piecewise polynomials of degree $\leqslant k-1$. The motivation comes from the desire to *eliminate the reduction operator* $\mathbf{P}_h$, and also is suggested by issues arising from approximation theory, in which it is natural to have the polynomials in W_h of one higher degree than those in $\boldsymbol{\Theta}_h$. Within this framework, various amounts of interelement continuity are possible, and we derive error estimates for several natural choices. These include fully discontinuous cases, and also the cases when W_h is continuous. In the former situation, W_h consists of all the piecewise polynomials of degree at most k for some $k \geqslant 2$, and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ is made of all the piecewise polynomials of degree $\leqslant k-1$. The element diagram in the lowest order case, $k=2$ is shown on the left of Fig. 1. In the case of when W_h is continuous, it coincides with the usual space of continuous piecewise polynomials of degree at most k, and the smallest of several possible choices for $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ is the rotated Brezzi–Douglas–Marini elements of order $k-1$, BDM^R_{k-1}, [13]. With $k=2$ this gives the element choice indicated on the right of Fig. 1. However, other choices of $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ are possible with the same choice of W_h. In fact any space which contains

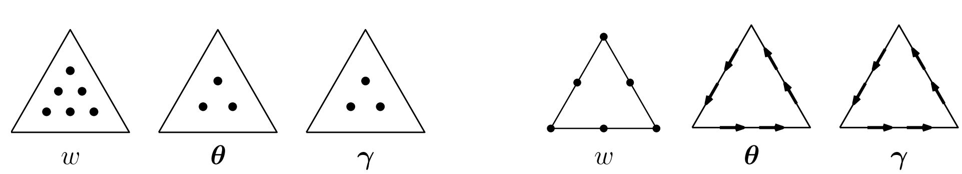

Fig. 1. Simplest elements with w discontinuous (left) and continuous (right).

BDM^R_{k-1}, e.g., the rotated Raviart–Thomas elements of order $k-1$ [21] (RT^R_{k-1}), or the space of the discontinuous piecewise polynomials of degree $\leqslant k-1$ could be used.

There are some differences between the fully discontinuous methods and the methods with continuous W_h, that become apparent in the derivation of error estimates. One difference is the regularity required on the solution to achieve a certain rate of convergence. This may have some added importance in the approximation of the Reissner–Mindlin plate problem, since the rotation vector has a boundary layer and thus higher norms are not bounded independently of the plate thickness t. For example, for the clamped plate, $\|\boldsymbol{\theta}\|_2$ is bounded, while $\|\boldsymbol{\theta}\|_3$ behaves like $t^{-1/2}$ as t tends to 0.

An outline of the paper is as follows: in the next section we introduce the notation for the spaces to be used, and recall some basic notation and useful formulae to deal with discontinuous approximations. In Section 3 we introduce the discretized problem and recall some known results concerning DG approximations. Specific methods are discussed in the last two sections. In particular, Section 4 deals with the cases in which functions in W_h are continuous, and Section 5 with the totally discontinuous case.

2. Notations and preliminaries

2.1. Functional spaces

We begin by adopting the notation employed in [4]. Let $\Omega\subset\mathbb{R}^2$ denote the domain occupied by the middle surface of the plate. For simplicity, we assume that Ω is a convex polygon.

We shall use the usual Sobolev spaces such as $H^s(T)$, with the corresponding seminorm and norm denoted by $|\cdot|_{s,T}$ and $\|\cdot\|_{s,T}$, respectively. When $T=\Omega$, we just write $|\cdot|_s$ and $\|\cdot\|_s$. By convention, we use boldface type for the vector-valued analogues: $\boldsymbol{H}^s(\Omega)=[H^s(\Omega)]^2$. Occasionally we shall use calligraphic type for symmetric-tensor-valued analogues: $\mathscr{H}^s(\Omega)=[H^s(\Omega)]^2_{\mathrm{sym}}$. We use parentheses $(\cdot,\cdot)$ to denote the inner product in any of the spaces $L^2(\Omega)$, $\boldsymbol{L}^2(\Omega)$, or $\mathscr{L}^2(\Omega)$.

We denote by $\mathscr{T}_h$ a decomposition of Ω into triangles T and by $\mathscr{E}_h$ the set of all the edges in $\mathscr{T}_h$. For piecewise polynomial spaces, we use the notation

$$\mathscr{L}^s_k(\mathscr{T}_h)=\{v\in H^s(\Omega):v|_T\in\mathscr{P}_k(T),T\in\mathscr{T}_h\},\tag{2.1}$$

with $\mathscr{P}_k(T)$ the set of polynomials of degree at most k on T. (Note that in (2.1), calligraphic font does not refer to tensor-valued quantities.)

Some of our finite elements will be discontinuous and so not contained in the space $H^1(\Omega)$, but rather in a piecewise Sobolev space

$$H^1(\mathscr{T}_h):=\{v\in L^2(\Omega):v|_T\in H^1(T),T\in\mathscr{T}_h\}.$$

Differential operators can be applied to this space only piecewise. We indicate this by a subscript h on the operator. Thus, for example, the piecewise gradient operator $\boldsymbol{\nabla}_h$ maps $H^1(\mathscr{T}_h)$ into $\boldsymbol{L}^2(\Omega)$ and the piecewise symmetric gradient (or *infinitesimal strain*) operator ε_h maps $\boldsymbol{H}^1(\mathscr{T}_h)$ into $\mathscr{L}^2(\Omega)$. The space $H^1(\mathscr{T}_h)$ is equipped with the seminorm $|v|_{1,h}=\|\boldsymbol{\nabla}_h v\|_0$ and the corresponding norm $\|v\|^2_{1,h}=|v|^2_{1,h}+\|v\|^2_0$. More generally, a subscript such as $\|\cdot\|_{s,h}$ will be used to indicate the *broken* (element by element) H^s-norm (for s a nonnegative integer).

A particular role will be played, for discontinuous approximations, by the set $\mathscr{E}_h$ of all the *edges* of the given decomposition $\mathscr{T}_h$. In particular, we shall use the symbol $\langle\cdot,\cdot\rangle$ to denote L^2-inner product (of functions or vectors) on $\mathscr{E}_h$. Hence, for instance, if ψ and χ are functions defined on $\mathscr{E}_h$ we have

$$\langle\psi,\chi\rangle:=\sum_{e\in\mathscr{E}_h}\int_e\psi\chi\,\mathrm{d}s.$$

2.2. Averages and jumps

As is usual in the DG approach, we define the jump and average of a function in $H^1(\mathscr{T}_h)$ as a function on the union of the edges of the triangulation. Let e be an internal edge of $\mathscr{T}_h$, shared by two elements T^+ and T^-, and let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ denote the unit normals to e, pointing outward from T^+ and T^-, respectively. If φ belongs to $H^1(\mathscr{T}_h)$ (or possibly the vector- or tensor-valued analogue), we define the average φ on e as usual:

$$\{\varphi\}=\frac{\varphi^++\varphi^-}{2}.$$

For a scalar function $\varphi\in H^1(\mathscr{T}_h)$ we define its jump on e as

$$[\![\varphi]\!]=\varphi^+\boldsymbol{n}^++\varphi^-\boldsymbol{n}^-,$$

which is a vector normal to e. The jump of a vector $\boldsymbol{\varphi}\in\boldsymbol{H}^1(\mathscr{T}_h)$ is the symmetric matrix-valued function given on e by

$$[\![\boldsymbol{\varphi}]\!]=\boldsymbol{\varphi}^+\odot\boldsymbol{n}^++\boldsymbol{\varphi}^-\odot\boldsymbol{n}^-,$$

where $\boldsymbol{\varphi}\odot\boldsymbol{n}=(\boldsymbol{\varphi}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{\varphi})/2$ is the symmetric part of the tensor product of $\boldsymbol{\varphi}$ and $\boldsymbol{n}$.

On a boundary edge, the average $\{\varphi\}$ is defined simply as the trace of φ, while for a scalar-valued function we define $[\![\varphi]\!]$ to be $\varphi\boldsymbol{n}$ (with $\boldsymbol{n}$ the outward unit normal), and for a vector-valued function we define $[\![\boldsymbol{\varphi}]\!]=\boldsymbol{\varphi}\odot\boldsymbol{n}$.

It is easy to check that

$$\sum_{T\in\mathscr{T}_h}\int_{\partial T}\boldsymbol{\varphi}\cdot\boldsymbol{n}_T v\,\mathrm{d}s=\langle\{\boldsymbol{\varphi}\},[\![v]\!]\rangle,\quad\boldsymbol{\varphi}\in\boldsymbol{H}^1(\Omega),\ v\in H^1(\mathscr{T}_h).\tag{2.2}$$

Similarly,

$$\sum_{T\in\mathscr{T}_h}\int_{\partial T}\mathscr{S}\boldsymbol{n}_T\cdot\boldsymbol{\eta}\,\mathrm{d}s=\langle\{\mathscr{S}\},[\![\boldsymbol{\eta}]\!]\rangle,\quad\mathscr{S}\in\mathscr{H}^1(\Omega),\boldsymbol{\eta}\in\boldsymbol{H}^1(\mathscr{T}_h).$$

It is not difficult to see that both the above relations hold in more general situations. For instance, (2.2) also holds for $\boldsymbol{\varphi} \in \boldsymbol{H}(\mathrm{div};\Omega)$, where $\boldsymbol{H}(\mathrm{div};\Omega)$ is the space of vectors $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ with $\mathrm{div}\boldsymbol{\varphi} \in L^2(\Omega)$.

2.3. The Reissner–Mindlin equations

Introducing the shear stress $\boldsymbol{\gamma} = \lambda t^{-2}(\nabla w - \boldsymbol{\theta})$, the Reissner–Mindlin plate problem may also be described by the Euler equations for the minimization of the plate energy. These are

$$-\mathrm{div}\mathsf{C}\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$-\mathrm{div}\boldsymbol{\gamma} = g \quad \text{in } \Omega, \tag{2.4}$$

$$\nabla w - \boldsymbol{\theta} - t^2\boldsymbol{\gamma} = 0 \quad \text{in } \Omega, \tag{2.5}$$

$$\boldsymbol{\theta} = 0, \quad w = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

Note that (2.5) should actually be $\nabla w - \boldsymbol{\theta} - \lambda^{-1}t^2\boldsymbol{\gamma} = 0$, where λ is the *shear correction factor*. Here however, to simplify the presentation, we set $\lambda = 1$. We are now going to introduce the variational formulation of Eqs. (2.3)–(2.6) (or, actually, of a more general case, that we shall need later on while applying a duality argument). We set, for $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ in $\boldsymbol{H}^1(\Omega)$

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) = (\mathsf{C}\varepsilon(\boldsymbol{\theta}), \varepsilon(\boldsymbol{\eta}))$$

and we consider the following problem:

Given $g \in L^2(\Omega)$ and $\boldsymbol{G} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{\theta} \in \boldsymbol{H}^1_0(\Omega)$, $w \in H^1_0(\Omega)$ and $\boldsymbol{\gamma} \in \boldsymbol{L}^2(\Omega)$ such that

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\eta}) = (g, v) + (\boldsymbol{G}, \boldsymbol{\eta}) \qquad \forall(\boldsymbol{\eta}, v) \in \boldsymbol{H}^1_0(\Omega) \times H^1_0(\Omega), \tag{2.7}$$

$$(\nabla w - \boldsymbol{\theta}, \boldsymbol{\tau}) - t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega). \tag{2.8}$$

It is clear that the Reissner–Mindlin equations (2.3), (2.1)–(2.6) are obtained for $\boldsymbol{G} = 0$. For the generalized problem (2.7) and (2.8), we recall the following result (see [5,6]).

Theorem 1. *Let Ω be a convex polygonal domain, and assume that the coefficient C is smooth. Then problem (2.7) and (2.8) has a unique solution that satisfies*

$$\|\boldsymbol{\theta}\|_2 + \|w\|_2 + \|\boldsymbol{\gamma}\|_0 + t\|\boldsymbol{\gamma}\|_1 \leqslant C(\|g\|_{-1} + t\|g\|_0 + \|\boldsymbol{G}\|_0), \tag{2.9}$$

where C is a constant depending only on Ω and on the coefficients in C.

3. Discontinuous Galerkin discretization

3.1. Discontinuous variational formulation of the continuous problem

To derive a finite element method for the Reissner–Mindlin system based on discontinuous elements, we test (2.3) against a test function $\boldsymbol{\eta} \in \boldsymbol{H}^2(\mathscr{T}_h)$ and (2.4) against a test function $v \in H^1(\mathscr{T}_h)$, integrate by parts, and add. Since $\boldsymbol{\eta}$ and v may be discontinuous across element boundaries, we obtain terms at the interelement boundaries that we manipulate using (2.2). The net result is

$$\begin{aligned}&(\mathsf{C}\varepsilon_h(\boldsymbol{\theta}), \varepsilon_h(\boldsymbol{\eta})) - \langle\{\mathsf{C}\varepsilon_h(\boldsymbol{\theta})\}, [\![\boldsymbol{\eta}]\!]\rangle + (\boldsymbol{\gamma}, \nabla_h v - \boldsymbol{\eta}) - \langle\{\boldsymbol{\gamma}\}, [\![v]\!]\rangle = (g, v),\\ &\quad(\boldsymbol{\eta}, v) \in \boldsymbol{H}^2(\mathscr{T}_h) \times H^1(\mathscr{T}_h),\\ &(\nabla_h w - \boldsymbol{\theta}, \boldsymbol{\tau}) - t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{H}^1(\mathscr{T}_h).\end{aligned} \tag{3.1}$$

Note that we could as well have written $\varepsilon(\boldsymbol{\theta})$ and ∇w instead of $\varepsilon_h(\boldsymbol{\theta})$ and $\nabla_h w$, respectively, due to the continuity properties of the exact solution. The second and fourth terms in (3.1) involve integrals over the edges and would not be present in conforming methods. They arise from the integration by parts and are necessary to maintain consistency.

We now proceed as is common for DG methods (for a different point of view on this type of derivation see [11]). First, we add terms to symmetrize this formulation so that it is adjoint-consistent as well. Second, to stabilize the method, we add *interior penalty* terms $p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta})$ and $p_W(w, v)$ in which the functions p_Θ and p_W will depend only on the jumps of their arguments. Since $[\![\boldsymbol{\theta}]\!] = 0$ and $[\![w]\!] = 0$, we find that $\boldsymbol{\theta}$, w, and $\boldsymbol{\gamma}$ satisfy

$$\begin{aligned}&(\mathsf{C}\varepsilon_h(\boldsymbol{\theta}), \varepsilon_h(\boldsymbol{\eta})) - \langle\{\mathsf{C}\varepsilon_h(\boldsymbol{\theta})\}, [\![\boldsymbol{\eta}]\!]\rangle - \langle[\![\boldsymbol{\theta}]\!], \{C\varepsilon_h(\boldsymbol{\eta})\}\rangle + (\boldsymbol{\gamma}, \nabla_h v - \boldsymbol{\eta})\\ &\quad - \langle\{\boldsymbol{\gamma}\}, [\![v]\!]\rangle + p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) + p_W(w, v) = (g, v), \quad (\boldsymbol{\eta}, v) \in \boldsymbol{H}^2(\mathscr{T}_h) \times H^1(\mathscr{T}_h),\\ &(\nabla_h w - \boldsymbol{\theta}, \boldsymbol{\tau}) - \langle[\![w]\!], \{\boldsymbol{\tau}\}\rangle - t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{H}^1(\mathscr{T}_h).\end{aligned} \tag{3.2}$$

3.2. Abstract discretization

To obtain a DG discretization, we have to choose finite dimensional subspaces $\boldsymbol{\Theta}_h \subset \boldsymbol{H}^2(\mathscr{T}_h)$, $W_h \subset H^1(\mathscr{T}_h)$, and $\boldsymbol{\Gamma}_h \subset \boldsymbol{H}^1(\mathscr{T}_h)$. The method then takes the form:

Find $(\boldsymbol{\theta}_h, w_h) \in \boldsymbol{\Theta}_h \times W_h$ and $\boldsymbol{\gamma}_h \in \boldsymbol{\Gamma}_h$ such that

$$\begin{aligned}&(\mathsf{C}\varepsilon_h(\boldsymbol{\theta}_h), \varepsilon_h(\boldsymbol{\eta})) - \langle\{\mathsf{C}\varepsilon_h(\boldsymbol{\theta}_h)\}, [\![\boldsymbol{\eta}]\!]\rangle - \langle[\![\boldsymbol{\theta}_h]\!], \{\mathsf{C}\varepsilon_h(\boldsymbol{\eta})\}\rangle\\ &\quad + (\boldsymbol{\gamma}_h, \nabla_h v - \boldsymbol{\eta}) - \langle\{\boldsymbol{\gamma}_h\}, [\![v]\!]\rangle + p_\Theta(\boldsymbol{\theta}_h, \boldsymbol{\eta})\\ &\quad + p_W(w_h, v) = (g, v), \quad (\boldsymbol{\eta}, v) \in \boldsymbol{\Theta}_h \times W_h,\end{aligned} \tag{3.3}$$

$$(\nabla_h w_h - \boldsymbol{\theta}_h, \boldsymbol{\tau}) - \langle[\![w_h]\!], \{\boldsymbol{\tau}\}\rangle - t^2(\boldsymbol{\gamma}_h, \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in \boldsymbol{\Gamma}_h. \tag{3.4}$$

For any choice of the finite element spaces $\boldsymbol{\Theta}_h$, W_h, and $\boldsymbol{\Gamma}_h$, and any interior penalty functions p_Θ and p_W depending only on the jumps of their arguments, this gives a consistent finite element method. Note that in contrast to the methods proposed in [4], we do not introduce a reduction operator $\boldsymbol{P}_h$.

To complete the specification of the method, we need only choose the finite element spaces $\boldsymbol{\Theta}_h$, W_h, and $\boldsymbol{\Gamma}_h$ and the interior penalty forms p_Θ and p_W. For the finite element spaces, the starting point for all our methods is to choose W_h to be either $\mathscr{L}^0_k$ or $\mathscr{L}^1_k$ (with $k \geqslant 2$), and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ to be subspaces of $\mathscr{L}^0_{k-1}$. As stated earlier, the motivation comes from the desire to eliminate the reduction operator $\mathbf{P}_h$ and also issues arising from approximation theory, in which it is natural to have the polynomials in W_h of one degree higher than those in $\boldsymbol{\Theta}_h$.

We make a standard choice for the interior penalty terms p_Θ and p_W:

$$p_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) = \sum_{e\in\mathscr{E}_h} \frac{\kappa^\Theta}{|e|}\int_e [\![\boldsymbol{\theta}]\!] : [\![\boldsymbol{\eta}]\!]\,\mathrm{d}s,$$

$$p_W(w,v) = \sum_{e\in\mathscr{E}_h} \frac{\kappa^W}{|e|}\int_e [\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s, \tag{3.5}$$

so that $p_\Theta(\boldsymbol{\eta},\boldsymbol{\eta})$, ($p_W(v,v)$, resp.) can be viewed as a measure of the deviation of $\boldsymbol{\eta}$ (v, resp.) from being continuous. The parameters κ^Θ and κ^W are positive constants to be chosen; they must be sufficiently large to ensure stability. In the case when W_h consists of continuous elements, the penalty term p_W will not be needed.

Throughout the paper, C will denote a generic constant that depends only on the minimum angle of the decomposition, on the degree k of the polynomials, and on the values of κ^Θ and κ^W (for discontinuous W_h).

3.3. DG norms and basic inequalities

For the error analysis which follows in the subsequent sections, it will be convenient to have additional notation. We first define norms

$$|||\boldsymbol{\eta}|||_\Theta^2 := \|\boldsymbol{\eta}\|_{1,h}^2 + \sum_{e\in\mathscr{E}_h}\left(\frac{1}{|e|}\|[\![\boldsymbol{\eta}]\!]\|_{0,e}^2 + |e|\|\{\mathsf{C}\varepsilon_h(\boldsymbol{\eta})\}\|_{0,e}^2\right),$$

$$\boldsymbol{\eta}\in\boldsymbol{H}^2(\mathscr{T}_h),$$

$$|||v|||_W^2 := |v|_{1,h}^2 + \sum_{e\in\mathscr{E}_h}\frac{1}{|e|}\|[\![v]\!]\|_{0,e}^2,\quad v\in H^1(\mathscr{T}_h),$$

$$|||\boldsymbol{\tau}|||_\Gamma^2 := \|\boldsymbol{\tau}\|_0^2 + \sum_{e\in\mathscr{E}_h}|e|\|\{\boldsymbol{\tau}\}\|_{0,e}^2,\quad \boldsymbol{\tau}\in\boldsymbol{H}^1(\mathscr{T}_h).$$

A useful result, that we will need in our analysis (see [1,2]) is the following: let T be a triangle, and let e be an edge of T. Then there exists a positive constant C only depending on the minimum angle of T such that

$$\|\varphi\|_{0,e}^2 \leqslant C(|e|^{-1}\|\varphi\|_{0,T}^2 + |e||\varphi|_{1,T}^2),\quad \varphi\in H^1(T). \tag{3.6}$$

Clearly, (3.6) also holds for vector-valued functions $\boldsymbol{\varphi}\in\boldsymbol{H}^1(\mathscr{T}_h)$. Using (3.6) it is not difficult to check that

$$|||\boldsymbol{\eta}|||_\Theta^2 \leqslant C\left(\sum_{T\in\mathscr{T}_h} h_T^{-2}\|\boldsymbol{\eta}\|_{0,T}^2 + |\boldsymbol{\eta}|_{1,T}^2 + h_T^2|\boldsymbol{\eta}|_{2,T}^2\right),$$

$$|||v|||_W^2 \leqslant C\left(\sum_{T\in\mathscr{T}_h} h_T^{-2}\|v\|_{0,T}^2 + |v|_{1,T}^2\right), \tag{3.7}$$

$$|||\boldsymbol{\tau}|||_\Gamma^2 \leqslant C\left(\sum_{T\in\mathscr{T}_h} \|\boldsymbol{\tau}\|_{0,T}^2 + h_T^2|\boldsymbol{\tau}|_{1,T}^2\right).$$

Let

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) = (\mathsf{C}\varepsilon_h(\boldsymbol{\theta}),\varepsilon_h(\boldsymbol{\eta})) - \langle\{\mathsf{C}\varepsilon_h(\boldsymbol{\theta})\},[\![\boldsymbol{\eta}]\!]\rangle - \langle[\![\boldsymbol{\theta}]\!],\{\mathsf{C}\varepsilon_h(\boldsymbol{\eta})\}\rangle + p_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}), \tag{3.8}$$

$$j(\boldsymbol{\tau},v) = \langle\{\boldsymbol{\tau}\},[\![v]\!]\rangle. \tag{3.9}$$

Clearly we have (see [2]) for $\boldsymbol{\theta},\boldsymbol{\eta}\in\boldsymbol{H}^2(\mathscr{T}_h)$, $v\in H^1(\mathscr{T}_h)$, and $\boldsymbol{\tau}\in\boldsymbol{H}^1(\mathscr{T}_h)$:

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) \leqslant C|||\boldsymbol{\theta}|||_\Theta|||\boldsymbol{\eta}|||_\Theta, \tag{3.10}$$

$$j(\boldsymbol{\tau},v) \leqslant C|||\boldsymbol{\tau}|||_\Gamma\left(\sum_{e\in\mathscr{E}_h}\frac{1}{|e|}\|[\![v]\!]\|_{0,e}^2\right)^{1/2} \leqslant C|||\boldsymbol{\tau}|||_\Gamma p_W(v,v)^{1/2} \leqslant C|||\boldsymbol{\tau}|||_\Gamma|||v|||_W. \tag{3.11}$$

Proofs of the two following lemmata, giving discrete Korn's inequality and a coercivity estimate, can be found in [9,4].

Lemma 1

$$\|\boldsymbol{\eta}\|_{1,h}^2 \leqslant C\left(\sum_{T\in\mathscr{T}_h}\|\varepsilon(\boldsymbol{\eta})\|_{0,T}^2 + \sum_{e\in\mathscr{E}_h}\frac{1}{|e|}\|[\![\boldsymbol{\eta}]\!]\|_{0,e}^2\right),\quad \boldsymbol{\eta}\in\boldsymbol{H}^1(\mathscr{T}_h).$$

Lemma 2. *There exist positive constants κ_0 and α depending only on the polynomial degree k and the shape regularity of the partition $\mathscr{T}_h$, such that: if the constant $\kappa^\Theta \geqslant \kappa_0$ (where κ^Θ is the penalty parameter appearing in (3.5)), then*

$$a_h(\boldsymbol{\eta},\boldsymbol{\eta}) \geqslant \alpha|||\boldsymbol{\eta}|||_\Theta^2,\quad \boldsymbol{\eta}\in\boldsymbol{\Theta}_h. \tag{3.12}$$

3.4. Compact formulation of the continuous and discretized problems

With the above notation, we may rewrite (3.2) as

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) + (\boldsymbol{\gamma},\boldsymbol{\nabla}_h v - \boldsymbol{\eta}) - j(\boldsymbol{\gamma},v) + p_W(w,v) = (g,v),\quad (\boldsymbol{\eta},v)\in\boldsymbol{H}^2(\mathscr{T}_h)\times H^1(\mathscr{T}_h), \tag{3.13}$$

$$(\boldsymbol{\nabla}_h w - \boldsymbol{\theta},\boldsymbol{\tau}) - j(\boldsymbol{\tau},w) - t^2(\boldsymbol{\gamma},\boldsymbol{\tau}) = 0,\quad \boldsymbol{\tau}\in\boldsymbol{H}^1(\mathscr{T}_h), \tag{3.14}$$

and (3.3) and (3.4) as

$$a_h(\boldsymbol{\theta}_h,\boldsymbol{\eta}) + (\boldsymbol{\gamma}_h,\boldsymbol{\nabla}_h v - \boldsymbol{\eta}) - j(\boldsymbol{\gamma}_h,v) + p_W(w_h,v) = (g,v),\quad (\boldsymbol{\eta},v)\in\boldsymbol{\Theta}_h\times W_h, \tag{3.15}$$

$$(\boldsymbol{\nabla}_h w_h - \boldsymbol{\theta}_h,\boldsymbol{\tau}) - j(\boldsymbol{\tau},w_h) - t^2(\boldsymbol{\gamma}_h,\boldsymbol{\tau}) = 0,\quad \boldsymbol{\tau}\in\boldsymbol{\Gamma}_h. \tag{3.16}$$

4. Continuous *w* and discontinuous *θ*

4.1. General setting of the methods with continuous *w*

In this section we shall consider methods in which the space $W_h\subset H_0^1(\Omega)$ and the spaces $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h\subset\boldsymbol{H}^1(\mathscr{T}_h)$ satisfy

$$\boldsymbol{\nabla}W_h \subseteq \boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h. \tag{4.1}$$

Note that (4.1) forbids the use of a space $\boldsymbol{\Theta}_h$ consisting of *continuous* functions. However, it allows choices where the tangential component is continuous (as well as choices where $\boldsymbol{\Theta}_h$ consists of totally discontinuous elements).

Since the space W_h is continuous, the general method given by Eqs. (3.15) and (3.16) simplifies to

$$a_h(\boldsymbol{\theta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}_h,\nabla v-\boldsymbol{\eta})=(g,v),\quad (\boldsymbol{\eta},v)\in\boldsymbol{\Theta}_h\times W_h, \tag{4.2}$$

$$(\nabla w_h-\boldsymbol{\theta}_h,\boldsymbol{\tau})-t^2(\boldsymbol{\gamma}_h,\boldsymbol{\tau})=0,\quad \boldsymbol{\tau}\in\boldsymbol{\Gamma}_h. \tag{4.3}$$

Note that, using (4.1), Eq. (4.3) can be written as

$$\boldsymbol{\gamma}_h=t^{-2}(\nabla w_h-\boldsymbol{\theta}_h). \tag{4.4}$$

We start by stating a basic abstract error estimate.

Theorem 2. *Assume that $W_h\subset H_0^1(\Omega)$ and that assumption (4.1) is satisfied. Let $(\boldsymbol{\theta},w,\boldsymbol{\gamma})$ be the solution of (2.3)–(2.6), and let $(\boldsymbol{\theta}_h,w_h,\boldsymbol{\gamma}_h)$ be the solution of (3.3) and (3.4). Let $\boldsymbol{\theta}^I$ and w^I be any elements in $\boldsymbol{\Theta}_h$ and $\boldsymbol{\Gamma}_h$ (respectively) and set*

$$\boldsymbol{\gamma}^I=t^{-2}(\nabla w^I-\boldsymbol{\theta}^I). \tag{4.5}$$

Then we have

$$|||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta+t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0\leqslant C(|||\boldsymbol{\theta}-\boldsymbol{\theta}^I|||_\Theta+t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_0). \tag{4.6}$$

Proof. For the choice of spaces in this section, and in particular by the continuity of W_h, Eq. (3.13) becomes

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta})+(\boldsymbol{\gamma},\nabla v-\boldsymbol{\eta})=(g,v)\quad\forall(\boldsymbol{\eta},v)\in\boldsymbol{\Theta}_h\times W_h.$$

Subtracting (4.2), we obtain the error equation

$$a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla v-\boldsymbol{\eta})=0\quad\forall(\boldsymbol{\eta},v)\in\boldsymbol{\Theta}_h\times W_h. \tag{4.7}$$

Choosing $\boldsymbol{\eta}=\boldsymbol{\theta}^I-\boldsymbol{\theta}_h$ and $v=w^I-w_h$ in (4.7), and using (4.4) and (4.5), this becomes

$$a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\theta}^I-\boldsymbol{\theta}_h)+t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\gamma}^I-\boldsymbol{\gamma}_h)=0. \tag{4.8}$$

Hence, adding and subtracting $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$, and then using (4.8) to cancel the first and third terms, we have

$$\begin{aligned}&a_h(\boldsymbol{\theta}_h-\boldsymbol{\theta}^I,\boldsymbol{\theta}_h-\boldsymbol{\theta}^I)+t^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I,\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I)\\&\quad=a_h(\boldsymbol{\theta}_h-\boldsymbol{\theta},\boldsymbol{\theta}_h-\boldsymbol{\theta}^I)+a_h(\boldsymbol{\theta}-\boldsymbol{\theta}^I,\boldsymbol{\theta}_h-\boldsymbol{\theta}^I)\\&\qquad+t^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I)+t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}^I,\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I)\\&\quad=a_h(\boldsymbol{\theta}-\boldsymbol{\theta}^I,\boldsymbol{\theta}_h-\boldsymbol{\theta}^I)+t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}^I,\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I).\end{aligned}$$

From this, (3.12), and (3.10), we easily obtain

$$|||\boldsymbol{\theta}_h-\boldsymbol{\theta}^I|||_\Theta^2+t^2\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}^I\|_0^2\leqslant C(|||\boldsymbol{\theta}-\boldsymbol{\theta}^I|||_\Theta^2+t^2\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_0^2).$$

The result (4.6) then follows by the triangle inequality. □

We now proceed to the choice of the spaces $\boldsymbol{\Theta}_h$, $\boldsymbol{\Gamma}_h$, and W_h and the interpolants $\boldsymbol{\theta}^I$ and w^I (which determine $\boldsymbol{\gamma}^I$). We shall then apply Theorem 2 to obtain error estimates.

4.2. Choice of W_h and w^I

For any k integer $\geqslant 2$, we take

$$W_h=\mathscr{L}_k^1, \tag{4.9}$$

where $\mathscr{L}_k^1$ is defined in (2.1). For the interpolant we shall use $w^I=\pi_W w$ where π_W is the natural projection onto W_h, i.e., classical choice for the interpolant on W_h, i.e., $\pi_W z\in W_h=\mathscr{L}_k^1$ is determined by

$$\begin{aligned}&\pi_W z(a_i)=z(a_i)\quad\forall\text{ vertices }a_i,\\&\int_e(z-\pi_W z)q\,\mathrm{d}s=0\ \forall q\in\mathscr{P}_{k-2}(e)\quad\forall e\in\mathscr{E}_h,\\&\int_T(z-\pi_W z)q\mathrm{d}x=0\ \forall q\in\mathscr{P}_{k-3}(T)\quad\forall T\in\mathscr{T}_h.\end{aligned} \tag{4.10}$$

It is well known that this standard interpolant satisfies the error estimate

$$\|w-w^I\|_{s,h}\leqslant Ch^{k+1-s}\|w\|_{k+1},\quad 0\leqslant s\leqslant k+1. \tag{4.11}$$

4.3. Choice of $\boldsymbol{\Theta}_h=\boldsymbol{\Gamma}_h$ and of the interpolants

With W_h given by (4.9), our first choice of $\boldsymbol{\Theta}_h=\boldsymbol{\Gamma}_h$ will be close to *the minimum choice that makes (4.1) hold true*. More precisely we take

$$\boldsymbol{\Theta}_h=\boldsymbol{\Gamma}_h=\mathrm{BDM}_{k-1}^R, \tag{4.12}$$

where BDM_{k-1}^R denotes the rotated Brezzi–Douglas–Marini space of degree $k-1$, i.e., the space of all piecewise polynomial vector fields of degree at most $k-1$ subject to interelement continuity of the tangential components. With this choice, the inclusion (4.1) is clearly satisfied.

We define $\boldsymbol{\theta}^I=\boldsymbol{\pi}_\Theta\boldsymbol{\theta}$, where $\boldsymbol{\pi}_\Theta:\boldsymbol{H}^1(\Omega)\to\boldsymbol{\Theta}_h$ is determined locally by the following degrees of freedom:

$$\int_e(\boldsymbol{\tau}-\boldsymbol{\pi}_\Theta\boldsymbol{\tau})\cdot tq\,\mathrm{d}s=0\quad\forall q\in\mathscr{P}_{k-1}(e), \tag{4.13}$$

$$\int_T(\boldsymbol{\tau}-\boldsymbol{\pi}_\Theta\boldsymbol{\tau})\cdot q\,\mathrm{d}x=0\quad\forall q\in\mathrm{RT}_{k-3}, \tag{4.14}$$

where RT_{k-3} is the usual (unrotated) Raviart–Thomas space of index $k-3$. In the framework of [7,8], $\boldsymbol{\pi}_\Theta$ is seen to be the natural projection into BDM_{k-1}^R (and, in particular, well-defined), although the degrees of freedom in (4.14) are not the ones which were used in the original reference (cf. [13]). Moreover, it is related to the natural projection operator π_W into W_h by the commutativity condition

$$\boldsymbol{\pi}_\Theta\nabla z=\nabla\pi_W z. \tag{4.15}$$

This can be checked by using the definition of the projection operators and integration by parts, and is a special case of the commutativity properties of projections presented, e.g., in [7,8].

As a consequence of the choices $w^I=\pi_W w$ and $\boldsymbol{\theta}^I=\boldsymbol{\pi}_\Theta\boldsymbol{\theta}$ and (4.15), we have

$$\begin{aligned}\boldsymbol{\gamma}^I&:=t^{-2}(\nabla w^I-\boldsymbol{\theta}^I)=t^{-2}(\nabla\pi_W w-\boldsymbol{\pi}_\Theta\boldsymbol{\theta})=t^{-2}\boldsymbol{\pi}_\Theta(\nabla w-\boldsymbol{\theta})\\&=\boldsymbol{\pi}_\Theta\boldsymbol{\gamma}.\end{aligned}$$

This puts us into the framework of [18] where the key condition is that $\boldsymbol{\gamma}^I:=t^{-2}(\nabla w^I-\boldsymbol{\theta}^I)$ is an interpolant of $\boldsymbol{\gamma}$.

Using standard techniques, we then have the following interpolation estimates:

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}^I\|_{s,h} \leqslant Ch^{l-s}\|\boldsymbol{\theta}\|_l, \quad \|\gamma-\gamma^I\|_{s,h} \leqslant Ch^{l-s}\|\gamma\|_l, \quad 0 \leqslant s \leqslant l, \; 1 \leqslant l \leqslant k. \tag{4.16}$$

4.4. Basic error estimates for θ and γ

We can now apply Theorem 2 to obtain the corresponding order of convergence estimates.

Theorem 3. *With the choices (4.9) and (4.12) for W_h and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$, let $(\boldsymbol{\theta}, w, \gamma)$ be the solution of (2.3)–(2.6), and let $(\boldsymbol{\theta}_h, w_h, \gamma_h)$ be the solution of (3.3) and (3.4). Then we have*

$$|||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta + t\|\gamma-\gamma_h\|_0 \leqslant Ch^{k-1}(\|\boldsymbol{\theta}\|_k + t\|\gamma\|_{k-1}).$$

Proof. This follows immediately from Theorem 2, (3.7), and (4.16). □

4.5. L^2 error estimates for $\boldsymbol{\theta}$ and w

In this section, we establish the following improved estimate for $\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0$ and also a basic estimate for $\|w-w_h\|_0$.

Theorem 4. *Under the assumptions of Theorem 3,*

$$\|w-w_h\|_0 + \|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 \leqslant Ch^k(\|\boldsymbol{\theta}\|_k + t\|\gamma\|_{k-1}).$$

Proof. We establish this result by a standard duality argument. Let $(\boldsymbol{\varphi}, z, \boldsymbol{\zeta}) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ be the solution of

$$a(\boldsymbol{\varphi}, \boldsymbol{\eta}) + (\boldsymbol{\zeta}, \nabla v - \boldsymbol{\eta}) = (\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\eta}) + (w-w_h, v) \quad \forall(\boldsymbol{\eta}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega), \tag{4.17}$$

$$(\nabla z - \boldsymbol{\varphi}, \boldsymbol{\tau}) - t^2(\boldsymbol{\zeta}, \boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega). \tag{4.18}$$

From the regularity result in Theorem 1, we have on convex polygons,

$$\|\boldsymbol{\varphi}\|_2 + \|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\mathrm{div})} + t\|\boldsymbol{\zeta}\|_1 \leqslant C(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0). \tag{4.19}$$

Using a derivation analogous to that used earlier, we get that $(\boldsymbol{\varphi}, z, \boldsymbol{\zeta})$ also satisfies:

$$a_h(\boldsymbol{\varphi}, \boldsymbol{\eta}) + (\boldsymbol{\zeta}, \nabla v - \boldsymbol{\eta}) = (\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\eta}) + (w-w_h, v) \quad \forall(\boldsymbol{\eta}, v) \in \boldsymbol{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h).$$

Choosing $\boldsymbol{\eta} = \boldsymbol{\theta}-\boldsymbol{\theta}_h$, $v = w-w_h$, and using the definitions of γ and γ_h, given by (2.5) and (4.4), we get

$$\begin{aligned}\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0^2 + \|w-w_h\|_0^2 &= a_h(\boldsymbol{\varphi}, \boldsymbol{\theta}-\boldsymbol{\theta}_h) + (\boldsymbol{\zeta}, \nabla(w-w_h) - (\boldsymbol{\theta}-\boldsymbol{\theta}_h)) \\ &= a_h(\boldsymbol{\varphi}, \boldsymbol{\theta}-\boldsymbol{\theta}_h) + t^2(\boldsymbol{\zeta}, \gamma-\gamma_h).\end{aligned} \tag{4.20}$$

Let $z^I = \pi_W z$ and $\boldsymbol{\varphi}^I = \boldsymbol{\pi}_\Theta \boldsymbol{\varphi}$. Then

$$|||\boldsymbol{\varphi}-\boldsymbol{\varphi}^I|||_\Theta \leqslant Ch\|\boldsymbol{\varphi}\|_2 \leqslant Ch(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0). \tag{4.21}$$

Defining $\boldsymbol{\zeta}^I = t^{-2}(\nabla z^I - \boldsymbol{\varphi}^I)$, we have $\boldsymbol{\zeta}^I = \boldsymbol{\pi}_\Theta \boldsymbol{\zeta}$, and applying (4.16) and the regularity result (4.19), we obtain

$$t\|\boldsymbol{\zeta}-\boldsymbol{\zeta}^I\|_0 \leqslant Ch(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0). \tag{4.22}$$

Now from (4.7) with $\boldsymbol{\eta} = \boldsymbol{\varphi}^I$, $v = z^I$, we then have (using the symmetry of the bilinear form a_h)

$$a_h(\boldsymbol{\varphi}^I, \boldsymbol{\theta}-\boldsymbol{\theta}_h) = -t^2(\gamma-\gamma_h, \boldsymbol{\zeta}^I).$$

Adding and subtracting $\boldsymbol{\varphi}^I$ in (4.20), we thus obtain

$$\begin{aligned}\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0^2 + \|w-w_h\|_0^2 &= a_h(\boldsymbol{\varphi}-\boldsymbol{\varphi}^I, \boldsymbol{\theta}-\boldsymbol{\theta}_h) + t^2(\boldsymbol{\zeta}-\boldsymbol{\zeta}^I, \gamma-\gamma_h) \\ &\leqslant C|||\boldsymbol{\varphi}-\boldsymbol{\varphi}^I|||_\Theta |||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta + t^2\|\boldsymbol{\zeta}-\boldsymbol{\zeta}^I\|_0\|\gamma-\gamma_h\|_0.\end{aligned}$$

Applying (4.21) and (4.22), we get

$$\|w-w_h\|_0 + \|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 \leqslant Ch(|||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta + t\|\gamma-\gamma_h\|_0).$$

The result now follows directly from Theorem 3. □

4.6. Error estimates for ∇w

We next obtain two error estimates for $\|\nabla(w-w_h)\|_0$.

Theorem 5. *Under the assumptions of Theorem 3,*

$$\|\nabla(w-w_h)\|_0 \leqslant \|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + t^2\|\gamma-\gamma_h\|_0 \leqslant C(h^k + th^{k-1})(\|\boldsymbol{\theta}\|_k + t\|\gamma\|_{k-1}). \tag{4.23}$$

$$\|\nabla(w-w_h)\|_0 \leqslant Ch^k(\|\boldsymbol{\theta}\|_k + t\|\gamma\|_{k-1} + \|w\|_{k+1}). \tag{4.24}$$

Proof. The first estimate is easily obtained through the relation $\nabla(w-w_h) = t^2(\gamma-\gamma_h) + (\boldsymbol{\theta}-\boldsymbol{\theta}_h)$ and the estimates for $\boldsymbol{\theta}$ and γ in Theorems 3 and 4.

In view of Theorem 4 and the interpolation estimate (4.11), to establish the second estimate it suffices to show that

$$\|\nabla(w-w_h)\|_0 \leqslant C(\|\nabla(w-w^I)\|_0 + \|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0). \tag{4.25}$$

From the error equation (4.7) with $\boldsymbol{\eta} = 0$ we have

$$(\gamma-\gamma_h, \nabla v) = 0 \quad \forall v \in W_h.$$

Consequently, from the definitions (2.5) and (4.4) of γ and γ_h (respectively), we get

$$(\nabla(w-w_h) - (\boldsymbol{\theta}-\boldsymbol{\theta}_h), \nabla v) = 0 \quad \forall v \in W_h. \tag{4.26}$$

Adding and subtracting ∇w, and then using (4.26) with $v = w^I - w_h$, we have

$$\begin{aligned}\|\nabla(w^I-w_h)\|_0^2 &= (\nabla(w^I-w), \nabla(w^I-w_h)) + (\nabla(w-w_h), \nabla(w^I-w_h)) \\ &= (\nabla(w^I-w), \nabla(w^I-w_h)) + (\boldsymbol{\theta}-\boldsymbol{\theta}_h, \nabla(w^I-w_h)),\end{aligned}$$

so

$$\|\nabla(w^I-w_h)\|_0 \leqslant \|\nabla(w-w^I)\|_0 + \|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0,$$

and (4.25) follows using the triangle inequality. □

Remark 1. Even for the lowest order case $k = 2$, estimate (4.24) involves $\|w\|_3$. Since from (2.4) and (2.5), it easily follow that w satisfies $\Delta w = \text{div}\theta - t^2 g$, on a smooth domain, standard a priori estimates for Poisson's equation and (1) give

$$\|w\|_3 \leqslant C(\|\phi\|_2 + t^2\|g\|_1) \leqslant C(\|g\|_{-1} + t\|g\|_0 + t^2\|g\|_1).$$

Hence, in this case, one obtains a uniform bound for $0 \leqslant t \leqslant 1$. On a convex polygon however, one can only expect H^2 – regularity for w. In this case, an alternative estimate is provided by (4.23).

Remark 2. We have shown that $\|\nabla(w - w_h)\|_0$ achieves the same order, k, of approximation as $\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_0$ and one order higher than $|||\boldsymbol{\theta} - \boldsymbol{\theta}_h|||_\Theta$. Although w^I converges to w with order $k + 1$, we have not been able to establish that higher order for the convergence of w_h.

4.7. Other possible choices

Still taking $W_h = \mathscr{L}_k^1$ as in (4.9), we have other possible choices for $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$. Indeed, we can take any finite element space which contains BDM_{k-1}^R, and continue to use for $\boldsymbol{\theta}^I$ the natural projection onto BDM_{k-1}^R (not onto the larger space $\boldsymbol{\Theta}_h$). This leaves unchanged the approximation results (4.11) and (4.16) and then the error estimates for the method.

Some reasonable such choices for $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ are $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h = \text{RT}_{k-1}^R$ or $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h = \mathscr{L}_{k-1}^0$ where RT_{k-1}^R denotes the rotated Raviart–Thomas spaces of degree $k - 1$, and $\mathscr{L}_{k-1}^0$ the space of discontinuous piecewise polynomials of degree $k - 1$. In the first choice, the space BDM_{k-1}^R is extended by adding local shape functions on each element. In the second, the space is extended by relaxing the interelement continuity.

The analysis can also extend to other choices of spaces W_h and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h$ for which $\nabla W_h \subset \boldsymbol{\Theta}_h$ and which admit projections satisfying

$$\nabla \pi_W z = \boldsymbol{\pi}_\Theta \nabla z.$$

One such possibility is to take W_h to be the space obtained by augmenting $\mathscr{L}_k^1$ by the bubble functions of degree $k + 1$, and choosing $\boldsymbol{\Theta}_h$ to be the Brezzi–Douglas–Fortin–Marini space of degree $k - 1$ [12,14]. It is not clear that using these larger spaces offers any advantages over the choice of $W_h = \mathscr{L}_k^1$ and $\boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h = \text{BDM}_{k-1}^R$, since they involve more degrees of freedom without producing higher convergence rates, and we will not pursue them here.

5. Discontinuous w and discontinuous θ

5.1. Choice of the spaces and of the interpolants

In this section we shall examine the choice of totally discontinuous elements, that is,

$$W_h = \mathscr{L}_k^0, \;\; \boldsymbol{\Theta}_h = \boldsymbol{\Gamma}_h = \mathscr{L}_{k-1}^0, \quad k \geqslant 2. \tag{5.1}$$

Our analysis will start from the totally discontinuous weak formulation of the continuous problem (3.13) and (3.14) and the corresponding formulation of the discrete problem (3.15) and (3.16).

In order to obtain γ_h in an explicit form from Eq. (3.16), it is convenient to introduce the *lifting* operator $J : \boldsymbol{H}^1(\mathscr{T}_h) \to \boldsymbol{\Gamma}_h$ defined (as in [3]) by

$$\int_\Omega J([\![v]\!]) \cdot \boldsymbol{\tau}\, \mathrm{d}x = -j(\boldsymbol{\tau}, v), \quad \boldsymbol{\tau} \in \boldsymbol{\Gamma}_h. \tag{5.2}$$

From (3.11) and the equivalence of the norms $\|\cdot\|_0$ and $|||\cdot|||_\Gamma$ on $\boldsymbol{\Gamma}_h$, it easily follows that

$$|||J([\![v]\!])|||_\Gamma^2 \leqslant C p_W(v, v) \quad v \in W_h. \tag{5.3}$$

Since the condition $\nabla_h W_h \subseteq \boldsymbol{\Gamma}_h$ is satisfied, we then have from (3.16):

$$t^2 \gamma_h = \nabla_h w_h - \boldsymbol{\theta}_h + J([\![w_h]\!]). \tag{5.4}$$

Although the space W_h imposes no interelement continuity, we shall use $w^I = \pi_W w$ where π_W is still the natural interpolant into the *continuous* finite element space $\mathscr{L}_k^1$ defined in (4.10). Similarly, since $\text{BDM}_{k-1}^R \subseteq \mathscr{L}_{k-1}^0$, we can choose $\boldsymbol{\theta}^I = \boldsymbol{\pi}_\Theta \boldsymbol{\theta}$ where $\boldsymbol{\pi}_\Theta$ is still the natural interpolant into BDM_{k-1}^R as defined in (4.13) and (4.14). We then continue to have

$$\gamma^I := t^{-2}(\nabla w^I - \boldsymbol{\theta}^I) = \boldsymbol{\pi}_\Theta \gamma. \tag{5.5}$$

In short, although we are using larger spaces W_h, $\boldsymbol{\Theta}_h$, and $\boldsymbol{\Gamma}_h$, than in the previous section, we use the same interpolants. As a result, the interpolation estimates (4.11) and (4.16) continue to hold.

5.2. Error estimates

Theorem 6. *Let* $(\boldsymbol{\theta}, w, \gamma)$ *be the solution of the continuous problem* (3.13) *and* (3.14), *and let* $(\boldsymbol{\theta}_h, w_h, \gamma_h)$ *be the solution of the discrete problem* (3.15) *and* (3.16) *with the choice of spaces* (5.1). *Then we have*

$$|||\boldsymbol{\theta} - \boldsymbol{\theta}_h|||_\Theta + t\|\gamma - \gamma_h\|_0 + [p_W(w - w_h, w - w_h)]^{1/2} \leqslant Ch^{k-1}(\|\boldsymbol{\theta}\|_k + \|\gamma\|_{k-1}), \tag{5.6}$$

$$|||w - w_h|||_W \leqslant Ch^{k-1}(\|\boldsymbol{\theta}\|_k + \|\gamma\|_{k-1} + \|w\|_k). \tag{5.7}$$

Proof. From (3.13) and (3.15), we immediately have the first error equation

$$a_h(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\eta}) + (\gamma - \gamma_h, \nabla_h v - \boldsymbol{\eta}) - j(\gamma - \gamma_h, v) - p_W(w_h, v) = 0 \quad \forall (\boldsymbol{\eta}, v) \in \boldsymbol{\Theta}_h \times W_h, \tag{5.8}$$

while subtracting (5.4) from (2.5), we have the second error equation

$$t^2(\gamma - \gamma_h) = \nabla_h(w - w_h) - (\boldsymbol{\theta} - \boldsymbol{\theta}_h) - \boldsymbol{J}([\![w_h]\!]). \tag{5.9}$$

Setting now

$$\boldsymbol{\theta}_\delta = \boldsymbol{\theta}_h - \boldsymbol{\theta}^I, \quad w_\delta = w_h - w^I, \quad \gamma_\delta = \gamma_h - \gamma^I,$$

and using (5.4) and (5.5) we immediately obtain

$$t^2\gamma_\delta = \nabla_h w_\delta - \theta_\delta + J([\![w_h]\!]). \tag{5.10}$$

Choosing $\eta = \theta_\delta$ and $v = w_\delta$ in (5.8) we have

$$a_h(\theta - \theta_h, \theta_\delta) + (\gamma - \gamma_h, \nabla_h w_\delta - \theta_\delta) - j(\gamma - \gamma_h, w_\delta) - p_W(w_h, w_\delta) = 0.$$

Using (5.10), and the continuity of w^I (in the penalty term and in $\mathbf{J}$), we then have

$$a_h(\theta - \theta_h, \theta_\delta) + t^2(\gamma - \gamma_h, \gamma_\delta) - (\gamma - \gamma_h, J([\![w_\delta]\!])) - j(\gamma - \gamma_h, w_\delta) - p_W(w_\delta, w_\delta) = 0. \tag{5.11}$$

Owing to the definition (5.2) of $\mathbf{J}$, and to the fact that $\gamma_h \in \mathbf{\Gamma}_h$, we have $(\gamma_h, J([\![w_\delta]\!])) + j(\gamma_h, w_\delta) = 0$. Using this in (5.11) we deduce

$$a_h(\theta - \theta_h, \theta_\delta) + t^2(\gamma - \gamma^I, \gamma_\delta) - t^2(\gamma_\delta, \gamma_\delta) - (\gamma, J([\![w_\delta]\!])) - j(\gamma, w_\delta) - p_W(w_\delta, w_\delta) = 0. \tag{5.12}$$

On the other hand, using (3.12) and adding and subtracting θ, we have

$$\alpha|||\theta_\delta|||_\Theta^2 \leqslant a_h(\theta_\delta, \theta_\delta) = a_h(\theta_h - \theta, \theta_\delta) + a_h(\theta - \theta^I, \theta_\delta). \tag{5.13}$$

Combining (5.12) and (5.13), we obtain

$$\alpha|||\theta_\delta|||_\Theta^2 + t^2\|\gamma_\delta\|_0^2 + p_W(w_\delta, w_\delta) \leqslant a_h(\theta - \theta^I, \theta_\delta) + t^2(\gamma - \gamma^I, \gamma_\delta) - (\gamma, J([\![w_\delta]\!])) - j(\gamma, w_\delta).$$

It will be convenient, also for future use, to isolate the most difficult term to bound in the above equation. We set

$$\mathcal{N} = (\gamma, J([\![w_\delta]\!])) + j(\gamma, w_\delta). \tag{5.14}$$

Using the continuity (3.10) of a_h and the arithmetic-geometric mean inequality, one easily obtains

$$|||\theta_\delta|||_\Theta^2 + t^2\|\gamma_\delta\|_0^2 + p_W(w_\delta, w_\delta) \leqslant C(|||\theta - \theta^I|||_\Theta^2 + t^2\|\gamma - \gamma^I\|_0^2 + |\mathcal{N}|). \tag{5.15}$$

In order to bound the term $\mathcal{N}$, we use again the definition (5.2) of J, and note that, for every $\tau \in \mathbf{\Gamma}_h$ we have

$$\mathcal{N} = (\gamma, J([\![w_\delta]\!])) + j(\gamma, w_\delta) = (\gamma - \tau, J([\![w_\delta]\!])) + j(\gamma - \tau, w_\delta). \tag{5.16}$$

Choosing $\tau = \gamma^I$ in (5.16), we easily have from (5.3) and (3.11)

$$|\mathcal{N}| \leqslant \|\gamma - \gamma^I\|_0 \|J([\![w_\delta]\!])\|_0 + |||\gamma - \gamma^I|||_\Gamma [p_W(w_\delta, w_\delta)]^{1/2} \leqslant C|||\gamma - \gamma^I|||_\Gamma [p_W(w_\delta, w_\delta)]^{1/2}.$$

Inserting this estimate in (5.15), and again using the arithmetic geometric mean inequality, we get

$$|||\theta_\delta|||_\Theta^2 + t^2\|\gamma_\delta\|_0^2 + p_W(w_\delta, w_\delta) \leqslant C(|||\theta - \theta^I|||_\Theta^2 + (1 + t^2)|||\gamma - \gamma^I|||_\Gamma^2), \tag{5.17}$$

and the estimate (5.6) follows from the triangle inequality and the interpolation bounds (4.16). Finally, to get estimate (5.7), we use first (5.10) and (5.3) to obtain

$$\|\nabla_h w_\delta\|_0 = \|t^2\gamma_\delta - J([\![w_\delta]\!]) + \theta_\delta\|_0 \leqslant C\{t^2|||\gamma_\delta|||_\Gamma + |||\theta_\delta|||_\Theta + [p_W(w_\delta, w_\delta)]^{1/2}\}. \tag{5.18}$$

Then (5.7) follows by (5.17) and the triangle inequality. □

5.3. Estimates of $\mathcal{N}$ using the Helmholtz decomposition

The estimates (5.6) and (5.7) obtained in the previous section have one undesirable feature, i.e., the norm $\|\gamma\|_{k-1}$ appearing on the right hand side of the estimates does not contain a factor of t, as was the case for the estimates obtained for continuous approximations of w. Since this norm behaves like $t^{-(k-3/2)}$ as $t \to 0$, the extra factor of t helps control the size of this term and for $k = 2$ insures that it remains bounded. In this subsection, we will show that error estimates with better regularity properties can be obtained if we assume the Helmholtz decomposition for γ is sufficiently smooth.

Looking at the derivation of error estimates in the previous section, we see that the problem comes from the estimation of the term $\mathcal{N}$ appearing in (5.14). We now show how use of the Helmholtz decomposition can lead to an improved estimate of this term. Since in the subsequent section we will introduce an appropriate dual problem to obtain L^2 estimates, and need to estimate a similar term, we work now in a more general framework and define, for any element $\chi \in H^1(\Omega)$, the quantity

$$\mathcal{N} = \mathcal{N}(\chi) := (\chi, J([\![w_\delta]\!])) + j(\chi, w_\delta).$$

We assume that χ has a smooth Helmholtz decomposition satisfying

$$\chi = \nabla s + \operatorname{curl} q, \quad s \in H^k(\Omega) \cap H_0^1(\Omega), \quad q \in H^k(\Omega)/\mathbb{R}. \tag{5.19}$$

We shall assume that

$$(\|s\|_k^2 + \|q\|_k^2)^{1/2} \leqslant C\|\chi\|_{H^{k-1}}, \quad (\|s\|_k^2 + \|q\|_{k-1}^2)^{1/2} \leqslant C\|\chi\|_{H^{k-2}(\mathrm{div})}, \tag{5.20}$$

where $H^{k-2}(\mathrm{div})$ is the space of vectors in $H^{k-2}(\Omega)$ having the divergence in $H^{k-2}(\Omega)$. Note that since $\Delta s = \operatorname{div}\chi$, (5.20) holds if we have H^k regularity for the Dirichlet problem for Poisson's equation, and so for Ω a convex polygon it holds at least for $k = 2$.

As in (5.16), the basic instrument for estimating $\mathcal{N}$ will be the property (based on the definition (5.2) of the operator J):

$$\mathcal{N} = (\chi, J([\![w_\delta]\!])) + j(\chi, w_\delta) = (\chi - \tau, J([\![w_\delta]\!])) + j(\chi - \tau, w_\delta), \quad \tau \in \mathbf{\Gamma}_h. \tag{5.21}$$

This time, however, we choose a different τ. Namely, we let $s^I \in \mathscr{L}_k^1 \cap H_0^1(\Omega)$ and $q^I \in \mathscr{L}_k^1/\mathbb{R}$ be interpolants of s and q, respectively, satisfying

$$\|s - s^I\|_0 + h|s - s^I|_1 \leqslant Ch^j|s|_j, \quad j = 1, \ldots, k, \tag{5.22}$$

$$\|q - q^I\|_0 + h|q - q^I|_1 \leqslant Ch^j|q|_j, \quad j = 1, \ldots, k. \tag{5.23}$$

We then define $\boldsymbol{\chi}^I \in \boldsymbol{\Gamma}_h$ as

$$\boldsymbol{\chi}^I = \nabla s^I + \operatorname{curl} q^I. \tag{5.24}$$

It follows immediately that

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}^I\|_0 \leqslant Ch^{k-1}(|s|_k + |q|_k) \leqslant Ch^{k-1}\|\boldsymbol{\chi}\|_{\boldsymbol{H}^{k-1}}. \tag{5.25}$$

Inserting $\tau = \boldsymbol{\chi}^I$ in (5.21), and using (5.10) to eliminate $\boldsymbol{J}([\![w_\delta]\!])$, we have

$$\mathscr{N} = t^2(\boldsymbol{\chi} - \boldsymbol{\chi}^I, \boldsymbol{\gamma}_\delta) + (\boldsymbol{\chi} - \boldsymbol{\chi}^I, \boldsymbol{\theta}_\delta) - (\boldsymbol{\chi} - \boldsymbol{\chi}^I, \nabla_h w_\delta) + j(\boldsymbol{\chi} - \boldsymbol{\chi}^I, w_\delta). \tag{5.26}$$

The first term in (5.26) is easily bounded using (5.25):

$$t^2|(\boldsymbol{\chi} - \boldsymbol{\chi}^I, \boldsymbol{\gamma}_\delta)| \leqslant t^2\|\boldsymbol{\chi} - \boldsymbol{\chi}^I\|_0\|\boldsymbol{\gamma}_\delta\|_0 \leqslant Ct^2\|\boldsymbol{\gamma}_\delta\|_0 h^{k-1}\|\boldsymbol{\chi}\|_{\boldsymbol{H}^{k-1}}. \tag{5.27}$$

The second term in (5.26), using the expression (5.24) for $\boldsymbol{\chi}^I$, becomes

$$(\boldsymbol{\chi} - \boldsymbol{\chi}^I, \boldsymbol{\theta}_\delta) = (\nabla(s - s^I) + \operatorname{curl}(q - q^I), \boldsymbol{\theta}_\delta). \tag{5.28}$$

All the terms appearing in (5.28) can be treated in the same way. For example, if ψ is in $H^2(\Omega)$ and θ is one of the two components of $\boldsymbol{\theta}_\delta$, we have

$$(\partial\psi/\partial x, \theta) = -\sum_{T\in\mathscr{T}_h}\left(\int_T \psi\partial\theta/\partial x\,\mathrm{d}x - \int_{\partial T}\psi\theta n_x\,\mathrm{d}s\right). \tag{5.29}$$

The first term in the right-hand side of (5.29) is easily bounded by $\|\psi\|_0\|\theta\|_{1,h}$. For the second term, recalling that ψ is continuous and that θ is one of the two components of $\boldsymbol{\theta}_\delta$, we have

$$\left|\sum_{T\in\mathscr{T}_h}\int_{\partial T}\psi\theta n_x\,\mathrm{d}s\right| \leqslant \sum_{e\in\mathscr{E}_h}(|e|^{1/2}\|\psi\|_{0,e})(|e|^{-1/2}\|[\![\boldsymbol{\theta}_\delta]\!]\|_{0,e}).$$

Using (3.6) we can collect the total estimate for (5.29) in the form

$$|(\partial\psi/\partial x, \theta)| \leqslant \left(\sum_{T\in\mathscr{T}_h}\|\psi\|_{0,T}^2 + h_T^2|\psi|_{1,T}^2\right)^{1/2}|||\boldsymbol{\theta}_\delta|||_\Theta.$$

Applying the same argument to all the terms and then using the approximation properties (5.22) and (5.23), we obtain

$$|(\nabla(s - s^I) + \operatorname{curl}(q - q^I), \boldsymbol{\theta}_\delta)| \leqslant Ch^{k-1}(|s|_{k-1} + |q|_{k-1})|||\boldsymbol{\theta}_\delta|||_\Theta. \tag{5.30}$$

The third and fourth terms in (5.26), always using the expression (5.24) for $\boldsymbol{\chi}^I$, become

$$-(\nabla(s - s^I) + \operatorname{curl}(q - q^I), \nabla_h w_\delta) + j(\nabla(s - s^I) + \operatorname{curl}(q - q^I), w_\delta). \tag{5.31}$$

Let us consider first the terms appearing in (5.31) and containing $s - s^I$. Using (5.18) and (3.11) we have

$$\begin{aligned}|(\nabla(s - s^I), \nabla_h w_\delta)| + |j(\nabla(s - s^I), w_\delta)| &\leqslant C(\|\nabla(s - s^I)\|_0\|\nabla_h w_\delta\|_0 + |||\nabla(s - s^I)|||_\Gamma[p_W(w_\delta, w_\delta)]^{1/2}) \\ &\leqslant C|||\nabla(s - s^I)|||_\Gamma\{t^2|||\boldsymbol{\gamma}_\delta|||_\Gamma + |||\boldsymbol{\theta}_\delta|||_\Theta + [p_W(w_\delta, w_\delta)]^{1/2}\}.\end{aligned} \tag{5.32}$$

To estimate the terms involving $\operatorname{curl}(q - q^I)$, we integrate by parts to obtain:

$$(\operatorname{curl}(q - q^I), \nabla_h w_\delta) = \sum_{T\in\mathscr{T}_h}\int_{\partial T}\operatorname{curl}(q - q^I)\cdot\boldsymbol{n}w_\delta\,\mathrm{d}s = \langle\{\operatorname{curl}(q - q^I)\}, [\![w_\delta]\!]\rangle.$$

It follows immediately from the definition (3.9) of j that

$$-(\operatorname{curl}(q - q^I), \nabla_h w_\delta) + j(\operatorname{curl}(q - q^I), w_\delta) = 0. \tag{5.33}$$

Collecting the estimates (5.27), (5.30), (5.32), and (5.33) of all the terms appearing in (5.26), and using the interpolation estimates, we obtain:

$$|\mathscr{N}(\boldsymbol{\chi})| \leqslant Ch^{k-1}(t\|\boldsymbol{\chi}\|_{\boldsymbol{H}^{k-1}} + \|\boldsymbol{\chi}\|_{\boldsymbol{H}^{k-2}(\mathrm{div})}) \times\{t\|\boldsymbol{\gamma}_\delta\|_0 + |||\boldsymbol{\theta}_\delta|||_\Theta + [p_W(w_\delta, w_\delta)]^{1/2}\}. \tag{5.34}$$

Inserting the above estimate for $\boldsymbol{\chi} = \boldsymbol{\gamma}$ into (5.15), we have then established the following theorem.

Theorem 7. *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of the continuous problem (3.13) and (3.14), and let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ be the solution of the discretized problem (3.15) and (3.16) with the choice of spaces (5.1). Assume further that we have the Helmholtz decomposition (5.19) for $\boldsymbol{\gamma}$. Then we have*

$$|||\boldsymbol{\theta} - \boldsymbol{\theta}_h|||_\Theta + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + [p_W(w - w_h, w - w_h)]^{1/2} \leqslant Ch^{k-1}(\|\boldsymbol{\theta}\|_k + t\|\boldsymbol{\gamma}\|_{k-1} + \|\boldsymbol{\gamma}\|_{\boldsymbol{H}^{k-2}(\mathrm{div})}), \tag{5.35}$$

$$|||w - w_h|||_W \leqslant Ch^{k-1}(\|\boldsymbol{\theta}\|_k + t\|\boldsymbol{\gamma}\|_{k-1} + \|\boldsymbol{\gamma}\|_{\boldsymbol{H}^{k-2}(\mathrm{div})} + \|w\|_k). \tag{5.36}$$

Remark 3. We point out that in our assumptions (and in particular for a convex domain Ω) the Helmholtz decomposition (5.19) for $\boldsymbol{\gamma}$ will always hold for $k = 2$. Hence, in particular, estimates (5.34), (5.35), and (5.36) will hold for $k = 2$.

5.4. L^2 error estimates

In this final section, we use a duality argument to derive an optimal L^2 estimate for $\boldsymbol{\theta} - \boldsymbol{\theta}_h$ and an improved estimate for $\|w-w_h\|_0$. We show that both of these are of order h^k provided the solution is sufficiently smooth.

To do so, we again use the dual problem of the previous section, i.e., in which $(\boldsymbol{\varphi}, z, \boldsymbol{\zeta})$ is the solution of (4.17) and (4.18) and hence satisfies the regularity estimate (4.19). As we did for the direct problem, we define the interpolants $z^I, \boldsymbol{\varphi}^I$ and $\boldsymbol{\zeta}^I$ by

$$\begin{aligned} &z^I = \pi_W z \in \mathscr{L}_2^1,\\ &\boldsymbol{\varphi}^I = \boldsymbol{\pi}_\Theta \boldsymbol{\varphi}, \\ &\boldsymbol{\zeta}^I = t^{-2}(\nabla z^I - \boldsymbol{\varphi}^I) = \boldsymbol{\pi}_\Theta \boldsymbol{\zeta}. \end{aligned} \tag{5.37}$$

From the regularity result (4.19), and the previous approximation properties (4.16), we easily obtain

$$t\|\boldsymbol{\zeta}-\boldsymbol{\zeta}^I\|_0 + |||\boldsymbol{\varphi}-\boldsymbol{\varphi}^I|||_\Theta \leqslant Ch(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0). \tag{5.38}$$

With a derivation analogous to that used previously, we see that $(\boldsymbol{\varphi}, z, \boldsymbol{\zeta})$ also satisfies, for all $(\boldsymbol{\eta}, v) \in \boldsymbol{H}^1(\mathscr{T}_h) \times H^1(\mathscr{T}_h)$,

$$a_h(\boldsymbol{\varphi}, \boldsymbol{\eta}) + (\boldsymbol{\zeta}, \nabla_h v - \boldsymbol{\eta}) - j(\boldsymbol{\zeta}, v) = (\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\eta}) + (w-w_h, v). \tag{5.39}$$

Taking $\boldsymbol{\eta} = \boldsymbol{\theta}-\boldsymbol{\theta}_h$, $v = w-w_h$, in (5.39), and using (5.9) we have

$$\begin{aligned} &\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0^2 + \|w-w_h\|_0^2 \\ &\quad = a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}) + (\nabla_h(w-w_h) - (\boldsymbol{\theta}-\boldsymbol{\theta}_h), \boldsymbol{\zeta}) - j(\boldsymbol{\zeta}, w_h) \\ &\quad = a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}) + t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\zeta}) - \mathscr{N}_d, \end{aligned} \tag{5.40}$$

where, in analogy with (5.14), we have set

$$\mathscr{N}_d \equiv \mathscr{N}_d(\boldsymbol{\zeta}) := (\boldsymbol{\zeta}, J([\![w_h]\!])) + j(\boldsymbol{\zeta}, w_h).$$

With the choice (5.37), from the error Eq. (5.8) for the *direct problem* with $\boldsymbol{\eta} = \boldsymbol{\varphi}^I, v = z^I$, we deduce:

$$\begin{aligned} a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}^I) &= -(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \nabla z^I - \boldsymbol{\varphi}^I) + j(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, z^I) + p_W(w_h, z^I) \\ &= -t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\zeta}^I). \end{aligned} \tag{5.41}$$

Adding and subtracting $\boldsymbol{\varphi}^I$ in (5.40), and then using (5.41) and the interpolation estimates (5.38), we obtain:

$$\begin{aligned} &\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0^2 + \|w-w_h\|_0^2 \\ &\quad = a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^I) + a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}^I) + t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\zeta}) - \mathscr{N}_d \\ &\quad = a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^I) + t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h, \boldsymbol{\zeta}-\boldsymbol{\zeta}^I) - \mathscr{N}_d \\ &\quad \leqslant Ch(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0)(|||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta + t|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h|||_0) - \mathscr{N}_d. \end{aligned} \tag{5.42}$$

At this point, we can use the estimates of the previous subsection. As already pointed out, estimate (5.34) will surely hold for $k = 2$. Using this and the regularity results (4.21) we have:

$$\begin{aligned} |\mathscr{N}_d| &\leqslant Ch(t\|\boldsymbol{\zeta}\|_1 + \|\boldsymbol{\zeta}\|_{\boldsymbol{H}(\mathrm{div})}) \\ &\qquad \times \{t\|\boldsymbol{\gamma}_\delta\|_0 + |||\boldsymbol{\theta}_\delta|||_\Theta + [p_W(w_\delta, w_\delta)]^{1/2}\} \\ &\leqslant Ch(\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0) \\ &\qquad \times \{t\|\boldsymbol{\gamma}_\delta\|_0 + |||\boldsymbol{\theta}_\delta|||_\Theta + [p_W(w_\delta, w_\delta)]^{1/2}\}. \end{aligned}$$

Hence, (5.42) becomes:

$$\begin{aligned} &\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0 \\ &\quad \leqslant Ch\{|||\boldsymbol{\theta}-\boldsymbol{\theta}_h|||_\Theta + t|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h|||_0 + t\|\boldsymbol{\gamma}_\delta\|_0 + |||\boldsymbol{\theta}_\delta|||_\Theta \\ &\qquad + [p_W(w_\delta, w_\delta)]^{1/2}\}. \end{aligned}$$

Applying our previous estimates, we immediately obtain the following result.

Theorem 8. *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of the continuous problem (3.13) and (3.14) and let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ be the solution of the discretized problem (3.15) and (3.16) with the choice of spaces (5.1). Then we have*

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0 \leqslant Ch^k(\|\boldsymbol{\theta}\|_k + \|\boldsymbol{\gamma}\|_{k-1}).$$

If moreover $\boldsymbol{\gamma}$ has a smooth Helmholtz decomposition of the type (5.19), then we have

$$\|\boldsymbol{\theta}-\boldsymbol{\theta}_h\|_0 + \|w-w_h\|_0 \leqslant Ch^k(\|\boldsymbol{\theta}\|_k + t\|\boldsymbol{\gamma}\|_{k-1} + \|\boldsymbol{\gamma}\|_{\boldsymbol{H}^{k-2}(\mathrm{div})}).$$

Remark 4. We remark that for the lowest order case ($k = 2$) all our error estimates, namely Theorems 3–5, and subsequent Remark, and Theorems 7 and 8, use norms of the exact solution $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ that are uniformly bounded with respect to t, according to the regularity results (2.9).

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