

Double Complexes and Local Cochain Projections

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The construction of projection operators, which commute with the exterior derivative and at the same time are bounded in the proper Sobolev spaces, represents a key tool in the recent stability analysis of finite element exterior calculus. These so-called bounded cochain projections have been constructed by combining a smoothing operator and the unbounded canonical projections defined by the degrees of freedom. However, an undesired property of these bounded projections is that, in contrast to the canonical projections, they are nonlocal. The purpose of this article is to discuss a recent alternative construction of bounded cochain projections, which also are local. A key tool for the new construction is the structure of a double complex, resembling the Čech-de Rham double complex of algebraic topology. © 2014 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 000: 000–000, 2014

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I. INTRODUCTION

The construction of projection operators onto finite element spaces which commute with the governing differential operators has always been a central feature of the analysis of mixed finite element methods, cf. Refs. [1, 2]. However, the fact that the canonical projections, defined from the degrees of freedom, are usually not bounded in the natural Sobolev norms has resulted in additional complexity of the stability arguments for such methods. In fact, for a long time, it was not known how to construct commuting projections which also are bounded in spaces like $H(\mathbf{curl})$ and $H(\mathbf{div})$, without making extra regularity assumptions. However, during the last decade rather general constructions of such projections have been given, leading to elegant and compact stability proofs.

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The first successful construction was given by Schöberl in [3], where a smoothing operator, constructed by perturbing the mesh, is combined with the canonical projection to obtain a commuting projection operator which also is bounded. In a related paper, Christiansen [4] proposed to use a more standard smoothing operator defined by a mollifier function. In the setting of finite element exterior calculus, variants of such constructions, all based on a proper combination of smoothing and canonical projections, are analyzed in [5, Section 5], [6, Section 5], and [7]. These operators are bounded in the appropriate norms and commute with the exterior derivative. However, these projections lack another key property of the canonical projections; they are not locally defined. In fact, it has not been clear if it is possible to construct bounded and commuting projections which are locally defined. In recent work, such projections were constructed in [8]. The construction is inspired by the well-known Clément operator [9], which is based on local projection operators defined on macroelements. In its original form, where the finite element space consists of continuous piecewise polynomial subspaces of the Sobolev space H^1 , the Clément operator is not a projection. However, by modifying the Clément operator, such that the local projections are defined with respect to piecewise polynomial spaces, a projection is easily obtained. The more challenging part, successfully addressed in Ref. [8], is to obtain commuting projections.

The construction proposed in Ref. [8] is discussed in the setting of exterior calculus and the de Rham complex on bounded domains in \mathbb{R}^n , and covers all orders of the basic finite element spaces described in Refs. [5, 6]. However, the lowest order case, corresponding to the Whitney elements [10], represents in many ways the most challenging part of the construction. A key difficulty is to relate the projection onto the space of Whitney k -forms, constructed from local projections on macroelements defined from subsimplexes of the mesh of dimension k , to the corresponding projection onto Whitney $(k + 1)$ -forms, constructed by local projections on the corresponding macroelements associated to $(k + 1)$ -dimensional subsimplexes. In Ref. [8], a double complex structure, which resembles the Čech-de Rham double complex, cf. Ref. [11], is introduced as a tool to handle this difficulty. Such structures were apparently first introduced by Weil in his proof of de Rham's theorem, cf. Ref. [12]. As far as we know, the construction given in Ref. [8] represents the first time a double complex has been utilized in numerical analysis. The purpose of the present article is to explain the construction given in [8] in the simplest possible setting. In particular, we want to motivate why double complexes seems to be a natural tool for generalizing Clément type operators, based on local projections on macroelements, to obtain bounded commuting projections. In the discussion below, we will therefore restrict the discussion to the lowest-order case of Whitney elements and to the case of two space dimensions.

II. THE DE RHAM COMPLEX AND ITS DISCRETIZATION

For a given polygon $\Omega \subset \mathbb{R}^2$, we consider the associated de Rham complex of the form

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\text{rot}, \Omega) \xrightarrow{\text{rot}} L^2(\Omega). \quad (2.1)$$

Here the operator rot is given by $\text{rot } \mathbf{v} := \partial_{x_2} v_1 - \partial_{x_1} v_2$, mapping a vector field \mathbf{v} into scalar fields. The space $\mathbf{H}(\text{rot}, \Omega)$ is the space of all vector fields $\mathbf{v} \in L^2(\Omega)^2$ with $\text{rot } \mathbf{v} \in L^2(\Omega)$, while $H^1(\Omega)$ is the corresponding Sobolev space consisting of all $u \in L^2(\Omega)$ with $\mathbf{grad } u \in L^2(\Omega)^2$. The sequence (2.1) is a complex since $\text{rot} \circ \mathbf{grad} = 0$. To enhance readability, we will continue to use bold typeface for vector-valued quantities and for operators returning vector fields.

We will consider the simplest finite element subsimplex of (2.1). Hence, for a given triangulation \mathcal{T}_h of Ω , we let $W(\mathcal{T}_h) \subset H^1(\Omega)$ be the corresponding space of continuous piecewise linear

functions and $\mathcal{P}_0(\mathcal{T}_h) \subset L^2(\Omega)$ the space of piecewise constants. The space $\mathcal{N}(\mathcal{T}_h) \subset \mathbf{H}(\text{rot}, \Omega)$ is frequently referred to as the rotated Raviart–Thomas space in the numerical analysis literature. It consists of all piecewise rigid motions with continuous tangential components on all edges of \mathcal{T}_h . Alternatively, the spaces $W(\mathcal{T}_h)$, $\mathcal{N}(\mathcal{T}_h)$, and $\mathcal{P}_0(\mathcal{T}_h)$ correspond to the space of Whitney forms of order 0, 1, and 2, respectively. It is a simple fact that

$$W(\mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{N}(\mathcal{T}_h) \xrightarrow{\text{rot}} \mathcal{P}_0(\mathcal{T}_h) \tag{2.2}$$

is a subcomplex of (2.1). In particular, $\text{grad}(W(\mathcal{T}_h)) \subset \mathcal{N}(\mathcal{T}_h)$ and $\text{rot}(\mathcal{N}(\mathcal{T}_h)) \subset \mathcal{P}_0(\mathcal{T}_h)$. Our goal is to define projection operators π_h^0 , π_h^1 , and π_h^2 such that the diagram

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\text{grad}} & \mathbf{H}(\text{rot}, \Omega) & \xrightarrow{\text{rot}} & L^2(\Omega) \\ \downarrow \pi_h^0 & & \downarrow \pi_h^1 & & \downarrow \pi_h^2 \\ W(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}(\mathcal{T}_h) & \xrightarrow{\text{rot}} & \mathcal{P}_0(\mathcal{T}_h) \end{array}$$

commutes. In addition, the projections π_h^i will be local and bounded in the norms of $H^1(\Omega)$, $\mathbf{H}(\text{rot}, \Omega)$, and $L^2(\Omega)$, respectively.

III. THE PROJECTION π_h^0

We will first define the projection π_h^0 . We let $\Delta_j(\mathcal{T}_h)$ be the set of subsimplexes of \mathcal{T}_h of dimension j . In other words, $\Delta_0(\mathcal{T}_h)$ is the set of vertices, $\Delta_1(\mathcal{T}_h)$ the set of edges, and $\Delta_2(\mathcal{T}_h)$ the set of triangles, while $\Delta(\mathcal{T}_h)$ is the union of the three sets. For each subset $f \in \Delta(\mathcal{T}_h)$, the associated macroelement Ω_f is given by

$$\Omega_f := \cup \{T \mid T \in \mathcal{T}_h, f \in \Delta(T)\}.$$

A vertex macroelement and an edge macroelement are shown in Fig. 1.

If $u \in H^1(\Omega)$, then the piecewise linear function $\pi_h^0 u$ is determined by its values at each vertex, since

$$\pi_h^0 u = \sum_{y \in \Delta_0(\mathcal{T}_h)} \pi_h^0 u(y) \lambda_y,$$

where λ_y is the piecewise linear hat function associated to the vertex y , that is, $\lambda_y(y) = 1$ and $\lambda_y \equiv 0$ on the complement of the macroelement Ω_y . Hence, to determine $\pi_h^0 u$, it is enough to

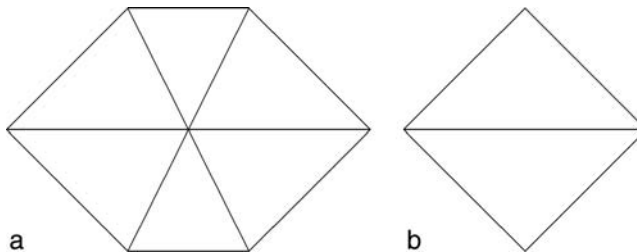


FIG. 1. (a) Vertex macroelement and (b) edge macroelement.

determine $\pi_h^0 u(y)$ for all $y \in \Delta_0(\mathcal{T}_h)$. However, since functions $u \in H^1(\Omega)$ in general do not have point values, we cannot take $\pi_h^0 u(y)$ to be $u(y)$. Instead, the idea of the Clément operator is to compute $\pi_h^0 u(y)$, for all $y \in \Delta_0(\mathcal{T}_h)$, from a local projection P_y applied to u . The local projections P_y are defined by solving discrete Laplace–Neumann problems on the macroelements Ω_y .

We let $W(\mathcal{T}_{y,h})$ be the restriction of the space $W(\mathcal{T}_h)$ to Ω_y , that is, $W(\mathcal{T}_{y,h})$ is the space of piecewise linear functions on Ω_y . The solution operator for the discrete Laplace–Neumann problem can naturally be separated into two parts, a mean value contribution and a second part which only depends on the gradient of the function u . More precisely, we define $P_y u$ by

$$P_y u = |\Omega_y|^{-1} \int_{\Omega_y} u \, dx + Q_y^0 u,$$

where $|\Omega_y| = \int_{\Omega_y} dx$. The operator Q_y^0 maps $H^1(\Omega_y)$ into $W(\mathcal{T}_{y,h})$ such that the mean value of $Q_y^0 v$ over Ω_y is zero, and

$$\int_{\Omega_y} \mathbf{grad}(Q_y^0 v - v) \cdot \mathbf{grad} w \, dx = 0, \quad w \in W(\mathcal{T}_{y,h}).$$

This defines $Q_y^0 v$ uniquely. The operator π_h^0 is then defined by the condition $\pi_h^0 u(y) = P_y u(y)$ for each $y \in \Delta_0(\mathcal{T}_h)$. Alternatively,

$$\pi_h^0 u = \sum_{y \in \Delta_0(\mathcal{T}_h)} P_y u(y) \lambda_y.$$

The operator π_h^0 is bounded in $H^1(\Omega)$. Furthermore, it is straightforward to check that it is a projection, that is, $\pi_h^0 u = u$ if $u \in W(\mathcal{T}_h)$, since we obviously have $P_y u(y) = u(y)$ in this case.

IV. THE PROJECTION π_h^1

In this section, we will construct the projection π_h^1 . In addition to the macroelements Ω_f , we will also need extended macroelements Ω_f^e given by

$$\Omega_f^e = \bigcup_{y \in \Delta_0(f)} \Omega_y.$$

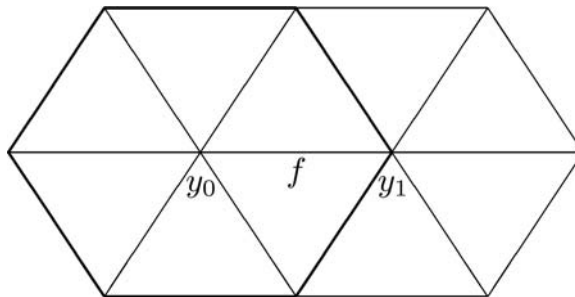


FIG. 2. The extended macroelement Ω_f^e for $f = [y_0, y_1]$.

Alternatively, Ω_f^e is the union of all triangles of \mathcal{T}_h which intersect f . An example of an extended macroelement is shown in Fig. 2.

In the special case that $\dim f = 0$, that is, f is a vertex, then $\Omega_f^e = \Omega_f$. In general, if $f, g \in \Delta(\mathcal{T}_h)$ with $g \in \Delta(f)$ then

$$\Omega_f \subset \Omega_g \quad \text{and} \quad \Omega_g^e \subset \Omega_f^e.$$

We let $\mathcal{T}_{f,h}^e$ be the restriction of \mathcal{T}_h to Ω_f^e . We will assume throughout that all the macroelements Ω_f^e are simply connected. On these macroelements, we utilize discrete complexes of the form

$$\mathring{W}(\mathcal{T}_{f,h}^e) \xrightarrow{\text{curl}} \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e) \xrightarrow{\text{div}} \mathring{\mathcal{P}}_0(\mathcal{T}_{f,h}^e). \tag{4.1}$$

Here **curl** denotes the two-dimensional curl-operator, that is, the operator which maps a scalar field u to the vector field $\mathbf{curl} u = (-\partial_{x_2} u, \partial_{x_1} u)$. The complex property follows since $\text{div} \circ \mathbf{curl} = 0$. The spaces $\mathring{W}(\mathcal{T}_{f,h}^e)$ and $\mathring{\mathcal{P}}_0(\mathcal{T}_{f,h}^e)$ consist of piecewise linear and piecewise constant functions on Ω_f^e , and restricted to vanishing boundary values in the piecewise linear case, and to vanishing integral over Ω_f^e in the piecewise constant case. Finally, the space $\mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e)$ is the lowest order Raviart–Thomas space on Ω_f^e with vanishing normal components on the boundary. Since the macroelements Ω_f^e are simply connected, it follows that the complex given in (4.1) is exact. This means that any element in $\mathring{\mathcal{P}}_0(\mathcal{T}_{f,h}^e)$ can be expressed as $\text{div } \mathbf{v}$, where $\mathbf{v} \in \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e)$, and any divergence free element of $\mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e)$ is equal to $\mathbf{curl} w$ for a unique $w \in \mathring{W}(\mathcal{T}_{f,h}^e)$. Recall that the projection π_h^1 is required to satisfy the commuting property

$$\pi_h^1 \mathbf{grad} u = \mathbf{grad} \pi_h^0 u, \quad u \in H^1(\Omega). \tag{4.2}$$

Since the left hand side of this identity only depends on $\mathbf{grad} u$, so must the right hand side. Furthermore, since we want π_h^1 to be local, we need to see that $\mathbf{grad} \pi_h^0 u$ depends locally on $\mathbf{grad} u$. We introduce the mean value operator, $M_h^0 : L^2(\Omega) \rightarrow W(\mathcal{T}_h)$, by

$$M_h^0 u = \sum_{y \in \Delta_0(\mathcal{T}_h)} \left(\int_{\Omega_y} u z_y^0 dx \right) \lambda_y,$$

where $z_y^0 = |\Omega_y|^{-1}$. Then we have that

$$\pi_h^0 u = M_h^0 u + \sum_{y \in \Delta_0(\mathcal{T}_h)} Q_y^0 u(y) \lambda_y. \tag{4.3}$$

From the definition of the operator Q_y^0 , we see that the second term here already depends on $\mathbf{grad} u$. Furthermore, we observe that

$$\mathbf{grad} M_h^0 u = \sum_{y \in \Delta_0(\mathcal{T}_h)} \left(\int_{\Omega_y} u z_y^0 dx \right) \mathbf{grad} \lambda_y.$$

Let $f = [y_0, y_1] \in \Delta_1(\mathcal{T}_h)$ be a fixed edge with vertices y_0 and y_1 , and consider the tangential component of $\mathbf{grad} M_h^0 u$ on f . We have

$$\int_f \mathbf{grad} M_h^0 u \cdot (y_1 - y_0) ds = |f| \left(\int_{\Omega_{y_1}} u z_{y_1}^0 dx - \int_{\Omega_{y_0}} u z_{y_0}^0 dx \right) = -|f| \int_{\Omega_f^e} u (\delta z^0)_f dx,$$

where $(\delta z^0)_f = z_{y_0}^0 - z_{y_1}^0$ and $|f|$ is the length of f . Here, we assume that the functions z_y^0 are extended by zero outside Ω_y , such that $(\delta z^0)_f$ is a piecewise constant function on the extended macroelement Ω_f^e , and with integral equal to zero. In other words, $(\delta z^0)_f$ belongs to the space $\mathring{P}_0(\mathcal{T}_{f,h}^e)$, and by the exactness of the complex (4.1), it follows that there is a unique function $z_f^1 \in \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e)$ such that

$$\operatorname{div} z_f^1 = (\delta z^0)_f, \quad \text{and} \quad \int_{\Omega_f^e} z_f^1 \cdot \operatorname{curl} w \, dx = 0, \quad w \in \mathring{W}(\mathcal{T}_{f,h}^e).$$

Hence, from integration by parts we obtain

$$\int_f \mathbf{grad} M_h^0 u \cdot (y_1 - y_0) \, ds = -|f| \int_{\Omega_f^e} u \operatorname{div} z_f^1 \, dx = |f| \int_{\Omega_f^e} \mathbf{grad} u \cdot z_f^1 \, dx.$$

Recall that functions in $\mathcal{N}(\mathcal{T}_h)$ are uniquely determined by the integrals of the tangential components over all edges. From the calculations above, we can therefore conclude that

$$\mathbf{grad} M_h^0 u = \sum_{f \in \Delta_1(\mathcal{T}_h)} \left(\int_{\Omega_f^e} \mathbf{grad} u \cdot z_f^1 \, dx \right) \phi_f, \tag{4.4}$$

where $\phi_f \in \mathcal{N}(\mathcal{T}_h)$ is the Whitney 1-form associated to the edge $f = [y_0, y_1]$, scaled such that $\int_f \phi_f \cdot (y_1 - y_0) \, ds = |f|$. In other words,

$$\phi_f = \lambda_{y_0} \mathbf{grad} \lambda_{y_1} - \lambda_{y_1} \mathbf{grad} \lambda_{y_0},$$

and any $v \in \mathcal{N}(\mathcal{T}_h)$ admits a representation of the form

$$v = \sum_{f=[y_0,y_1] \in \Delta_1(\mathcal{T}_h)} |f|^{-1} \int_f v \cdot (y_1 - y_0) \, ds \phi_f.$$

For any $v \in L^2(\Omega)^2$, we now define $M_h^1 v \in \mathcal{N}(\mathcal{T}_h)$ by

$$M_h^1 v = \sum_{f \in \Delta_1(\mathcal{T}_h)} \left(\int_{\Omega_f^e} v \cdot z_f^1 \, dx \right) \phi_f.$$

The identity $\mathbf{grad} M_h^0 u = M_h^1 \mathbf{grad} u$ follows from (4.4). For each $y \in \Delta_0(\mathcal{T}_h)$, we introduce the operator $Q_{y,-}^1 : L^2(\Omega_y)^2 \rightarrow W(\mathcal{T}_{y,h})$ defined by

$$\int_{\Omega_y} (\mathbf{grad} (Q_{y,-}^1 v) - v) \cdot \mathbf{grad} w \, dx = 0, \quad w \in W(\mathcal{T}_{y,h}),$$

with the mean value of $Q_{y,-}^1 v$ set to zero. Hence, by construction, we have

$$Q_{y,-}^1 \mathbf{grad} u = Q_y^0 u. \tag{4.5}$$

Therefore, if we define an operator $S_h^1 : L^2(\Omega)^2 \rightarrow \mathcal{N}(\mathcal{T}_h)$ by

$$S_h^1 \mathbf{v} = M_h^1 \mathbf{v} + \sum_{y \in \Delta_0(\mathcal{T}_h)} Q_{y,-}^1 \mathbf{v}(y) \mathbf{grad} \lambda_y,$$

then the desired commuting relation $\mathbf{grad} \pi_h^0 u = S_h^1 \mathbf{grad} u$ follows, cf. (4.3) and (4.5). However, the operator S_h^1 will in general not be a projection onto the space $\mathcal{N}(\mathcal{T}_h)$. Therefore, the operator π_h^1 will instead be of the form

$$\pi_h^1 \mathbf{v} = S_h^1 \mathbf{v} + \sum_{f=[y_0,y_1] \in \Delta_1(\mathcal{T}_h)} \left(|f|^{-1} \int_f (I - S_h^1) \mathbf{Q}_f^1 \mathbf{v} \cdot (\mathbf{y}_1 - \mathbf{y}_0) ds \right) \phi_f,$$

where \mathbf{Q}_f^1 is a local projection defined with respect to the extended macroelement Ω_f^e . The operator $\mathbf{Q}_f^1 : \mathbf{H}(\text{rot}, \Omega_f^e) \rightarrow \mathcal{N}(\mathcal{T}_{f,h}^e)$ is defined by

$$\begin{aligned} \int_{\Omega_f^e} (\mathbf{Q}_f^1 \mathbf{v} - \mathbf{v}) \cdot \mathbf{grad} w \, dx &= 0, \quad w \in W(\mathcal{T}_{f,h}^e), \\ \int_{\Omega_f^e} \text{rot}(\mathbf{Q}_f^1 \mathbf{v} - \mathbf{v}) \text{rot} \psi \, dx &= 0, \quad \psi \in \mathcal{N}(\mathcal{T}_{f,h}^e). \end{aligned} \tag{4.6}$$

These conditions determine $\mathbf{Q}_f^1 \mathbf{v} \in \mathcal{N}(\mathcal{T}_{f,h}^e)$ uniquely as a consequence of the exactness of the complex (2.2) restricted to the domain Ω_f^e . Furthermore, the operator \mathbf{Q}_f^1 is a local projection. In fact, the operator \mathbf{Q}_f^1 admits the decomposition

$$\mathbf{Q}_f^1 \mathbf{v} = \mathbf{grad} \mathbf{Q}_f^0 \mathbf{v} + \mathbf{Q}_f^2 \text{rot} \mathbf{v}, \tag{4.7}$$

where the operators \mathbf{Q}_f^0 and \mathbf{Q}_f^2 are defined by subsystems of (4.6). More precisely, $\mathbf{Q}_f^0 \mathbf{v} \in W(\mathcal{T}_{f,h}^e)$ has integral zero over Ω_f^e and satisfies

$$\int_{\Omega_f^e} (\mathbf{grad} \mathbf{Q}_f^0 \mathbf{v} - \mathbf{v}) \cdot \mathbf{grad} w \, dx = 0, \quad w \in W(\mathcal{T}_{f,h}^e),$$

while $\mathbf{Q}_f^2 u \in \mathcal{N}(\mathcal{T}_{f,h}^e)$ is determined by

$$\begin{aligned} \int_{\Omega_f^e} \mathbf{Q}_f^2 u \cdot \mathbf{grad} w \, dx &= 0, \quad w \in W(\mathcal{T}_{f,h}^e), \\ \int_{\Omega_f^e} (\text{rot} \mathbf{Q}_f^2 u - u) \text{rot} \psi \, dx &= 0, \quad \psi \in \mathcal{N}(\mathcal{T}_{f,h}^e), \end{aligned}$$

for any $u \in L^2(\Omega_f^e)$.

The operator π_h^1 is a projection onto $\mathcal{N}(\mathcal{T}_{f,h}^e)$, since for any $\mathbf{v} \in \mathcal{N}(\mathcal{T}_{f,h}^e)$, we have

$$\pi_h^1 \mathbf{v} = \sum_{f=[y_0,y_1] \in \Delta_1(\mathcal{T}_h)} \left(|f|^{-1} \int_f \mathbf{Q}_f^1 \mathbf{v} \cdot (\mathbf{y}_1 - \mathbf{y}_0) ds \right) \phi_f = \mathbf{v}.$$

On the other hand, it follows from (4.7) that for $w \in H^1(\Omega_f^e)$, $\mathcal{Q}_f^1 \mathbf{grad} w = \mathbf{grad} \mathcal{Q}_f^0 \mathbf{grad} w$, and this implies that on f ,

$$\begin{aligned} S_h^1 \mathcal{Q}_f^1 \mathbf{grad} w &= S_h^1 \mathbf{grad} \mathcal{Q}_f^0 \mathbf{grad} w = \mathbf{grad} \pi_h^0 \mathcal{Q}_f^0 \mathbf{grad} w \\ &= \mathbf{grad} \mathcal{Q}_f^0 \mathbf{grad} w = \mathcal{Q}_f^1 \mathbf{grad} w. \end{aligned}$$

Therefore, it follows that for any $f = [y_0, y_1] \in \Delta_1(\mathcal{T}_h)$ and $w \in H^1(\Omega_f^e)$,

$$(I - S_h^1) \mathcal{Q}_f^1 \mathbf{grad} w \cdot (y_1 - y_0) = 0 \quad \text{on } f, \tag{4.8}$$

and as a consequence,

$$\pi_h^1 \mathbf{grad} w = S_h^1 \mathbf{grad} w = \mathbf{grad} \pi_h^0 w.$$

We have therefore seen that π_h^1 is a projection operator, which is local and satisfies the desired commuting relation (4.2).

V. THE DOUBLE COMPLEX

In the construction of the projection π_h^1 above, we have already implicitly used a double complex structure. To see this more clearly, consider the direct sum over all $y \in \Delta_0(\mathcal{T}_h)$ of the complexes of the form (4.1). This gives the exact complex

$$\bigoplus_{y \in \Delta_0(\mathcal{T}_h)} \mathring{W}(\mathcal{T}_{f,h}^e) \xrightarrow{\mathbf{curl}} \bigoplus_{y \in \Delta_0(\mathcal{T}_h)} \mathcal{RT}(\mathcal{T}_{f,h}^e) \xrightarrow{\mathbf{div}} \bigoplus_{y \in \Delta_0(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{f,h}^e).$$

Here the differential operators **curl** and **div** are applied to each component in the sum. The operator $\delta = \delta_0$, introduced in the construction of the operator π_h^1 above, represents another operator naturally acting on these spaces. The operator δ_0 is of the form

$$\delta_0 : \bigoplus_{y \in \Delta_0(\mathcal{T}_h)} V_0(y) \rightarrow \bigoplus_{f \in \Delta_1(\mathcal{T}_h)} V_1(f), \quad (\delta_0 u)_f = u_{y_0} - u_{y_1} \quad \text{if } f = [y_0, y_1].$$

Here the space $V_m(f)$ is a substitute for any of the local spaces $\mathring{W}(\mathcal{T}_{f,h}^e)$, $\mathcal{RT}(\mathcal{T}_{f,h}^e)$, or $\mathring{P}_0(\mathcal{T}_{f,h}^e)$ for $f \in \Delta_m(\mathcal{T}_h)$. Furthermore, it is implicitly assumed that the functions u_y are extended by zero outside Ω_y . In fact, if $u \in \bigoplus_{y \in \Delta_0(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{y,h})$ and all the components of $u = \{u_y \mid y \in \Delta_0(\mathcal{T}_h)\}$ have the same mean value with respect to Ω_y , the function $\delta_0 u$ will be in $\bigoplus_{f \in \Delta_1(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{f,h}^e)$. This was exactly what we utilized above to define $z^1 = \{z_f^1\} \in \bigoplus_{f \in \Delta_1(\mathcal{T}_h)} \mathcal{RT}(\mathcal{T}_{f,h}^e)$ such that $\mathbf{div} z^1 = \delta z^0$. We similarly define an operator $\delta = \delta_1 : \bigoplus_{g \in \Delta_1(\mathcal{T}_h)} V_1(g) \rightarrow \bigoplus_{f \in \Delta_2(\mathcal{T}_h)} V_2(f)$ by

$$(\delta_1 u)_f = u_{[y_1, y_2]} - u_{[y_0, y_2]} + u_{[y_0, y_1]} \quad \text{if } f = [y_0, y_1, y_2],$$

where the notation $[\cdot, \dots, \cdot]$ is used to denote convex combination. We obtain a double complex of the form

$$\begin{array}{ccccc}
 \bigoplus_{f \in \Delta_0(\mathcal{T}_h)} \dot{W}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{curl}} & \bigoplus_{f \in \Delta_0(\mathcal{T}_h)} \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{div}} & \bigoplus_{f \in \Delta_0(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{f,h}^e) \\
 \downarrow \delta_0 & & \downarrow \delta_0 & & \downarrow \delta_0 \\
 \bigoplus_{f \in \Delta_1(\mathcal{T}_h)} \dot{W}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{curl}} & \bigoplus_{f \in \Delta_1(\mathcal{T}_h)} \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{div}} & \bigoplus_{f \in \Delta_1(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{f,h}^e) \\
 \downarrow \delta_1 & & \downarrow \delta_1 & & \downarrow \delta_1 \\
 \bigoplus_{f \in \Delta_2(\mathcal{T}_h)} \dot{W}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{curl}} & \bigoplus_{f \in \Delta_2(\mathcal{T}_h)} \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e) & \xrightarrow{\text{div}} & \bigoplus_{f \in \Delta_2(\mathcal{T}_h)} \mathring{P}_0(\mathcal{T}_{f,h}^e)
 \end{array}$$

This is a double complex in the sense that each row and each column is a complex. In particular, $\delta_1 \circ \delta_0 = 0$. Furthermore, the operators δ_0 and δ_1 commute with the differential operators **curl** and **div**, that is,

$$\delta_i \mathbf{curl} = \mathbf{curl} \circ \delta_i \quad \text{and} \quad \delta_i \mathbf{div} = \mathbf{div} \circ \delta_i.$$

In particular, consider the function $\delta_1 z^1 \in \bigoplus_{f \in \Delta_2(\mathcal{T}_h)} \mathring{\mathcal{RT}}(\mathcal{T}_{f,h}^e)$, where $z^1 = \{z_f^1\}_{f \in \Delta_1(\mathcal{T}_h)}$ is the function introduced above to define the operator M_h^1 . This function is divergence free, since

$$\mathbf{div} \delta_1 z^1 = \delta_1 \mathbf{div} z^1 = \delta_1 \circ \delta_0 z^0 = 0.$$

As a consequence of the exactness of the last row of the double complex above, we conclude that there is a unique $z^2 \in \bigoplus_{f \in \Delta_0(\mathcal{T}_h)} \dot{W}(\mathcal{T}_{f,h}^e)$ such that $\mathbf{curl} z^2 = \delta_1 z^1$. The components of the function z^2 will be utilized in the construction of the operator π_h^2 below.

VI. THE PROJECTION π_h^2

It remains to construct the locally defined projection π_h^2 onto $\mathcal{P}_0(\mathcal{T}_h)$ such that

$$\pi_h^2 \mathbf{rot} v = \mathbf{rot} \pi_h^1 v.$$

We start by computing $\mathbf{rot} S_h^1 v = \mathbf{rot} M_h^1 v$. We have

$$\mathbf{rot} M_h^1 v = \sum_{f \in \Delta_1(\mathcal{T}_h)} \left(\int_{\Omega_f^e} v \cdot z_f^1 dx \right) \mathbf{rot} \phi_f.$$

Let $T = [y_0, y_1, y_2] \in \Delta_2(\mathcal{T}_h)$, where the vertices are ordered counter clockwise. The function $\mathbf{rot} M_h^1$ is a constant on T and the only nonzero contributions in the sum above on T arise from the three edges $[y_1, y_2]$, $[y_0, y_2]$, and $[y_0, y_1]$. From the edge $f = [y_1, y_2]$ we have

$$\int_T \mathbf{rot} \phi_{[y_1,y_2]} dx = |f|^{-1} \int_{[y_1,y_2]} \phi_{[y_1,y_2]} \cdot (y_2 - y_1) ds = 1.$$

Similar calculations show

$$\int_T \mathbf{rot} \phi_{[y_0,y_2]} dx = -1, \quad \text{and} \quad \int_T \mathbf{rot} \phi_{[y_0,y_1]} dx = 1.$$

We therefore obtain that

$$\int_T \operatorname{rot} \mathbf{M}_h^1 \mathbf{v} \, dx = \int_{\Omega_T^e} \mathbf{v} \cdot (\delta_1 \mathbf{z}^1)_T \, dx = \int_{\Omega_T^e} \mathbf{v} \cdot \operatorname{curl} z_T^2 \, dx = \int_{\Omega_T^e} (\operatorname{rot} \mathbf{v}) z_T^2 \, dx.$$

For any $u \in L^2(\Omega)$, we now define $M_h^2 u$ by

$$M_h^2 u = \sum_{T \in \Delta_2(\mathcal{T}_h)} \int_{\Omega_T^e} u z_T^2 \, dx.$$

The identity $M_h^2 \operatorname{rot} \mathbf{v} = \operatorname{rot} \mathbf{M}_h^1 \mathbf{v} = \operatorname{rot} \mathbf{S}_h^1 \mathbf{v}$ is a consequence of the calculations above. From the definition of the operator π_h^1 , we now obtain

$$\begin{aligned} \operatorname{rot} \pi_h^1 \mathbf{v} &= M_h^2 \operatorname{rot} \mathbf{v} + \sum_{f=[y_0, y_1] \in \Delta_1(\mathcal{T}_h)} \left(|f|^{-1} \int_f (I - \mathbf{S}_h^1) \mathbf{Q}_f^1 \mathbf{v} \cdot (\mathbf{y}_1 - \mathbf{y}_0) \, ds \right) \operatorname{rot} \phi_f \\ &= M_h^2 \operatorname{rot} \mathbf{v} + \sum_{f=[y_0, y_1] \in \Delta_1(\mathcal{T}_h)} \left(|f|^{-1} \int_f (I - \mathbf{S}_h^1) \mathbf{Q}_f^2 \operatorname{rot} \mathbf{v} \cdot (\mathbf{y}_1 - \mathbf{y}_0) \, ds \right) \operatorname{rot} \phi_f, \end{aligned}$$

where the last identity follows by combining (4.7) and (4.8). Hence, if we define $S_h^2 : L^2(\Omega) \rightarrow \mathcal{P}_0(\mathcal{T}_h)$ by

$$S_h^2 u = M_h^2 u + \sum_{f=[y_0, y_1] \in \Delta_1(\mathcal{T}_h)} \left(|f|^{-1} \int_f (I - \mathbf{S}_h^1) \mathbf{Q}_f^2 u \cdot (\mathbf{y}_1 - \mathbf{y}_0) \, ds \right) \operatorname{rot} \phi_f,$$

then the identity $S_h^2 \operatorname{rot} \mathbf{v} = \operatorname{rot} \pi_h^1 \mathbf{v}$ follows by construction. However, in the present case, the operator S_h^2 is also a projection. To see this, let $u \in \mathcal{P}_0(\mathcal{T}_h)$. It is enough to show that

$$\int_T S_h^2 u \, dx = \int_T u \, dx, \quad T \in \Delta_2(\mathcal{T}_h). \tag{6.1}$$

However, due to the exactness of the complex (2.2), restricted to Ω_T^e , there is a $\mathbf{v} \in \mathcal{N}(\mathcal{T}_{T,h}^e)$ such that $\operatorname{rot} \mathbf{v} = u$ on Ω_T^e . Therefore, by the projection property of the operator π_h^1 we obtain

$$\int_T S_h^2 u \, dx = \int_T S_h^2 \operatorname{rot} \mathbf{v} \, dx = \int_T \operatorname{rot} \pi_h^1 \mathbf{v} \, dx = \int_T \operatorname{rot} \mathbf{v} \, dx = \int_T u \, dx.$$

We therefore define π_h^2 to be the operator S_h^2 .

VII. CONCLUSIONS

We have constructed Clément-type projection operators for discretization of the de Rham complex in two space dimensions. Only the lowest-order finite element spaces, that is, the Whitney forms, are considered. The projections are locally defined, they commute with the differential operators of the de Rham complex, and they are bounded in the natural Sobolev norms. A discussion in the general case, covering higher-order piecewise polynomial spaces and arbitrary space dimensions, can be found in the recent paper [8].

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