A FINITE ELEMENT METHOD FOR THE APPROXIMATION OF THE INCOMPRESSIBLE, LINEARIZED EULER EQUATIONS

Richard S. Falk
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903

Gerard R. Richter
Department of Computer Science
Rutgers University
New Brunswick, NJ 08903

Abstract. We present a finite element method for the transient, linearized, incompressible Euler equations in two space dimensions. The velocity equations are discretized via the discontinuous Galerkin method over a space-time mesh of tetrahedrons. The mesh is assumed to have been constructed in such a way that the tetrahedrons can be ordered explicitly with respect to velocity evolution. For $n \geq 0$, the method yields a discontinuous piecewise polynomial approximation of degree n for velocity and a continuous approximation of degree $n+1$ for pressure. We derive error estimates of order $h^{n+1/2}$ for velocity and $h^{n-1/2}$ for the pressure gradient.

1. Introduction.

We present a finite element method, based on the discontinuous Galerkin method, for a linearized model of the incompressible Euler equations in two space dimensions:

$$\begin{aligned} \mathbf{u}_t + \mathbf{w} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \qquad &\text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 \qquad &\text{in } Q, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad &\text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $Q = \Omega \times (0,T]$ where Ω is a domain in R^2 with unit outer normal $\mathbf{n}$. The desired velocity and pressure are denoted by $\mathbf{u}$ and p, and $\operatorname{div} \mathbf{u}_0 = 0$. In addition, we assume

$$\operatorname{div} \mathbf{w} = 0 \qquad \text{in } Q, \qquad \mathbf{w} \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega \times (0,T].$$

The velocity field $\mathbf{w}$ may be thought of as corresponding to a Newton iterate $\mathbf{u}^{(k)}$ about which the Euler equations have been linearized. A term $\nabla \mathbf{w} \cdot \mathbf{u}$ also arises in the linearization; we have omitted it for the sake of simplicity. Our treatment of this linearized problem is intended to elucidate some aspects of finite element approximation of the (nonlinear) Euler equations.

Our finite element method for (1.1) uses a mesh of tetrahedrons in space-time. The mesh is constructed by dividing the time interval $[0,T]$ into subintervals $[t_m, t_{m+1}], m = 0, 1, \cdots M-1$, of (quasiuniform) width k, and then subdividing each layer $S_m \equiv \Omega \times [t_m, t_{m+1}]$ into a set τ_h^m of tetrahedrons. We assume this is done in such a way that:

(i) all vertices in τ_h^m lie on $t = t_m$ or $t = t_{m+1}$, and on these planes the mesh reduces to a triangulation of (quasiuniform) side length h with minimum angle bounded away from zero.

Note that the characteristics of (1.1) have direction $\mathbf{W} = (\mathbf{w}, 1)$ in $\mathbf{x}, t$-space. Thus the direction of flow across a given tetrahedron face with unit outer normal $\mathbf{N} = (\mathbf{n}_x, n_t)$ is determined by the sign of

$\mathbf{W}\cdot\mathbf{N} = \mathbf{w}\cdot\mathbf{n}_x + n_t$. A negative (positive) sign corresponds to an inflow (outflow) face of the tetrahedron. We also assume:

(ii) $|\mathbf{w}\cdot\mathbf{n}_x| \le |n_t|$ on all tetrahedron faces, and the domain of dependence of each tetrahedron in τ_h^m includes at most a bounded number (independent of h and k) of other tetrahedrons in τ_h^m.

Such a mesh can be constructed in various ways if the Courant number $\lambda \equiv \frac{k}{h}||\mathbf{w}||_{\infty,Q}$ is sufficiently small, i.e., k is chosen sufficiently small relative to h. Condition (ii) implies unidirectional flow, in the direction of increasing t, across each face of each tetrahedron. If ∇p in (1.1) were known, $\mathbf{u}$ could be developed "explicitly", from one tetrahedron to another. In fact, $\mathbf{u}$ can be viewed as evolving through τ_h^m in a front-like fashion, in a bounded number of parallel steps.

Before defining our finite element approximation, we need some more notation. Let $P_n(T)$ denote the set of polynomials in $\mathbf{x}$ and t of degree at most n on T. Define

$$\mathbf{V}_h^m = \{\mathbf{v} \in [L^2(S_m)]^2 : \mathbf{v}|_T \in [P_n(T)]^2, \text{ for all } T \in \tau_h^m\},$$
$$Q_h^m = \{q \in H^1(S_m) : q|_T \in P_{n+1}(T), \text{ for all } T \in \tau_h^m\},$$

and $\mathbf{V}_h$, Q_h to be the extensions of these spaces to all of Q, i.e,

$$\mathbf{V}_h = \{\mathbf{v} \in [L^2(Q)]^2 : \mathbf{v}|_{S_m} \in \mathbf{V}_h^m\}, \qquad Q_h = \{q \in L^2(Q) : q|_{S_m} \in Q_h^m\}.$$

Thus, the functions in Q_h are continuous in $\mathbf{x}$, but possibly discontinuous in t across time levels t_m.

We shall use the notation:

$$(\mathbf{u},\mathbf{v})_T = \int_T \mathbf{u}\cdot\mathbf{v}\,dx, \qquad (\mathbf{u},\mathbf{v})_{S_m} = \sum_{T\in\tau_h^m} (\mathbf{u},\mathbf{v})_T$$

to denote the inner product over a single tetraheron T and the sum of the integrals over all tetrahedrons comprising the mesh τ_h^m, respectively.

Given a domain $D \subset Q$, we denote the boundary of D by $\Gamma(D)$. The inflow portion of $\Gamma(D)$, $\Gamma_{\text{in}}(D)$, is characterized by $\mathbf{W}\cdot\mathbf{N} < 0$; analogously for $\Gamma_{\text{out}}(D)$. For $\mathbf{v} \in [H^k(D)]^2, q \in H^k(D)$, we shall denote the corresponding norms by $||\mathbf{v}||_{k,D}$ and $||q||_{k,D}$, with k omitted when it has value zero. For a subset of $\Gamma(D)$, e.g., $\Gamma_{\text{in}}(D)$, we define

$$< \mathbf{u},\mathbf{v} >_{\Gamma_{\text{in}}(D)} = \int_{\Gamma_{\text{in}}(D)} \mathbf{u}\cdot\mathbf{v}\,|\mathbf{W}\cdot\mathbf{N}|\,ds$$

and $|\mathbf{v}|_{\Gamma_{\text{in}}(D)} = \sqrt{< \mathbf{v},\mathbf{v} >_{\Gamma_{\text{in}}(D)}}$. We shall also use the notation

$$B_T(\mathbf{w};\mathbf{u},\mathbf{v}) \equiv (\mathbf{u}_t + \mathbf{w}\cdot\nabla\mathbf{u},\mathbf{v})_T + < \mathbf{u}^+ - \mathbf{u}^-, \mathbf{v} >_{\Gamma_{\text{in}}(T)},$$

where T is an individual tetrahedron, and for a point $P = (\mathbf{x},t) \in \Gamma(T)$, $\mathbf{u}_h^{\pm}(P) \equiv \lim_{\epsilon\to 0+} \mathbf{u}_h(P \pm \epsilon\,\mathbf{W})$.

We now define the approximate problem as follows. Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in Q_h$ satisfying for $m = 0, 1, \cdots, M-1$:

$$B_T(\mathbf{w};\mathbf{u}_h,\mathbf{v}_h) + (\nabla p_h,\mathbf{v}_h)_T = (\mathbf{f},\mathbf{v}_h)_T \qquad \text{for all } T \in \tau_h^m,\ \mathbf{v}_h \in [P_n(T)]^2, \tag{1.2}$$

$$(\mathbf{u}_h + k^2(\mathbf{u}_h)_t, \nabla q_h)_{S_m} = 0 \qquad \text{for all } q_h \in Q_h^m. \tag{1.3}$$

A potential advantage of this formulation is that for known p_h, (1.2) can be solved explicitly, element by element, for $\mathbf{u}_h$. This could be useful in designing an iterative method for computing the solution. In addition, the method can be readily extended to deal with an $O(h)$ diffusion term (cf. [2]).

In this paper we will derive error estimates of order $h^{n+1/2}$ for $\mathbf{u}_h$ and $h^{n-1/2}$ for ∇p_h assuming $\mathbf{u} \in [H^{n+1}(Q)]^2$ and $p \in H^{n+3/2}(Q)$. In a related work, Johnson and Saranen [1] analyzed a discontinuous Galerkin method for the nonlinear incompressible Euler equations, and obtained a velocity estimate of the same order. Their method uses a divergence-free velocity space, leading to an implicit velocity approximation.

2. Stability.

Our objective here will be to derive a global stability result for the finite element method (1.2) - (1.3).

We first note that

$$B_T(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2}\left(|\mathbf{v}^-|^2_{\Gamma_{\text{out}}(T)} - |\mathbf{v}^-|^2_{\Gamma_{\text{in}}(T)} + |\mathbf{v}^+ - \mathbf{v}^-|^2_{\Gamma_{\text{in}}(T)}\right). \tag{2.1}$$

Thus

$$B_m(\mathbf{w};\mathbf{v},\mathbf{v}) \equiv \sum_{T\in S_m} B_T(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2}\left(|\mathbf{v}^-|^2_{\Gamma_{\text{out}}(S_m)} - |\mathbf{v}^-|^2_{\Gamma_{\text{in}}(S_m)} + \sum_{T\in S_m} |\mathbf{v}^+ - \mathbf{v}^-|^2_{\Gamma_{\text{in}}(T)}\right).$$

Moreover, since $\mathbf{W}\cdot\mathbf{N} = 1$ on $\Gamma_{\text{out}}(S_m)$ and -1 on $\Gamma_{\text{in}}(S_m)$, the above norms over $\Gamma_{\text{out}}(S_m)$ and $\Gamma_{\text{in}}(S_m)$ are in fact unweighted.

Lemma 2.1. *The following hold in individual tetrahedrons T:*

$$\frac{1}{2}\left(|\mathbf{u}_h^-|^2_{\Gamma_{\text{out}}(T)} - |\mathbf{u}_h^-|^2_{\Gamma_{\text{in}}(T)} + |\mathbf{u}_h^+ - \mathbf{u}_h^-|^2_{\Gamma_{\text{in}}(T)}\right) + (\nabla p_h, \mathbf{u}_h)_T \le \epsilon\|\mathbf{u}_h\|_T^2 + C\epsilon^{-1}\|\mathbf{f}\|_T^2, \tag{2.2}$$

$$\|(\mathbf{u}_h)_t\|_T^2 + \|\nabla p_h\|_T^2 + 2(\nabla p_h, (\mathbf{u}_h)_t)_T \le C\left(\lambda^2 k^{-2}\|\mathbf{u}_h\|_T^2 + k^{-1}|\mathbf{u}_h^+ - \mathbf{u}_h^-|^2_{\Gamma_{\text{in}}(T)} + \|\mathbf{f}\|_T^2\right), \tag{2.3}$$

where $\epsilon > 0$ is arbitrary.

Proof.

(i) To prove (2.2), we take $\mathbf{v}_h = \mathbf{u}_h$ in (1.2), then use (2.1), and apply the Schwarz and arithmetic-geometric mean (AGM) inequalities to $(\mathbf{f}, \mathbf{u}_h)_T$.

(ii) Before proving (2.3), we first note that

$$\begin{aligned}\left|(\mathbf{w}\cdot\nabla\mathbf{u}_h, \mathbf{v}_h)_T + < \mathbf{u}_h^+ - \mathbf{u}_h^-, \mathbf{v}_h >_{\Gamma_{\text{in}}(T)}\right| &\le C\left\{h^{-1}\|w\|_{\infty,T}\|\mathbf{u}_h\|_T\|\mathbf{v}_h\|_T + |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}\ |\mathbf{v}_h|_{\Gamma_{\text{in}}(T)}\right\} \\ &\le C\left\{\lambda k^{-1}\|\mathbf{u}_h\|_T + k^{-1/2}|\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}\right\}\|\mathbf{v}_h\|_T. \end{aligned} \tag{2.4}$$

In obtaining this bound, inverse inequalities were applied to $\|\nabla\mathbf{u}_h\|_T$ and $|\mathbf{v}_h|_{\Gamma_{\text{in}}(T)}$.

We now take $\mathbf{v}_h = (\mathbf{u}_h)_t + \nabla p_h$ in (1.2) and apply (2.4) to obtain

$$\|(\mathbf{u}_h)_t + \nabla p_h\|_T^2 \le \left\{C\left(\lambda k^{-1}\|\mathbf{u}_h\|_T + k^{-1/2}|\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}\right) + \|\mathbf{f}\|_T\right\}\|(\mathbf{u}_h)_t + \nabla p_h\|_T.$$

By an appropriate application of the AGM inequality, we then get

$$\|(\mathbf{u}_h)_t + \nabla p_h\|_T^2 \le C\left(\lambda^2 k^{-2}\|\mathbf{u}_h\|_T^2 + k^{-1}|\mathbf{u}_h^+ - \mathbf{u}_h^-|^2_{\Gamma_{\text{in}}(T)} + \|\mathbf{f}\|_T^2\right),$$

which is equivalent to (2.3).

□

LEMMA 2.2. *If k is sufficiently small, then for any $\mathbf{v}_h \in \mathbf{V}_h^m$,*

$$\|\mathbf{v}_h\|_{S_m}^2 \le C'' \left(k|\mathbf{v}_h^-|_{\Gamma_{\text{in}}(S_m)}^2 + k \sum_{T \in S_m} |\mathbf{v}_h^+ - \mathbf{v}_h^-|_{\Gamma_{\text{in}}(T)}^2 + k^2 \|(\mathbf{v}_h)_t\|_{S_m}^2 \right). \tag{2.5}$$

Proof. This result follows from assumption (ii) by a scaling argument.

□

LEMMA 2.3. *For h, k, and λ sufficiently small,*

$$|\mathbf{u}_h^-|_{\Gamma_{\text{out}}(S_m)}^2 + \xi_1 \sum_{T \in S_m} |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + \xi_2 \|\mathbf{u}_h\|_{S_m}^2 + \xi_3 k^2 \|\nabla p_h\|_{S_m}^2 \le (1 + Ck)|\mathbf{u}_h^-|_{\Gamma_{\text{in}}(S_m)}^2 + C\|\mathbf{f}\|_{S_m}^2, \tag{2.6}$$

where $\xi_i, i = 1, \cdots, 4$ are positive.

Proof. We add twice (2.2) to k^2 times (2.3). Summing over $T \in S_m$, then applying (1.3) yields

$$\begin{aligned} |\mathbf{u}_h^-|_{\Gamma_{\text{out}}(S_m)}^2 - |\mathbf{u}_h^-|_{\Gamma_{\text{in}}(S_m)}^2 + \sum_{T \in S_m} |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + k^2 \|(\mathbf{u}_h)_t\|_{S_m}^2 + k^2 \|\nabla p_h\|_{S_m}^2 \\ \le (2\epsilon + C\lambda^2)\|\mathbf{u}_h\|_{S_m}^2 + C \left\{ k \sum_{T \in S_m} |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + (\epsilon^{-1} + k^2)\|\mathbf{f}\|_{S_m}^2 \right\}. \end{aligned}$$

To the above we then add $\frac{1}{C''}$ times (2.5) with $\mathbf{v}_h = \mathbf{u}_h$ to get

$$\begin{aligned} |\mathbf{u}_h^-|_{\Gamma_{\text{out}}(S_m)}^2 - |\mathbf{u}_h^-|_{\Gamma_{\text{in}}(S_m)}^2 \sum_{T \in S_m} + |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + \frac{1}{C''}\|\mathbf{u}_h\|_{S_m}^2 + k^2 \|\nabla p_h\|_{S_m}^2 \\ \le (2\epsilon + C\lambda^2)\|\mathbf{u}_h\|_{S_m}^2 + C \left\{ k|\mathbf{u}_h^-|_{\Gamma_{\text{in}}(S_m)}^2 + k \sum_{T \in S_m} |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + (\epsilon^{-1} + k^2)\|\mathbf{f}\|_{S_m}^2 \right\}. \end{aligned}$$

□

This then leads to the following global stability result.

THEOREM 2.4. *For h, k, and λ sufficiently small,*

$$|\mathbf{u}_h^-|_{\Gamma_{\text{out}}(Q)}^2 + \sum_{T \in Q} |\mathbf{u}_h^+ - \mathbf{u}_h^-|_{\Gamma_{\text{in}}(T)}^2 + \|\mathbf{u}_h\|_Q^2 + \|\nabla p_h\|_Q^2 \le C \left(|\mathbf{u}_h^-|_{\Gamma_{\text{in}}(Q)}^2 + \|\mathbf{f}\|_Q^2 \right). \tag{2.7}$$

3. Error estimates.

Our goal in this section is to prove the following error estimate:

THEOREM 3.1. *Let* $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ *and* $\epsilon = p - p_h$. *Then if* $\mathbf{u} \in [H^{n+1}(Q)]^2$, $p \in H^{n+3/2}(Q)$ *and* h, k, *and* λ *are sufficiently small,*

$$|\mathbf{e}^-|^2_{\Gamma_{\rm out}(Q)} + \|\mathbf{e}\|^2_Q + \sum_{m=0}^{M-1} \sum_{T \in \tau_h^m} |\mathbf{e}^+ - \mathbf{e}^-|^2_{\Gamma_{\rm in}(T)} + k^2 \sum_{m=0}^{M-1} \|\nabla \epsilon\|^2_{S_m} \leq C h^{2n+1} \left(\|\mathbf{u}\|^2_{n+1,Q} + \|p\|^2_{n+3/2,Q} \right).$$

Proof. From the definitions of $\mathbf{u}, p$ and $\mathbf{u}_h, p_h$ it easily follows that

$$B_T(\mathbf{w}; \mathbf{e}, \mathbf{v}) + (\nabla \epsilon, \mathbf{v})_T = 0 \qquad \text{for all } \mathbf{v} \in \mathbf{V}_h^m, \tag{3.1}$$

$$(\mathbf{e} + k^2 \mathbf{e}_t, \nabla q_h)_{S_m} = 0 \qquad \text{for all } q_h \in Q_h^m. \tag{3.2}$$

Now let $\mathbf{u}_I$ be the $[L^2]^2$ projection of $\mathbf{u}$ into $\mathbf{V}_h$ and p_I the L^2 projection of p into Q_h, and define $\mathbf{e}_h = \mathbf{u}_I - \mathbf{u}_h$, $\epsilon_h = p_I - p_h$. Using (3.1) and the definition of $\mathbf{u}_I$, we obtain

$$\begin{aligned} B_T(\mathbf{w}; \mathbf{e}, \mathbf{e}) + (\nabla \epsilon_h, \mathbf{e})_T &= B_T(\mathbf{w}; \mathbf{e}, \mathbf{u} - \mathbf{u}_I) + B_T(\mathbf{w}; \mathbf{e}, \mathbf{e}_h) \\ &\quad + (\nabla \epsilon_h, \mathbf{u} - \mathbf{u}_I)_T + (\nabla \epsilon, \mathbf{e}_h)_T + (\nabla [p_I - p], \mathbf{e}_h)_T \\ &= B_T(\mathbf{w}; \mathbf{e}, \mathbf{u} - \mathbf{u}_I) + (\nabla [p_I - p], \mathbf{e}_h)_T. \end{aligned} \tag{3.3}$$

We estimate the terms on the right side of (3.3) as follows: First, we write for any $\mathbf{v} \in \mathbf{V}_h^m$

$$\begin{aligned} B_T(\mathbf{w}; \mathbf{e}, \mathbf{u} - \mathbf{u}_I) &= (\mathbf{u}_t - [\mathbf{u}_t]_I + \mathbf{w} \cdot \nabla(\mathbf{u} - \mathbf{u}_I), \mathbf{u} - \mathbf{u}_I)_T + ([\mathbf{w} - P_0 \mathbf{w}] \cdot \nabla(\mathbf{u}_I - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_I)_T \\ &\quad + ([\mathbf{u}_t]_I - [\mathbf{u}_h]_t + P_0 \mathbf{w} \cdot \nabla(\mathbf{u}_I - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_I)_T + < \mathbf{e}^+ - \mathbf{e}^-, \mathbf{u}^+ - \mathbf{u}_I^+ >_{\Gamma_{\rm in}(T)}, \end{aligned}$$

and observe that the next to last term in the above identity is zero, since $[\mathbf{u}_t]_I - [\mathbf{u}_h]_t + P_0 \mathbf{w} \cdot \nabla(\mathbf{u}_I - \mathbf{u}_h) \in [P_n(T)]^2$, and hence is orthogonal to $\mathbf{u} - \mathbf{u}_I$. Applying standard estimates, we further obtain

$$\begin{aligned} (\mathbf{u}_t - [\mathbf{u}_t]_I + \mathbf{w} \cdot \nabla[\mathbf{u} - \mathbf{u}_I], \mathbf{u} - \mathbf{u}_I)_T &\leq (\|\mathbf{u}_t - [\mathbf{u}_t]_I\|_T + \|\mathbf{w}\|_{\infty,T} \|\nabla(\mathbf{u} - \mathbf{u}_I)\|_T) \|\mathbf{u} - \mathbf{u}_I\|_T \\ &\leq C \left(h \|\mathbf{u}_t - [\mathbf{u}_t]_I\|^2_T + h \|\nabla(\mathbf{u} - \mathbf{u}_I)\|^2_T + h^{-1} \|\mathbf{u} - \mathbf{u}_I\|^2_T \right), \end{aligned}$$

$$\begin{aligned} ([\mathbf{w} - P_0 \mathbf{w}] \cdot \nabla(\mathbf{u}_I - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_I)_T &\leq \|\mathbf{w} - P_0 \mathbf{w}\|_{\infty,T} \|\nabla(\mathbf{u}_I - \mathbf{u}_h)\|_T \|\mathbf{u} - \mathbf{u}_I\|_T \\ &\leq C \|\mathbf{w}\|_{1,\infty,T} \|\mathbf{u}_I - \mathbf{u}_h\|_T \|\mathbf{u} - \mathbf{u}_I\|_T \\ &\leq \frac{h}{2} \|\mathbf{e}_h\|^2_T + C h^{-1} \|\mathbf{u} - \mathbf{u}_I\|^2_T, \end{aligned}$$

and

$$\left| < \mathbf{e}^+ - \mathbf{e}^-, \mathbf{u}^+ - \mathbf{u}_I^+ >_{\Gamma_{\rm in}(T)} \right| \leq \frac{1}{4} |\mathbf{e}^+ - \mathbf{e}^-|^2_{\Gamma_{\rm in}(T)} + |(\mathbf{u} - \mathbf{u}_I)^+|^2_{\Gamma_{\rm in}(T)}.$$

Furthermore, for arbitrary $\delta > 0$,

$$(\nabla [p_I - p], \mathbf{e}_h)_T \leq \|\nabla [p_I - p]\|_T \|\mathbf{e}_h\|_T \leq \frac{1}{2\delta} \|\nabla [p_I - p]\|^2_T + \frac{\delta}{2} \|\mathbf{e}_h\|^2_T.$$

Combining these results, we obtain

$$\begin{aligned}(3.4)\quad B_T(\mathbf{w};\mathbf{e},\mathbf{e})+(\nabla\epsilon_h,\mathbf{e})_T \le\ & C(h\|\mathbf{u}_t-[\mathbf{u}_t]_I\|_T^2+h\|\nabla(\mathbf{u}-\mathbf{u}_I)\|_T^2+h^{-1}\|\mathbf{u}-\mathbf{u}_I\|_T^2\\ &+\frac{1}{2\delta}\|\nabla[p_I-p]\|_T^2)+\frac{h+\delta}{2}\|\mathbf{e}_h\|_T^2+\frac{1}{4}|\mathbf{e}^+-\mathbf{e}^-|^2_{\Gamma_{\rm in}(T)}+|(\mathbf{u}-\mathbf{u}_I)^+|^2_{\Gamma_{\rm in}(T)}.\end{aligned}$$

Now, we also have using (3.1) that:

$$\begin{aligned}([\mathbf{e}_h]_t,[\mathbf{e}_h]_t)_T+2(\nabla\epsilon_h,\mathbf{e}_t)_T+(\nabla\epsilon_h,\nabla\epsilon_h)_T &= ([\mathbf{e}_h]_t+\nabla\epsilon_h,[\mathbf{e}_h]_t+\nabla\epsilon_h)_T+2(\nabla\epsilon_h,[\mathbf{u}-\mathbf{u}_I]_t)_T\\ &=(\mathbf{e}_t+\nabla\epsilon,[\mathbf{e}_h]_t+\nabla\epsilon_h)_T+([\mathbf{u}_I-\mathbf{u}]_t+\nabla[p_I-p],[\mathbf{e}_h]_t+\nabla\epsilon_h)_T+2(\nabla\epsilon_h,[\mathbf{u}-\mathbf{u}_I]_t)_T\\ &=([\mathbf{u}_I-\mathbf{u}]_t+\nabla[p_I-p],[\mathbf{e}_h]_t+\nabla\epsilon_h)_T+2(\nabla\epsilon_h,[\mathbf{u}-\mathbf{u}_I]_t)_T\\ &\quad-(\mathbf{w}\cdot\nabla[\mathbf{u}-\mathbf{u}_I]+\mathbf{w}\cdot\nabla\mathbf{e}_h,[\mathbf{e}_h]_t+\nabla\epsilon_h)_T-<\mathbf{e}^+-\mathbf{e}^-,([\mathbf{e}_h]_t+\nabla\epsilon_h)^+>_{\Gamma_{\rm in}(T)}.\end{aligned}$$

Applying the Schwarz and arithmetic-geometric mean inequalities and an inverse inequality, it is not difficult to show that there exists a constant C (depending on $\mathbf{w}$) such that

$$\begin{aligned}(3.5)\quad \frac{1}{2}\|[\mathbf{e}_h]_t\|_T^2+2(\nabla\epsilon_h,\mathbf{e}_t)_T+\frac{1}{2}\|\nabla\epsilon_h\|_T^2 \le C\{&\|[\mathbf{u}-\mathbf{u}_I]_t\|_T^2+\|\nabla[p-p_I]\|_T^2+\|\nabla[\mathbf{u}-\mathbf{u}_I]\|_T^2\\ &+\lambda^2k^{-2}\|\mathbf{e}_h\|_T^2+k^{-1}|\mathbf{e}^+-\mathbf{e}^-|^2_{\Gamma_{\rm in}(T)}\}.\end{aligned}$$

To (3.4), we add $k^2/2$ times (3.5). Observing that $|\mathbf{W}\cdot\mathbf{N}|=1$ on $\Gamma_{\rm out}(S_m)$ and $\Gamma_{\rm in}(S_m)$ and after summing over all $T\in\tau_h^m$, application of (3.2) then yields

$$\begin{aligned}&\frac{1}{2}|\mathbf{e}^-|^2_{\Gamma_{\rm out}(S_m)}+\left(\frac{1}{4}-\frac{Ck}{2}\right)\sum_{T\in\tau_h^m}|\mathbf{e}^+-\mathbf{e}^-|^2_{\Gamma_{\rm in}(S_m)}+\frac{k^2}{4}\|(\mathbf{e}_h)_t\|^2_{S_m}+\frac{k^2}{4}\|\nabla\epsilon_h\|^2_{S_m}\\ &\le\frac{1}{2}|\mathbf{e}^-|^2_{\Gamma_{\rm in}(S_m)}+C\left(h\|\mathbf{u}_t-[\mathbf{u}_t]_I\|^2_{S_m}+k^2\|[\mathbf{u}-\mathbf{u}_I]_t\|^2_{S_m}\right.\\ &\quad+(h+k^2)\|\nabla(\mathbf{u}-\mathbf{u}_I)\|^2_{S_m}+h^{-1}\|\mathbf{u}-\mathbf{u}_I\|^2_{S_m}+(\delta^{-1}+k^2)\|\nabla[p_I-p]\|^2_{S_m})\\ (3.6)\qquad &\quad\left.+(\delta+h+\lambda^2)\|\mathbf{e}_h\|^2_{S_m}\right)+\sum_{T\in\tau_h^m}|(\mathbf{u}-\mathbf{u}_I)^+|^2_{\Gamma_{\rm in}(T)}.\end{aligned}$$

Now from Lemma 2, we have

$$\begin{aligned}\|\mathbf{e}_h\|^2_{S_m}&\le C''\left(k|\mathbf{e}_h^-|^2_{\Gamma_{\rm in}(S_m)}+k\sum_{T\in\tau_h^m}|\mathbf{e}_h^+-\mathbf{e}_h^-|^2_{\Gamma_{\rm in}(T)}+k^2\|(\mathbf{e}_h)_t\|^2_{S_m}\right)\\ &\le C''\left(2k|\mathbf{e}^-|^2_{\Gamma_{\rm in}(S_m)}+2k\sum_{T\in\tau_h^m}|\mathbf{e}^+-\mathbf{e}^-|^2_{\Gamma_{\rm in}(T)}+k^2\|(\mathbf{e}_h)_t\|^2_{S_m}\right.\\ (3.7)\qquad &\quad\left.+2k|\mathbf{u}^--\mathbf{u}_I^-|^2_{\Gamma_{\rm in}(S_m)}+2k\sum_{T\in\tau_h^m}|(\mathbf{u}-\mathbf{u}_I)^+-(\mathbf{u}-\mathbf{u}_I)^-|^2_{\Gamma_{\rm in}(T)}\right).\end{aligned}$$

Adding $1/(8C'')$ times (3.7) to (3.6), and combining terms, we get

$$\begin{aligned}
&\frac{1}{2}|\mathbf{e}^-|^2_{\Gamma_{\rm out}(S_m)} + \left(\frac{1}{4} - \frac{Ck}{2} - \frac{k}{4}\right) \sum_{T\in\tau_h^m} |\mathbf{e}^+ - \mathbf{e}^-|^2_{\Gamma_{\rm in}(T)} \\
&\qquad + \frac{k^2}{8}\|(\mathbf{e}_h)_t\|^2_{S_m} + \frac{k^2}{4}\|\nabla\epsilon_h\|^2_{S_m} + \left(\frac{1}{8C''} - C(\delta + h + \lambda^2)\right)\|\mathbf{e}_h\|^2_{S_m} \\
&\le \left(\frac{1}{2} + \frac{k}{4}\right)|\mathbf{e}^-|^2_{\Gamma_{\rm in}(S_m)} + C(h\|\mathbf{u}_t - [\mathbf{u}_t]_I\|^2_{S_m} + k^2\|[\mathbf{u}-\mathbf{u}_I]_t\|^2_{S_m} \\
&\qquad + (h+k^2)\|\nabla(\mathbf{u}-\mathbf{u}_I)\|^2_{S_m} + h^{-1}\|\mathbf{u}-\mathbf{u}_I\|^2_{S_m} + (\delta^{-1}+k^2)\|\nabla[p_I - p]\|^2_{S_m}) \\
&\qquad + \frac{k}{4}|\mathbf{u}^- - \mathbf{u}_I^-|^2_{\Gamma_{\rm in}(S_m)} + \frac{k}{4}\sum_{T\in\tau_h^m} |(\mathbf{u}-\mathbf{u}_I)^+ - (\mathbf{u}-\mathbf{u}_I)^-|^2_{\Gamma_{\rm in}(T)}.
\end{aligned}$$

Using standard approximation theory estimates, we further obtain for λ, δ, h, and k sufficiently small that for some constant $\xi > 0$,

$$\begin{aligned}
&|\mathbf{e}^-|^2_{\Gamma_{\rm out}(S_m)} + \xi\sum_{T\in\tau_h^m}|\mathbf{e}^+ - \mathbf{e}^-|^2_{\Gamma_{\rm in}(T)} + \xi k^2\|(\mathbf{e})_t\|^2_{S_m} + \xi k^2\|\nabla\epsilon\|^2_{S_m} + \xi\|\mathbf{e}\|^2_{S_m} \\
&\qquad\qquad \le (1+Ck)|\mathbf{e}^-|^2_{\Gamma_{\rm in}(S_m)} + Ch^{2n+1}\left(\|\mathbf{u}\|^2_{n+1,S_m} + \|p\|^2_{n+3/2,S_m}\right).
\end{aligned}$$

Iterating this inequality, we obtain

$$\begin{aligned}
&|\mathbf{e}^-|^2_{\Gamma_{\rm out}(Q)} + \xi\sum_{m=0}^{M-1}\sum_{T\in\tau_h^m}|\mathbf{e}^+ - \mathbf{e}^-|^2_{\Gamma_{\rm in}(T)} + \xi k^2\sum_{m=0}^{M-1}\|(\mathbf{e}_h)_t\|^2_{S_m} + \xi k^2\sum_{m=0}^{M-1}\|\nabla\epsilon_h\|^2_{S_m} + \xi\|\mathbf{e}_h\|^2_Q \\
&\qquad\qquad \le C|\mathbf{e}^-|^2_{\Gamma_{\rm in}(Q)} + Ch^{2n+1}\left(\|\mathbf{u}\|^2_{n+1,Q} + \|p\|^2_{n+3/2,Q}\right).
\end{aligned}$$

The result follows immediately from standard estimates for $\mathbf{e}$ on $\Gamma_{\rm in}(Q)$ and the triangle inequality.

□

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