C^m EIGENFUNCTIONS OF PERRON-FROBENIUS OPERATORS AND A NEW APPROACH TO NUMERICAL COMPUTATION OF HAUSDORFF DIMENSION

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Abstract. We develop a new approach to the computation of the Hausdorff dimension of the invariant set of an iterated function system or IFS. In the one dimensional case, our methods require only C^3 regularity of the maps in the IFS. The key idea, which has been known in varying degrees of generality for many years, is to associate to the IFS a parametrized family of positive, linear, Perron-Frobenius operators L_s. The operators L_s can typically be studied in many different Banach spaces. Here, unlike most of the literature, we study L_s in a Banach space of real-valued, C^k functions, $k \geq 2$; and we note that L_s is not compact, but has a strictly positive eigenfunction v_s with positive eigenvalue λ_s equal to the spectral radius of L_s. Under appropriate assumptions on the IFS, the Hausdorff dimension of the invariant set of the IFS is the value $s = s_*$ for which $\lambda_s = 1$. This eigenvalue problem is then approximated by a collocation method using continuous piecewise linear functions (in one dimension) or bilinear functions (in two dimensions). Using the theory of positive linear operators and explicit a priori bounds on the derivatives of the strictly positive eigenfunction v_s, we give rigorous upper and lower bounds for the Hausdorff dimension s_*, and these bounds converge to s_* as the mesh size approaches zero.

1. Introduction

Our interest in this paper is in finding rigorous estimates for the Hausdorff dimension of invariant sets for (possibly infinite) iterated function systems or IFS's. The case of graph directed IFS's (see [40] and [39]) is also of great interest and can be studied by our methods, but for simplicity we shall restrict attention here to the IFS case.

Let $D \subset \mathbb{R}^n$ be a nonempty compact set, ρ a metric on D which gives the topology on D, and $\theta_j : D \to D$, $1 \leq j \leq m$, a contraction mapping, i.e., a Lipschitz mapping (with respect to ρ) with Lipschitz constant $\mathrm{Lip}(\theta_j)$, satisfying $\mathrm{Lip}(\theta_j) := c_j < 1$. If $m < \infty$ and the above assumption holds, it is known that there exists a unique, compact, nonempty set $C \subset D$ such that $C = \cup_{j=1}^m \theta_j(C)$. The set C is called the invariant set for the IFS $\{\theta_j \,|\, 1 \leq j \leq m\}$. If $m = \infty$ and $\sup\{c_j \,|\, 1 \leq j \leq m\} = c < 1$, there is a naturally defined nonempty invariant set

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$C \subset D$ such that $C = \cup_{j=1}^{\infty} \theta_j(C)$, but C need not be compact. It is useful to note that the Lipschitz condition of the IFS can be weakened, and we address this in a subsequent section (cf. (H6.1) in Section 6).

Although we shall eventually specialize, it may be helpful to describe initially some function analytic results in the generality of the previous paragraph. Let H be a bounded, open, mildly regular subset of $\mathbb{R}^n$ and let $C^k(\bar{H})$ denote the real Banach space of C^k real-valued maps, all of whose partial derivatives of order $\nu \leq k$ extend continuously to $\bar{H}$. For a given positive integer N, assume that $b_j : \bar{H} \to (0, \infty)$ are strictly positive C^N functions for $1 \leq j \leq m < \infty$ and $\theta_j : \bar{H} \to \bar{H}$, $1 \leq j \leq m$, are C^N maps and contractions. For $s > 0$ and integers k, $0 \leq k \leq N$, one can define a bounded linear map $L_{s,k} : C^k(\bar{H}) \to C^k(\bar{H})$ by the formula

$$(L_{s,k} f)(x) = \sum_{j=1}^{m} [b_j(x)]^s f(\theta_j(x)). \tag{1.1}$$

Linear maps like $L_{s,k}$ are sometimes called positive transfer operators or Perron-Frobenius operators and arise in many contexts other than computation of Hausdorff dimension: see, for example, [1]. If $r(L_{s,k})$ denotes the spectral radius of $L_{s,k}$, then $\lambda_s = r(L_{s,k})$ is positive and independent of k for $0 \leq k \leq N$; and λ_s is an algebraically simple eigenvalue of $L_{s,k}$ with a corresponding unique, normalized strictly positive eigenfunction $v_s \in C^N(\bar{H})$. Furthermore, the map $s \mapsto \lambda_s$ is continuous. If $\sigma(L_{s,k}) \subset \mathbb{C}$ denotes the spectrum of the complexification of $L_{s,k}$, $\sigma(L_{s,k})$ depends on k, but for $1 \leq k \leq N$,

$$\sup\{|z| : z \in \sigma(L_{s,k}) \setminus \{\lambda_s\}\} < \lambda_s. \tag{1.2}$$

If $k = 0$, the strict inequality in (1.2) may fail. A more precise version of the above result in stated in Theorem 5.1 of this paper and Theorem 5.1 is a special case of results in [46]. The method of proof involves ideas from the theory of positive linear operators, particularly generalizations of the Kreĭn-Rutman theorem to noncompact linear operators; see [32], [3], [53], [44], and [37]. We do not use the thermodynamic formalism (see [49]) and often our operators cannot be studied in Banach spaces of analytic functions.

The linear operators which are relevant for the computation of Hausdorff dimension comprise a small subset of the transfer operators described in (1.1), but the analysis problem which we shall consider here can be described in the generality of (1.1) and is of interest in this more general context. We want to find rigorous methods to estimate $r(L_{s,k})$ accurately and then use these methods to estimate s_*, where, in our applications, s_* will be the unique number $s \geq 0$ such that $r(L_{s,k}) = 1$. Under further assumptions, we shall see that s_* equals $\dim_H(C)$, the Hausdorff dimension of the invariant set associated to the IFS. This observation about Hausdorff dimension has been made, in varying degrees of generality by many authors. See, for example, [6], [7], [5], [9], [10], [13], [18], [20], [22], [21], [24], [25], [26], [27], [39], [38], [47], [49], [50], [51], and [54].

In the applications in this paper, H will always be a bounded open subset of $\mathbb{R}^n$ for $n = 1$ or 2. When $n = 1$, we shall assume that H is a finite union of bounded open intervals, that $\theta_j : \bar{H} \to \bar{H}$ is a C^N contraction mapping, where $N \geq 3$, (or more generally satisfies (H6.1)) and $\theta_j'(x) \neq 0$ for all $x \in \bar{H}$. In the notation of (1.1), we define $b_j(x) = |\theta_j'(x)|$. When $n = 2$, we assume that H is a bounded, open mildly regular subset of $\mathbb{R}^2 = \mathbb{C}$ and that θ_j, $1 \le j \le m$ are analytic or conjugate analytic contraction maps (or more generally satisfy (H6.1)), defined on an open neighborhood of $\bar{H}$ and satisfying $\theta_j(H) \subset H$. We define $D\theta_j(z) = \lim_{h\to 0} |[\theta_j(z+h) - \theta_j(z)]/h|$, where $h \in \mathbb{C}$ in the limit, and we assume that $D\theta_j(z) \neq 0$ for $z \in \bar{H}$. In this case, $L_{s,k}$ is defined by (1.1), with x replaced by z, and $b_j(z) = |D\theta_j(z)|^s$.

Given the existence of a strictly positive C^N eigenfunction v_s for (1.1) when $H \subset R$, we show in Section 6 for $p = 1$ and $p = 2$, that one can obtain explicit upper and lower bounds for the quantity $D^p v_s(x)/v_s(x)$ for $x \in \bar{H}$, where D^p denotes the p-th derivative of v_s. Such bounds can also be obtained for $p = 3$ and $p = 4$, but the arguments and calculations are more complicated. When $H \subset \mathbb{R}^2$, it is also possible to obtain explicit upper and lower bounds for $D_1^p v_s(x_1, x_2))/v_s(x_1, x_2)$ and $D_2^p v_s(x_1, x_2))/v_s(x_1, x_2)$, where $D_1 = \partial/\partial x_1$ and $D_2 = \partial/\partial x_2$. However, for simplicity we restrict ourselves to the choice $\theta_j(z) = (z + \beta_j)^{-1}$, where $\beta_j \in \mathbb{C}$ and $\mathrm{Re}(\beta_j) > 0$. In this case we obtain in Section 7 explicit upper and lower bounds for $D_k^p v_s(x_1, x_2))/v_s(x_1, x_2)$ for $1 \le p \le 4$, $1 \le k \le 2$, and $x_1 > 0$. In both the one and two dimensional cases, these estimates play a crucial role in allowing us to obtain rigorous upper and lower bounds for the Hausdorff dimension.

The basic idea of our numerical scheme is to cover $\bar{H}$ by nonoverlapping intervals of length h if $H \subset R$ or by nonoverlapping squares of side h if $H \subset R^2$. We then approximate the strictly positive, C^2 eigenfunction v_s by a continuous piecewise linear function (if $H \subset \mathbb{R}$) or a continuous piecewise bilinear function (if $H \subset \mathbb{R}^2$). Using the explicit bounds on the unmixed derivatives of v_s of order 2, we are then able to associate to the operator $L_{s,k}$, square matrices A_s and B_s, which have nonnegative entries and also have the property that $r(A_s) \le \lambda_s \le r(B_s)$. A key role here is played by an elementary fact which is not as well known as it should be. If M is a nonnegative matrix and v is a strictly positive vector and $Mv \le \lambda v$, (coordinate-wise), then $r(M) \le \lambda$. An analogous statement is true if $Mv \ge \lambda v$. We emphasize that our approach is robust and allows us to study the case $H \subset \mathbb{R}$ when $\theta_j(\cdot)$, $1 \le j \le m$, is only C^3.

If s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, so that s_* is the Hausdorff dimension of the invariant set for the IFS under study, we proceed as follows. If we can find a number s_1 such that $r(B_{s_1}) \le 1$, then, since the map $s \mapsto \lambda_s$ is decreasing, $\lambda_{s_1} \le r(B_{s_1}) \le 1$, and we can conclude that $s_* \le s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \ge 1$, then $\lambda_{s_2} \ge r(A_{s_2}) \ge 1$, and we can conclude that $s_* \ge s_2$. By choosing the mesh size for our approximating piecewise polynomials to be sufficiently small, we can make $s_1 - s_2$ small, providing a good estimate for s_*. For a given s, $r(A_s)$ and $r(B_s)$ are easily found by variants of the power method for eigenvalues, since (see Section 8) the largest eigenvalue has multiplicity one and is the only eigenvalue of its modulus. When the IFS is infinite, the procedure is somewhat more complicated, and we include the necessary theory to deal with this case.

If the coefficients $b_j(\cdot)$ and the maps $\theta_j(\cdot)$ in (1.1) are C^N with $N > 2$, it is natural to approximate $v_s(\cdot)$ by piecewise polynomials of degree $N - 1$ when

$H \subset \mathbb{R}$ and by corresponding higher order approximations when $H \subset \mathbb{R}^2$. The corresponding matrices A_s and B_s may no longer have all nonnegative entries and the arguments of this paper are no longer directly applicable. However, we hope to prove in a future paper that the inequality $r(A_s) \le \lambda_s \le r(B_s)$ remains true and leads to much improved upper and lower bounds for $r(L_s)$. Heuristic evidence for this assertion is given in Table 3.2 of Section 3.2.

We illustrate our new approach by first considering in Section 3 the computation of the Hausdorff dimension of invariant sets in $[0, 1]$ arising from classical continued fraction expansions. In this much studied case, one defines $\theta_m = 1/(x + m)$, for m a positive integer and $x \in [0, 1]$; and for a subset $\mathcal{B} \subset \mathbb{N}$, one considers the IFS $\{\theta_m \,|\, m \in \mathcal{B}\}$ and seeks estimates on the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for this IFS. This problem has previously been considered by many authors. See [4], [6], [7], [18], [20], [22], [21], [25], [26], and [19]. In this case, (1.1) becomes

$$(L_{s,k}v)(x) = \sum_{m\in\mathcal{B}} \Big(\frac{1}{x+m}\Big)^{2s} v\Big(\frac{1}{x+m}\Big), \qquad 0 \le x \le 1,$$

and one seeks a value $s \ge 0$ for which $\lambda_s := r(L_{s,k}) = 1$. Table 3.1 in Section 3.2 gives upper and lower bounds for the value s such that $\lambda_s = 1$ for various sets $\mathcal{B}$. Jenkinson and Pollicott [26] use a completely different method and obtain, when $|\mathcal{B}|$ is small, high accuracy estimates for $\dim_H(C(\mathcal{B}))$, in which successive approximations converge at a super-exponential rate. It is less clear (see [25]) how well the approximation scheme in [26] or [25] works when $|\mathcal{B}|$ is moderately large or when different real analytic functions $\hat{\theta}_j : [0, 1] \to [0, 1]$ are used. Here, in the one dimensional case, we present an alternative approach with much wider applicability that only requires the maps in the IFS to be C^3. As an illustration, we consider in Section 3.3 perturbations of the IFS for the middle thirds Cantor set for which the corresponding contraction maps are C^3, but not C^4.

In Section 4, we consider the computation of the Hausdorff dimension of some invariant sets arising for complex continued fractions. Suppose that $\mathcal{B}$ is a subset of $I_1 = \{m + ni \,|\, m \in \mathbb{N}, n \in \mathbb{Z}\}$, and for each $b \in \mathcal{B}$, define $\theta_b(z) = (z + b)^{-1}$. Note that θ_b maps $\bar{G} = \{z \in \mathbb{C} \,|\, |z - 1/2| \le 1/2\}$ into itself. We are interested in the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for the IFS $\{\theta_b \,|\, b \in \mathcal{B}\}$. This is a two dimensional problem and we allow the possibility that $\mathcal{B}$ is infinite. In general (contrast work in [26] and [25]), it does not seem possible in this case to replace $L_{s,k}$, $k \ge 2$, by an operator Λ_s acting on a Banach space of analytic functions of one complex variable and satisfying $r(\Lambda_s) = r(L_{s,k})$. Instead, we work in $C^2(\bar{G})$ and apply our methods to obtain rigorous upper and lower bounds for the Hausdorff dimension $\dim_H(C(\mathcal{B}))$ for several examples. The case $\mathcal{B} = I_1$ has been of particular interest and is one motivation for this paper. In [16], Gardner and Mauldin proved that $d := \dim_H(C(I_1)) < 2$, in [38], Mauldin and Urbanski proved that $d \le 1.885$, and in [48], Priyadarshi proved that $d \ge 1.78$. In Section 4.2, we prove that $1.85550 \le d \le 1.85589$.

The square matrices A_s and B_s mentioned above and described in more detail in Section 3 have nonnegative entries and satisfy $r(A_s) \le \lambda_s \le r(B_s)$. To apply standard numerical methods, it is useful to know that all eigenvalues $\mu \ne r(A_s)$ of A_s satisfy $|\mu| < r(A_s)$ and that $r(A_s)$ has algebraic multiplicity one and that corresponding results hold for $r(B_s)$. Such results are proved in Section 8 in the one dimensional case when the mesh size, h, is sufficiently small, and a similar argument can be used in the two dimensional case. Note that this result does not follow from the standard theory of nonnegative matrices, since A_s and B_s typically have zero columns and are not primitive. We also prove that $r(A_s) \leq r(B_s) \leq (1 + C_1 h^2) r(A_s)$, where the constant C_1 can be explicitly estimated. In Section 9, we prove that the map $s \mapsto \lambda_s$ is log convex and strictly decreasing; and the same result is proved for $s \mapsto r(M_s)$, where M_s is a naturally defined matrix such that $A_s \leq M_s \leq B_s$.

Although many of the key results in the paper are described above, the paper is long and an outline summarizing the sections may be helpful. In Section 2, we recall the definition of Hausdorff dimension and present some mathematical preliminaries. In Sections 3 and 4, we present the details of our approximation scheme for Hausdorff dimension, explain the crucial role played by estimates on derivatives of order ≤ 2 of v_s, and give the aforementioned estimates for Hausdorff dimension. We emphasize that this is a feasibility study. We have limited the accuracy of our approximations to what is easily found using the standard precision of *Matlab* and have run only a limited number of examples, using mesh sizes that allow the programs to run fairly quickly. In addition, we have not attempted to exploit the special features of our problems, such as the fact that our matrices are sparse. Thus, it is clear that one could write a more efficient code that would also speed up the computations. However, the *Matlab* programs we have developed are available on the web at `www.math.rutgers.edu/~falk/hausdorff/codes.html`, and we hope other researchers will run other examples of interest to them.

The theory underlying the work in Sections 3 and 4 is deferred to Sections 5–9. In Section 5 we describe some results concerning existence of C^m positive eigenfunctions for a class of positive (in the sense of order-preserving) linear operators. We remark that Theorem 5.1 in Section 5 was only proved in [46] for finite IFS's. As a result, some care is needed in dealing with infinite IFS's: see Theorem 5.2 and Corollary 5.3. In Section 6, we derive explicit bounds on the derivatives of the eigenfunction v_s of L_s in the one-dimensional case and in Section 7, we derive explicit bounds on the derivatives of eigenfunctions of operators in which the mappings θ_β are given by Möbius transformations which map a given bounded open subset H of $\mathbb{C} := \mathbb{R}^2$ into H. In Section 8, we verify some spectral properties of the approximating matrices which justify standard numerical algorithms for computing their spectral radii. Finally, in Section 9, we show the log convexity of the spectral radius $r(L_s)$, which we exploit in our numerical approximation scheme.

2. Preliminaries

We recall the definition of the Hausdorff dimension, $\dim_H(K)$, of a subset $K \subset \mathbb{R}^N$. To do so, we first define for a given $s \geq 0$ and each set $K \subset \mathbb{R}^N$,

$$H^s_\delta(K) = \inf\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta \text{ cover of } K\},$$

where $|U|$ denotes the diameter of U and a countable collection $\{U_i\}$ of subsets of $\mathbb{R}^N$ is a δ-cover of $K \subset \mathbb{R}^N$ if $K \subset \cup_i U_i$ and $0 < |U_i| < \delta$. We then define the

s-dimensional Hausdorff measure

$$H^s(K) = \lim_{\delta \to 0+} H^s_\delta(K).$$

Finally, we define the Hausdorff dimension of K, $\dim_H(K)$, as

$$\dim_H(K) = \inf\{s : H^s(K) = 0\}.$$

We now state the main result connecting Hausdorff dimension to the spectral radius of the map defined by (1.1). To do so, we first define the concept of an *infinitesimal similitude.* Let (S, d) be a compact, perfect metric space. If $\theta : S \to S$, then θ is an infinitesimal similitude at $t \in S$ if for any sequences $(s_k)_k$ and $(t_k)_k$ with $s_k \neq t_k$ for $k \geq 1$ and $s_k \to t$, $t_k \to t$, the limit

$$\lim_{k\to\infty} \frac{d(\theta(s_k), \theta(t_k))}{d(s_k, t_k)} =: (D\theta)(t)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. Furthermore, θ is an infinitesimal similitude on S if θ is an infinitesimal similitude at t for all $t \in S$.

This concept generalizes the concept of affine linear similitudes, which are affine linear contraction maps θ satisfying for all $x, y \in \mathbb{R}^n$

$$d(\theta(x), \theta(y)) = cd(x, y), \quad c < 1.$$

In particular, the examples discussed in this paper, such as maps of the form $\theta(x) = 1/(x+m)$, with m a positive integer, are infinitesimal similitudes. More generally, if S is a compact subset of $\mathbb{R}^1$ and $\theta : S \to S$ extends to a C^1 map defined on an open neighborhood of S in $\mathbb{R}^1$, then θ is an infinitesimal similitude. If S is a compact subset of $\mathbb{R}^2 := \mathbb{C}$ and $\theta : S \to S$ extends to an analytic or conjugate analytic map defined on an open neighborhood of S in $\mathbb{C}$, θ is an infinitesimal similitude.

Theorem 2.1. *(Theorem 1.2 of* [47]*.) Let $\theta_i : S \to S$ for $1 \leq i \leq N$ be infinitesimal similitudes and assume that the map $t \mapsto (D\theta_i)(t)$ is a strictly positive Hölder continuous function on S. Assume that θ_i is a Lipschitz map with Lipschitz constant $c_i \leq c < 1$ and let C denote the unique, compact, nonempty invariant set such that*

$$C = \cup_{i=1}^N \theta_i(C).$$

Further, assume that θ_i satisfy

$$\theta_i(C) \cap \theta_j(C) = \emptyset, \text{ for } 1 \leq i, j \leq N.\ i \neq j$$

and are one-to-one on C. Then the Hausdorff dimension of C is given by the unique σ_0 such that $r(L_{\sigma_0}) = 1$.

3. Examples in one dimension

3.1. **Continued fraction Cantor sets.** We first consider the problem of computing the Hausdorff dimension of some Cantor sets arising from continued fraction expansions. More precisely, given any number $0 < x < 1$, we can consider its continued fraction expansion

$$x = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_1, a_2, a_3, \ldots \in \mathbb{N}$. We then consider the Cantor set $E_{[m_1,\ldots,m_p]}$, of all points in $[0,1]$ where we restrict the coefficients a_i to the values $m_1, \ldots, m_p$. A number of papers (e.g., [6], [7], [18], [20], [22], [26]) have considered this problem in the case of the set $E_{1,2}$, consisting of all points in $[0,1]$ for which each a_i has the value 1 or 2. In [26], a method is presented that computes this dimension to 25 decimal places. Computations are also presented in that paper and in [25] for other choices of the values $m_1, \ldots, m_p$. In [4], the Hausdorff dimension of the Cantor set $E_{2,4,6,8,10}$ is computed to three decimal places (0.517).

Corresponding to the choices of m_i, we associate contraction maps $\theta_m(x) = 1/(x+m)$. A key fact is that the Cantor sets we consider can be generated as limit points of sequences of these contraction maps. For example, the set $E_{1.2}$ can be generated using the maps $\theta_1(x) = 1/(x+1)$ and $\theta_2(x) = 1/(x+2)$ as the set of limit points of sequences $\theta_{m_1} \ldots \theta_{m_n}(0)$, for $m_1, m_2, \ldots \in \{1, 2\}$.

For $v \in C[0,1]$, we define

$$(L_s v)(x) = \sum_{j=1}^{p} \left|\theta'_{m_j}(x)\right|^s v(\theta_{m_j}(x)). \tag{3.1}$$

Our computations are based on the following result, which we shall prove in subsequent sections.

Theorem 3.1. *For all $s > 0$, L_s has a unique strictly positive eigenvector v_s with $L_s v_s = \lambda_s v_s$, where $\lambda_s > 0$ and $\lambda_s = r(L_s)$, the spectral radius of L_s. Furthermore, the map $s \mapsto \lambda_s$ is strictly decreasing and continuous, and for all $p > 0$, $(-1)^p D^{(p)} v_s(x) > 0$ for all $x \in [0,1]$ and*

$$|D^{(p)} v_s(x)| \le (2s)(2s+1)\cdots(2s+p-1)(\gamma^{-p}) v_s(x), \tag{3.2}$$

where $\gamma = \min_j m_j$. Finally, the Hausdorff dimension of the Cantor set generated from the maps θ_{m_1}, …, θ_{m_p} is the unique value of s with $\lambda_s = 1$.

Note that it follows easily from (3.2) when $p = 1$ and $x_1, x_2 \in [0,1]$, that

$$v_s(x_2) \le v_s(x_1) \exp(2s|x_2 - x_1|/\gamma). \tag{3.3}$$

To see this, write

$$\log \frac{v_s(x_2)}{v_s(x_1)} = \log v_s(x_2) - \log v_s(x_1) = \int_{x_1}^{x_2} \frac{d}{dx} \log v_s(x)\, dx = \int_{x_1}^{x_2} \frac{v_s'(x)}{v_s(x)}\, dx,$$

apply the bound in (3.2), and exponentiate the result.

To obtain approximations of the dimension of the Cantor sets described in this section, we first approximate a function $f \in C^2[0,1]$ by a continuous, piecewise linear function defined on a mesh of interval size h on $[0,1]$. More specifically, we approximate $f(x)$, $x_k \le x \le x_{k+1}$ by its piecewise linear interpolant $f^I(x)$ given by

$$f^I(x) = \frac{x_{k+1} - x}{h} f(x_k) + \frac{x - x_k}{h} f(x_{k+1}), \quad x_k \le x \le x_{k+1},$$

where the mesh points x_k satisfy $0 = x_0 < x_1, \cdots < x_N = 1$, with $x_{k+1} - x_k = h = 1/N$. The goal is to reduce the infinite dimensional eigenvalue problem to a finite dimensional one. Standard results for the error in linear interpolation on an interval $[a, b]$ assert that

$$f^I(x) - f(x) = \frac{1}{2}(b - x)(x - a) f''(\xi)$$

for some $\xi \in [a, b]$. If $x_{r_j} \le \theta_{m_j}(x) \le x_{r_j+1}$, we get

$$v_s^I(\theta_{m_j}(x)) = \frac{[x_{r_j+1} - \theta_{m_j}(x)]}{h} v_s(x_{r_j}) + \frac{[\theta_{m_j}(x) - x_{r_j}]}{h} v_s(x_{r_j+1}).$$

We can also use the properties in Theorem 3.1 to bound the interpolation error. Letting $f(x) = v_s(x)$, we obtain from Theorem 3.1 that

$$0 < v_s''(\theta_{m_j}(x)) \le 2s(2s + 1)\gamma^{-2} v_s(\theta_{m_j}(x)).$$

Using the interpolation error estimate and (3.3), we get for $x_{r_j} \le \theta_{m_j}(x) \le x_{r_j+1}$,

$$\begin{aligned} 0 &< v_s^I(\theta_{m_j}(x)) - v_s(\theta_{m_j}(x)) \\ &\le [x_{r_j+1} - \theta_{m_j}(x)][\theta_{m_j}(x) - x_{r_j}] s(2s + 1)\gamma^{-2} \max_{[x_{r_j}, x_{r_j+1}]} v_s(\xi). \\ &\le [x_{r_j+1} - \theta_{m_j}(x)][\theta_{m_j}(x) - x_{r_j}] s(2s + 1)\gamma^{-2} \exp(2sh/\gamma)\, v_s^I(\theta_{m_j}(x)), \end{aligned}$$

since the point at which the maximum occurs is within h of either of the two endpoints of the subinterval.

Using this estimate, we have precise upper and lower bounds on the error in the interval $[x_{r_j}, x_{r_j+1}]$ that only depend on the function values of v_s at x_{r_j} and x_{r_j+1}. Letting

$$\mathrm{err}_j(x) = [x_{r_j+1} - \theta_{m_j}(x)][\theta_{m_j}(x) - x_{r_j}] s(2s + 1)\gamma^{-2} \exp(2sh/\gamma),$$

we have for each mesh point x_k, with $x_{r_j} \le \theta_{m_j}(x_k) \le x_{r_j+1}$,

$$[1 - \mathrm{err}_j(x_k)] v_s^I(\theta_{m_j}(x_k)) \le v_s(\theta_{m_j}(x_k)) \le v_s^I(\theta_{m_j}(x_k)).$$

Since for each mesh point x_k, $r(L_s)v_s(x_k) = (L_s v_s)(x_k)$, we can use (3.1) and the above result to to see that

$$\begin{aligned} r(L_s)v_s(x_k) = L_s v_s(x_k) &= \sum_{j=1}^{p} \left|\theta'_{m_j}(x_k)\right|^s v_s(\theta_{m_j}(x_k)) \\ &\le \sum_{j=1}^{p} \left|\theta'_{m_j}(x_k)\right|^s v_s^I(\theta_{m_j}(x_k)) \end{aligned}$$

and

$$r(L_s)v_s(x_k) \ge \sum_{j=1}^{p} \left|\theta'_{m_j}(x_k)\right|^s [1 - \mathrm{err}_j(x_k)] v_s^I(\theta_{m_j}(x_k)).$$

Let w_s be a vector with $(w_s)_k = v_s(x_k)$, $k = 0, \dots N$. Define $(N+1) \times (N+1)$ matrices B_s and A_s by

$$(B_s w_s)_k = \sum_{j=1}^{p} \left| \theta'_{m_j}(x_k) \right|^s w_s^I(\theta_{m_j}(x_k)),$$

$$(A_s w_s)_k = \sum_{j=1}^{p} \left| \theta'_{m_j}(x_k) \right|^s [1 - \mathrm{err}_j(x_k)] w_s^I(\theta_{m_j}(x_k)),$$

where, if $x_{r_j} \le \theta_{m_j}(x) \le x_{r_j+1}$, we define

$$w_s^I(\theta_{m_j}(x)) = \frac{[x_{r_j+1} - \theta_{m_j}(x)]}{h} (w_s)_{r_j} + \frac{[\theta_{m_j}(x) - x_{r_j}]}{h} (w_s)_{r_j+1}.$$

Note that all of the entries of B_s will be nonnegative and since $\mathrm{err}_j(x) = O(h^2)$, this is true for A_s as well, provided h is sufficiently small.

Since $v_s(x_k) > 0$ for $k = 0, \dots, N$, we can apply the following result about nonnegative matrices to see that

$$r(A_s) \le r(L_s) \le r(B_s).$$

Lemma 3.2. *Let M be an $(N+1) \times (N+1)$ matrix with non-negative entries and w an $N+1$ vector with strictly positive components.*

$$\text{If } (Mw)_k \ge \lambda w_k, \quad k = 0, \dots N, \qquad \text{then } r(M) \ge \lambda,$$
$$\text{If } (Mw)_k \le \lambda w_k, \quad k = 0, \dots N, \qquad \text{then } r(M) \le \lambda.$$

Since this result is crucial to our approximation scheme, we supply the proof below to keep our presentation self-contained. Note, however, that Lemma 3.2 is actually a special case of much more general results concerning order-preserving, homogeneous cone mappings: see Lemma 2.2 in [34] and Theorem 2.2 in [36]. If we let D denote the positive diagonal $(N+1) \times (N+1)$ matrix with diagonal entries w_j, $1 \le j \le N+1$, $r(M) = r(D^{-1}MD)$; and Lemma 3.2 can also be obtained by applying Theorem 1.1 on page 24 of [41] to $D^{-1}MD$.

Proof. If $(Mw)_k \ge \lambda w_k$, $k = 0, \dots N$, it easily follows that $(M^n w)_k \ge \lambda^n w_k$ and so $\|M^n w\|_\infty \ge \lambda^n \|w\|_\infty$. Let e be vector with all $e_i = 1$. Then

$$\|M^n\|_\infty = \|M^n e\|_\infty \ge \|M^n w\|_\infty / \|w\|_\infty \ge \lambda^n.$$

Hence,

$$r(M) = \lim_{n \to \infty} \|M^n\|_\infty^{1/n} \ge \lambda.$$

If $(Mw)_k \le \lambda w_k$, $k = 0, \dots N$, it easily follows that $(M^n w)_k \le \lambda^n w_k$. Let k be chosen so that $\|M^n\|_\infty = \sum_j (M^n)_{k,j}$. Since $[r(M)]^n = r(M^n) \le \|M^n\|_\infty$,

$$\min_j w_j [r(M)]^n \le \min_j w_j \sum_j (M^n)_{k,j} \le \sum_j (M^n)_{k,j} w_j = (M^n w)_k \le \lambda^n w_k.$$

So,

$$\min_j w_j \le [\lambda / r(M)]^n w_k.$$

If $r(M) > \lambda$, then letting $n \to \infty$, we get that $\min_j w_j \le 0$, which contradicts the fact that all $w_j > 0$. Hence, $r(M) \le \lambda$. ☐

As described in Section 1, if s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, then s_* is the Hausdorff dimension of the set $E_{[m_1,\ldots,m_p]}$. If we can find a number s_1 such that $r(B_{s_1}) \leq 1$, then $r(L_{s_1}) \leq r(B_{s_1}) \leq 1$, and we can conclude that $s_* \leq s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \geq 1$, then $r(L_{s_2}) \geq r(A_{s_2}) \geq 1$, and we can conclude that $s_* \geq s_2$. By choosing the mesh sufficiently fine, we can make $s_1 - s_2$ small, providing a good estimate for s_*.

We can also reduce the number of computations by first iterating the maps θ_{m_i} to produce a smaller initial domain that we need to approximate. For example, if we seek the Hausdorff dimension of the set $E_{1,2}$, since $\theta_1([0,1]) = [1/2,1]$ and $\theta_2([0,1]) = [1/3,1/2]$, the maps θ_1 and θ_2 map $[1/3,1] \mapsto [1/3,1]$, so we can restrict the problem to this subinterval. Further iterating, we see that $\theta_1([1/3,1]) = [1/2,3/4]$ and $\theta_2([1/3,1]) = [1/3,3/7]$. Hence the maps θ_1 and θ_2 map $[1/3,3/7] \cup [1/2,3/4]$ to itself and we can further restrict the problem to this domain.

3.2. **Continued fraction Cantor sets – numerical results.** In this section, we report in Table 3.1 the results of the application of the algorithm described above to the computation of the Hausdorff dimension of a sample of continued fraction Cantor sets. Where the true value was known to sufficient accuracy, it is not hard to check that the rate of convergence as h is refined is $O(h^2)$. Although the theory developed above does not apply to higher order piecewise polynomial approximation, since one cannot guarantee that the approximate matrices have nonnegative entries, we also report in Table 3.2 and Table 3.3 the results of higher order piecewise polynomial approximation to demonstrate the promise of this approach. In this case, we only provide the results for B_s, which does not contain any corrections for the interpolation error. In a future paper we hope to prove that rigorous upper and lower bounds for the Hausdorff dimension can also be obtained when higher order piecewise polynomial approximations are used.

The errors are computed based on the results reported in [26]. For the last five entries, we do not have independent results for the true solution correct to a sufficient number of decimal places to compute the error.

In the computations shown using higher order piecewise polynomials, since the number of unknowns for a continuous, piecewise polynomial of degree k on N uniformly spaced subintervals of width h is given by $kN+1$, to get a fair comparison, we have adjusted the mesh sizes so that each computation involves the same number of unknowns. For this problem, the eigenfunction v_s is smooth and the computations show a dramatic increase in the accuracy of the approximation as the degree of the approximating piecewise polynomial is increased.

3.3. **An example with less regularity.** For $0 \leq \lambda \leq 1$, we consider the maps

$$\theta_1(x) = \frac{1}{3+2\lambda}(x + \lambda x^{7/2}), \qquad \theta_2(x) = \frac{1}{3+2\lambda}(x + \lambda x^{7/2}) + \frac{2+\lambda}{3+2\lambda}, \tag{3.4}$$

which map the unit interval to itself. Both these maps $\in C^3([0,1]$, but $\notin C^4([0,1]$. We note that because of the lack of regularity, the methods of [26] and [25] cannot be applied. When $\lambda = 0$, these maps become

$$\theta_1(x) = \frac{x}{3}, \qquad \theta_2(x) = \frac{x}{3} + \frac{2}{3},$$

TABLE 3.1. Computation of Hausdorff dimension s of some continued fraction Cantor sets.

Set	h	lower s	upper s	lower err	upper err
E[1,2]	.0001	0.531280505099895	0.531280506539767	1.18e-09	2.63e-10
	.00005	0.531280505981423	0.531280506343388	2.96e-10	6.62e-11
E[1,3]	.0001	0.454489076859422	0.454489077843624	8.02e-10	1.82e-10
	.00005	0.454489077459035	0.454489077707546	2.03e-10	4.57e-11
E[1,4]	.0001	0.411182724095752	0.411182724934834	6.79e-10	1.60e-10
	.00005	0.411182724603313	0.411182724815117	1.71e-10	4.03e-11
E[2,3]	.0001	0.337436780744847	0.337436780851139	6.12e-11	4.51e-11
	.00005	0.337436780790228	0.337436780817793	1.58e-11	1.17e-11
E[2,4]	.0001	0.306312767993699	0.306312768092506	5.91e-11	3.97e-11
	.00005	0.306312768039239	0.306312768061760	1.35e-11	8.98e-12
E[3,4]	.0001	0.263737482885901	0.263737482913807	1.15e-11	1.64e-11
	.00005	0.263737482894486	0.263737482901574	2.94e-12	4.15e-12
E[10,11]	.0002	0.146921235390446	0.146921235393309	3.37e-13	2.53e-12
	.00005	0.146921235390764	0.146921235390925	1.95e-14	1.42e-13
E[100,10000]	.0004	0.052246592638657	0.052246592638662	1.88e-15	3.12e-15
	.0001	0.052246592638659	0.052246592638659	1.25e-16	1.25e-16
E[2,4,6,8,10]	.0001	0.517357030830725	0.517357030987649		
	.00005	0.517357030911231	0.517357030949266		
E[1,. . . ,10]	.0001	0.925737589218857	0.925737591547918		
	.00005	0.925737590664670	0.925737591246997		
E[1,3, 5, . . . , 33]	.0001	0.770516007582087	0.770516008987138		
	.00005	0.770516008433225	0.770516008784885		
E[2, 4, 6, . . . , 34]	.0001	0.633471970121772	0.633471970288076		
	.00005	0.633471970211609	0.633471970252711		
E[1, . . . ,34]	.0001	0.980419623378987	0.980419625624112		
	.00005	0.980419624765058	0.980419625326256		

TABLE 3.2. Computation of Hausdorff dimension s of E[1,2] using higher order piecewise polynomials.

degree	h	s	error
1	.01	0.531282991861209	2.49 e-06
2	.02	0.531280509905739	3.63 e-09
4	.04	0.531280506277708	5.03 e-13
5	.05	0.531280506277197	7.99 e-15

and the corresponding Cantor set has Hausdorff dimension $\ln 2/\ln 3$ $\approx 0.630929753571458$.

Our computations, shown in Table 3.4, are based on the following result, which we shall prove in subsequent sections.

Theorem 3.3. *Let*

$$(L_s v)(x) = \sum_{j=1}^{2} |\theta_j'(x)|^s v(\theta_j(x)),$$

where θ_1 and θ_2 are given by (3.4). For all $s > 0$, L_s has a unique (up to normalization) strictly positive C^2 eigenvector v_s with $L_s v_s = \lambda_s v_s$, where $\lambda_s > 0$ and

Table 3.3. Computation of Hausdorff dimension s of E[2,4,6,8,10] using piecewise cubic polynomials.

h	s
0.1	0.517357031893604
.05	0.517357031040156
.02	0.517357030941730
.01	0.517357030937108
.005	0.517357030937029
.002	0.517357030937018
.001	0.517357030937018

$\lambda_s = r(L_s)$, the spectral radius of L_s. Furthermore, the map $s \mapsto \lambda_s$ is strictly decreasing and continuous, and for all $x_1, x_2 \in [0,1]$, we have the estimate

$$0 < \frac{D^2 v_s(x)}{v_s(x)} \le s^2 [C_1(\lambda)]^2 \Big(\frac{6+4\lambda}{4-3\lambda}\Big) + s\frac{(6+4\lambda)^2}{(4-3\lambda)(8+11\lambda)}\left[C_2(\lambda) + C_1(\lambda) M_0(\lambda)\frac{(6+4\lambda)}{(4-3\lambda)}\right],$$

where C_1, C_2, and M_0 are defined by (6.6), (6.23), *and* (6.14), *respectively. Finally, the Hausdorff dimension of the Cantor set generated from the maps θ_1 and θ_2 is the unique value of s with $\lambda_s = r(L_s) = 1$.*

Table 3.4. Computation of Hausdorff dimension s of less regular examples.

λ	h	lower s	upper s	upper s - lower s
0.0	.0001	0.630929753571458	0.630929753571458	0
0.25	.0001	0.691029102085966	0.691029110502743	$8.4168e-09$
0.5	.0001	0.733474587362570	0.733474622222681	$3.4860e-08$
0.75	.0001	0.767207161950980	0.767207292955634	$1.3100e-07$
1.0	.0001	0.796727161816835	0.796727861914653	$7.0010e-07$

4. Examples in two dimensions

4.1. **The problems.** Let $H = \{(x,y) \in \mathbb{R}^2 : (x-1/2)^2 + y^2 \le 1/4, y \ge 0\}$. Writing $z = x + iy$, we can consider H as a subset of the complex plane.

Let $C_{\mathbb{R}}(H)$ denote the Banach space of real-valued, continuous functions $f : H \to \mathbb{R}$ in the sup norm. Let $I_1 = \{b = m + ni : m \in \mathbb{N}, n \in \mathbb{Z}\}$ and for $b \in I_1$ and $z \in \mathbb{C}$, let $\theta_b(z) = 1/(z+b)$. If $D = \{z \in \mathbb{C} : |z - 1/2| \le 1/2\}$, it is known that for $b \in I_1$, $\theta_b(D) \subset D$ and $\theta_{b_1}(D \setminus \{0\}) \cap \theta_{b_2}(D \setminus \{0\}) = \emptyset$ for $b_1, b_2 \in I_1$, $b_1 \ne b_2$. Clearly, $\theta_b(D) \subset D \setminus \{0\}$ for $b \in I_1$. If we identify H with $\{z \in D : \mathrm{Im}(z) \ge 0\}$, and if $b \in I_1$ and $\mathrm{Im}(b) \ge 0$, $\theta_b(H) \subset \{z \in D : \mathrm{Im}(z) \le 0\}$. Hence $\overline{1/(z+b)} \in H$ if $z \in H$, $b \in I_1$, and $\mathrm{Im}(b) \ge 0$. If $z \in H$, $b \in I_1$, and $\mathrm{Im}(b) < 0$, one can show that $\theta_b(z) \in \{z \in D : \mathrm{Im}(z) > 0\}$.

Let $C_{\mathbb{R}}(D)$ denote the Banach space of real-valued, continuous functions $v : D \to \mathbb{R}$ and let $\mathcal{B}$ denote a subset of I_1. If $\mathcal{B}$ is a finite set and $s \geq 0$, one can define a bounded linear map $L_s : C_{\mathbb{R}}(D) \to C_{\mathbb{R}}(D)$ by

$$(L_s v)(z) = \sum_{b \in \mathcal{B}} \Big| \frac{d}{dz} \theta_b(z) \Big|^s v(\theta_b(z)) = \sum_{b \in \mathcal{B}} \frac{v(\theta_b(z))}{|z+b|^{2s}}. \tag{4.1}$$

If $\mathcal{B}$ is infinite, one can prove (see Section 5 of [42]) that if, for some $s > 0$, the infinite series $\sum_{b \in \mathcal{B}} [1/|z+b|^{2s}]$ converges for some $z \in D$, then it converges for all $z \in D$ and $z \mapsto [1/|z+b|^{2s}]$ is a continuous function on D. It then follows with the aid of Dini's theorem that L_s given by (4.1) defines a bounded linear map of $C_{\mathbb{R}}(D) \to C_{\mathbb{R}}(D)$.

If we define $\sigma = \sigma(\mathcal{B}) := \inf\{s > 0 \,|\, \exists z \in D \text{ such that } \sum_{b \in \mathcal{B}} [1/|z+b|^{2s}] < \infty\}$, it follows from the above remarks that for all $s > \sigma(\mathcal{B})$, L_s defined by (4.1) gives a bounded linear map of $C_{\mathbb{R}}(D) \to C_{\mathbb{R}}(D)$. If $s = \sigma$, it may or may not happen that $\sum_{b \in \mathcal{B}} [1/|z+b|^{2s}] < \infty$ for some $z \in D$. In any event, it is not hard to prove that if $s > 1$, $\sum_{b \in \mathcal{B}} [1/|z+b|^{2s}] < \infty$ for all $z \in D$.

Our computational results are based on the following theorems, which are special cases of results which we shall prove in subsequent sections of the paper.

Theorem 4.1. *Let $\mathcal{B}$ be a subset of I_1, and for $s > \sigma(\mathcal{B}) = \sigma$, let $L_s : C_{\mathbb{R}}(D) \to C_{\mathbb{R}}(D)$ be defined by (4.1). For each $s > \sigma(\mathcal{B})$, there exists a unique (to within scalar multiples) strictly positive Lipschitz eigenvector v_s of L_s, i.e., $L_s v_s = \lambda_s v_s$, where $\lambda_s > 0$ and $\lambda_s = r(L_s)$, the spectral radius of L_s defined by $r(L_s) := \lim_{k \to \infty} \|L_s^k\|^{1/k}$. If $\bar{\mathcal{B}} := \{\bar{b} \,|\, b \in \mathcal{B}\}$, then $v_s(\bar{z}) = v_s(z)$ for all $z \in D$. If $\mathcal{B}$ is finite, $v_s(x, y)$ is C^∞ on D and $x \mapsto v_x(x, y)$ is decreasing for $(x, y) \in D$.*

If $\mathcal{B} \subset I_1$, let $\mathcal{B}_\infty = \{\omega = (b_1, \ldots, b_k, \ldots) \,|\, b_j \in \mathcal{B} \ \forall j \geq 1\}$. Given $z \in D$ and $\omega(b_1, \ldots, b_k, \ldots) \in \mathcal{B}_\infty$, one can prove that $\lim_{k \to \infty} (\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k})(z) := \pi(\omega) \in D$ exists and is independent of z. Define $K = \{\pi(\omega) \,|\, \omega \in \mathcal{B}_\infty\}$. It is not hard to prove that $K = \cup_{b \in \mathcal{B}} \theta_b(K)$. In general K is not compact, but if $\mathcal{B}$ is finite, K is compact and is the unique compact, nonempty set K such that $K = \cup_{b \in \mathcal{B}} \theta_b(K)$. We shall call K the invariant set associated to $\mathcal{B}$.

Theorem 4.2. *Let $\mathcal{B}$ be a subset of I_1 and let K be the invariant set associated to $\mathcal{B}$. The Hausdorff dimension s_* of K is given by $s_* = \inf\{s > 0 \,|\, r(L_s) = \lambda_s < 1\}$ and $r(L_{s_*}) = 1$ if $\mathcal{B}$ is finite or L_{s_*} is defined. The map $s \mapsto \lambda_s$, $s > 1$, is a continuous, strictly decreasing function for $s > \sigma(\mathcal{B})$.*

In all examples which we shall consider, L_s is a bounded linear map of $C_{\mathbb{R}}(D) \to C_{\mathbb{R}}(D)$ for $s = s_*$ and $r(L_{s_*}) = 1$.

Theorems 4.1 and 4.2 essentially reduce the problem of estimating the Hausdorff dimension of the invariant set K for $\mathcal{B} \in I_1$ to the problem of estimating the value of s for which $r(L_s) = 1$. If $\bar{\mathcal{B}} = \mathcal{B}$ and if we use the fact that $v_s(\bar{z}) = v_s(z)$ for $z \in H$, we find that

$$\lambda_s v_s(z) = \sum_{\substack{b \in \mathcal{B}, |b| \leq R \\ \mathrm{Im}(b) \geq 0}} \frac{1}{|z+b|^{2s}} v_s(1/(\bar{z} + \bar{b})) \tag{4.2}$$

$$+ \sum_{\substack{b\in\mathcal{B},|b|\le R\\ \mathrm{Im}(b)<0}} \frac{1}{|z+b|^{2s}} v_s(1/(z+b)) + \sum_{b\in\mathcal{B},|b|>R} \frac{1}{|z+b|^{2s}} v_s(1/(z+b)).$$

If $\mathcal{B} = I_1$, it was stated in [38] that the Hausdorff dimension of the invariant set K is ≤ 1.885 and in [48], it was shown that the Hausdorff dimension of the set K is ≥ 1.78. We shall give much sharper estimates below. We shall also give estimates for the Hausdorff dimension of the invariant set of $\mathcal{B} \subset I_1$, for some other choices of $\mathcal{B}$, e.g., $\mathcal{B} = I_2 := \{b = m + ni : m \in \mathbb{N}, n \in \mathbb{N} \cup 0\}$ and $\mathcal{B} = I_3 := \{b = m + ni : m \in \{1, 2\}\}, n \in \{0, \pm 1, \pm 2\}\}$.

4.2. **Numerical Method.** For an integer $N > 0$, we define a mesh domain $D_h \supset D$, consisting of squares of sides $h = 1/N$. D_h is chosen to have the property that if $(x, y) \in D$, then the four corners of the mesh square of side h containing (x, y) are mesh points in D_h. Although we could simply choose D_h to be the rectangle $[0, 1] \times [0, 1/2]$, that choice would add unknowns we do not use. We also note that in the case $\mathcal{B} = I_3$, there is a smaller domain $E \subset D$ such that $\theta_b(E) \subset E \setminus \{0\}$ and although we have not done so, we could have reduced the size of the approximate problem by using a mesh domain $E_h \supset E$.

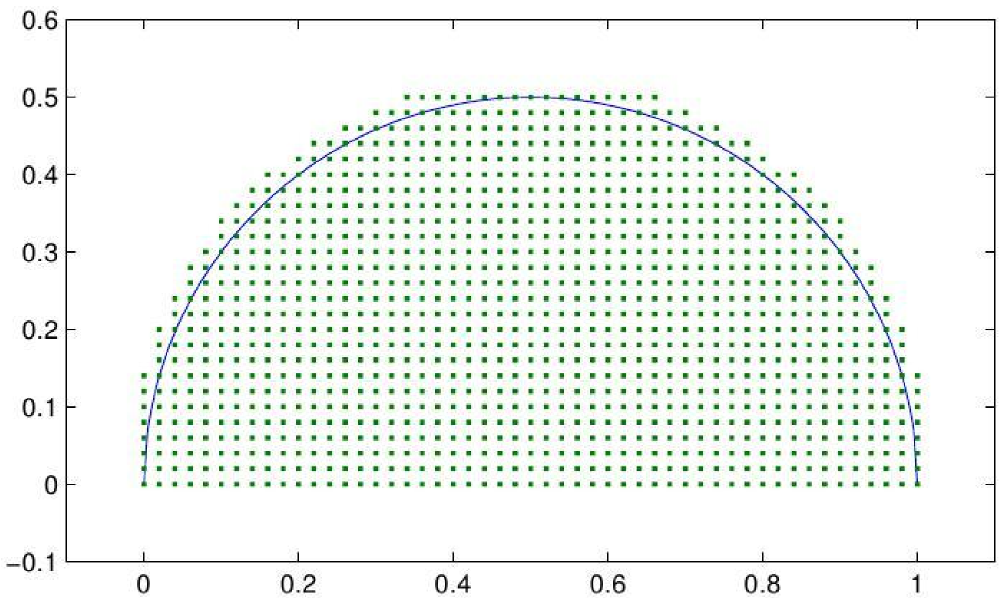

FIGURE 4.1. Domain D and mesh domain D_h

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We then approximate the function v_s by a piecewise bilinear function defined on the mesh D_h so that we can approximate the infinite dimensional eigenvalue problem by a finite dimensional one. In order to obtain a finite dimensional problem, we also need to restrict the range of b to the set $|b| \le R$ for a suitably chosen value of R for which the error in restricting the sum can be given a precise bound which is sufficiently small.

More precisely, our goal is to again define matrices A_s and B_s such that

$$r(A_s) \leq r(L_s) \leq r(B_s), \quad s > 1.$$

We then use the same procedure as for the one-dimensional problems. If s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, then s_* is the Hausdorff dimension of the set K. If we can find a number s_1 such that $r(B_{s_1}) \leq 1$, then $r(L_{s_1}) \leq r(B_{s_1}) \leq 1$, and we can conclude that $s_* \leq s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \geq 1$, then $r(L_{s_2}) \geq r(A_{s_2}) \geq 1$, and we can conclude that $s_* \geq s_2$. By choosing the mesh sufficiently fine, we can make $s_1 - s_2$ small, providing a good estimate for s_*.

We next describe how to construct the matrices A_s and B_s, once we have defined the mesh D_h. To do this, we use the following results (proved in Section 7).

$$v_s(z_1) \leq v_s(z_2) \exp(\sqrt{5} s |z_1 - z_2|), \quad z_1, z_2 \in D, \tag{4.3}$$

$$-\frac{s}{4\gamma^2(s+1)} v_s(x,y) \leq D_{xx} v_s(x,y) \leq \frac{2s(2s+1)}{\gamma^2} v_s(x,y), \tag{4.4}$$

$$-\frac{2s}{\gamma^2} v_s(x,y) \leq D_{yy} v_s(x,y) \leq \frac{2s(2s+1)}{4\gamma^2} v_s(x,y). \tag{4.5}$$

Here we suppose that $v_s(z)$ is as in (4.2) and that $\operatorname{Re}(b) \geq \gamma > 0$ for all $v \in \mathcal{B}$.

We also use some standard results about bilinear interpolation. On the mesh square

$$R_{k,l} = \{(x,y) : x_k \leq x \leq x_{k+1}, y_l \leq y \leq y_{l+1}\},$$

where $x_{k+1} - x_k = y_{l+1} - y_l = h$, the bilinear interpolant $f^I(x,y)$ of a function $f(x,y)$ is given by:

$$\begin{aligned} f^I(x,y) = &\Big[\frac{x_{k+1} - x}{h}\Big]\Big[\frac{y_{l+1} - y}{h}\Big] f(x_k, y_l) + \Big[\frac{x - x_k}{h}\Big]\Big[\frac{y_{l+1} - y}{h}\Big] f(x_{k+1}, y_l) \\ &+ \Big[\frac{x_{k+1} - x}{h}\Big]\Big[\frac{y - y_l}{h}\Big] f(x_k, y_{l+1}) + \Big[\frac{x - x_k}{h}\Big]\Big[\frac{y - y_l}{h}\Big] f(x_{k+1}, y_{l+1}). \end{aligned}$$

The error in bilinear interpolation satisfies for all $(x,y) \in R_{k,l}$ and some points (a_k, b_l) and $(c_k, d_l) \in R_{k,l}$,

$$\begin{aligned} f^I(x,y) - f(x,y) = 1/2)\Big[&(x_{k+1} - x)(x - x_k)(D_{xx} f)(a_k, b_l) \\ &+ (y_{l+1} - y)(y - y_l)(D_{yy} f)(c_k, d_l)\Big]. \end{aligned}$$

For $z = x + iy$, let $f(x,y) = v_s(\theta_b(z))$. Further let $z_{k,l} = x_k + i y_l$. If $(\tilde{x}, \tilde{y}) = (\operatorname{Re}\theta_b(z), \operatorname{Im}\theta_b(z)) \in R_{k,l}$, (which we will sometimes abbreviate by $\theta_b(z) \in R_{k,l}$), we get

$$\begin{aligned} v_s^I(\theta_b(z)) = &\Big[\frac{x_{k+1} - \tilde{x}}{h}\Big]\Big[\frac{y_{l+1} - \tilde{y}}{h}\Big] v_s(z_{k,l}) + \Big[\frac{\tilde{x} - x_k}{h}\Big]\Big[\frac{y_{l+1} - \tilde{y}}{h}\Big] v_s(z_{k+1,l}) \\ &+ \Big[\frac{x_{k+1} - \tilde{x}}{h}\Big]\Big[\frac{\tilde{y} - y_l}{h}\Big] v_s(z_{k,l+1}) + \Big[\frac{\tilde{x} - x_k}{h}\Big]\Big[\frac{\tilde{y} - y_l}{h}\Big] v_s(z_{k+1,l+1}). \end{aligned}$$

Defining

$$\Psi_b(z) = 1/(\bar{z} + \bar{b}),$$

we have an analogous formula for $v_s^I(\Psi_b(z))$, with $(\tilde{x}, \tilde{y}) = (\operatorname{Re}\Psi_b(z), \operatorname{Im}\Psi_b(z))$.

We next use inequalities (4.3), (4.4), and (4.5) to obtain bounds on the interpolation error. By (4.4) and (4.5), we get for $\theta_b(z) = \tilde{x} + i\tilde{y}$ where $(\tilde{x}, \tilde{y}) \in R_{k,l}$,

$$\begin{aligned}
-\left[\frac{s}{8\gamma^2(s+1)} + \frac{s}{\gamma^2}\right]&([x_{k+1} - \tilde{x}][\tilde{x} - x_k]v_s(a_k, b_l) + [y_{l+1} - \tilde{y}][\tilde{y} - y_l]v_s(c_k, d_l)) \\
&\le v_s^I(\theta_b(z)) - v_s(\theta_b(z)) \\
&\le \frac{s(2s+1)}{\gamma^2}([x_{k+1} - \tilde{x}][\tilde{x} - x_k]v_s(a_k, b_l) + [y_{l+1} - \tilde{y}][\tilde{y} - y_l]v_s(c_k, d_l)).
\end{aligned}$$

Applying (4.3), we then obtain

$$\begin{aligned}
-\left[\frac{s}{8\gamma^2(s+1)} + \frac{s}{\gamma^2}\right]&([x_{k+1} - \tilde{x}][\tilde{x} - x_k] + [y_{l+1} - \tilde{y}][\tilde{y} - y_l])\exp(\sqrt{10}sh)v_s^I(\theta_b(z)) \\
&\le v_s^I(\theta_b(z)) - v_s(\theta_b(z)) \\
&\le \frac{s(2s+1)}{\gamma^2}([x_{k+1} - \tilde{x}][\tilde{x} - x_k] + [y_{l+1} - \tilde{y}][\tilde{y} - y_l])\exp(\sqrt{10}sh)v_s^I(\theta_b(z)).
\end{aligned}$$

since any point in $R_{k,l}$ is within $\sqrt{2}h$ of each of the four corners of the square $R_{k,l}$. An analogous result holds for $v_s(\Psi_b(z))$.

Using this estimate, we have precise upper and lower bounds on the error in the mesh square $R_{k,l}$ that only depend on the function values of v_s at the four corners of the square and the value of b. Letting

$$\begin{aligned}
\operatorname{err}_b^1(\theta_b(z)) &= \Big([x_{k+1} - \tilde{x}][\tilde{x} - x_k] + [y_{l+1} - \tilde{y}][\tilde{y} - y_l]\Big)\frac{s(2s+1)}{\gamma^2}\exp(\sqrt{10}sh), \\
\operatorname{err}_b^2(\theta_b(z)) &= \Big([x_{k+1} - \tilde{x}][\tilde{x} - x_k] + [y_{l+1} - \tilde{y}][\tilde{y} - y_l]\Big)\frac{s}{\gamma^2}\left[\frac{9+8s}{8+8s}\right]\exp(\sqrt{10}sh),
\end{aligned}$$

(where again $\theta_b(z) = \tilde{x} + i\tilde{y}$), we have for each mesh point $z_{i,j} = x_i + iy_j$, with $\theta_b(z_{i,j}) \in R_{k,l}$,

$$[1 - \operatorname{err}_b^1(z_{i,j})]v_s^I(\theta_b(z_{i,j})) \le v_s(\theta_b(z_{i,j})) \le [1 + \operatorname{err}_b^2(z_{i,j})]v_s^I(\theta_b(z_{i,j})).$$

Again, the analogous result holds for $v_s(\Psi_b(z))$. Before using this result as in the one dimensional examples to find upper and lower matrices that can be used to find upper and lower bounds on the Hausdorff dimension of the set K, we must first deal with the final expression in (4.2) where the sum is taken over $|b| > R$.

Lemma 4.3. *For $s > 1$, we have*

$$\begin{aligned}
\sum_{b\in I_1, |b|>R} \frac{1}{|z+b|^{2s}}v_s(\theta_b(z)) &\le \exp\Big(\frac{2s}{\sqrt{R^2 - R}}\Big)\Big(\frac{R}{R-1}\Big)^s \\
&\quad\cdot\left[\Big(\frac{1}{2s-1}\Big)\Big(\frac{1}{R-1}\Big)^{2s-1} + \Big(\frac{\pi}{2}\Big)\Big(\frac{1}{s-1}\Big)\Big(\frac{1}{R-\sqrt{2}}\Big)^{2s-2}\right]v_s(0).
\end{aligned}$$

$$\sum_{b\in I_2, |b|>R} \frac{1}{|z+b|^{2s}}v_s(\theta_b(z)) \le \exp\Big(\frac{2s}{\sqrt{R^2 - R}}\Big)\Big(\frac{R}{R-1}\Big)^s$$

$$\cdot\left[\Big(\frac{1}{2s-1}\Big)\Big(\frac{1}{R-1}\Big)^{2s-1}+\Big(\frac{\pi}{4}\Big)\Big(\frac{1}{s-1}\Big)\Big(\frac{1}{R-\sqrt{2}}\Big)^{2s-2}\right]v_s(0).$$

Proof. Using (4.3), we have

$$v_s(\theta_b(z))\le \exp(2s|\theta_b(z)|)v_s(0).$$

Now for $z=x+iy\in D$ and $b=m+in\in I_1$, we have

$$\begin{aligned}\min_{(x-1/2)^2+y^2\le 1/4}(x+m)^2+(y+n)^2 &\ge \min_{0\le x\le 1}(x+m)^2+\min_{|y|\le 1/2}(y+n)^2\\ &\ge m^2+(|n|-1/2)^2\ge m^2+n^2-|n|.\end{aligned}$$

Hence, for $z\in D$,

$$\frac{1}{|z+b|^2}=\frac{1}{(x+m)^2+(y+n)^2}\le\frac{1}{m^2+n^2-|n|}.$$

Also, it is easy to check that if $m^2+n^2\ge R^2>1$,

$$\frac{1}{m^2+n^2-|n|}\le\frac{R}{R-1}\frac{1}{m^2+n^2}\le\frac{1}{R^2-R}.$$

Hence, for $m^2+n^2\ge R^2>1$ and $z\in D$,

$$\exp(2s|\theta_b(z)|)\le\exp\Big(\frac{2s}{\sqrt{m^2+n^2-|n|}}\Big)\le\exp\Big(\frac{2s}{\sqrt{R^2-R}}\Big).$$

It follows that

$$\begin{aligned}&\sum_{b\in I_1,|b|>R}\frac{1}{|z+b|^{2s}}\exp(2s\theta_b(z))\\ &\qquad\le\exp\Big(\frac{2s}{\sqrt{R^2-R}}\Big)\Big(\frac{R}{R-1}\Big)^s\sum_{b\in I_1,|b|>R}\Big(\frac{1}{m^2+n^2}\Big)^s.\end{aligned}$$

Now for $n=0$ and $m\ge R$,

$$\sum_{m\ge R}\frac{1}{m^{2s}}\le\int_{R-1}^{\infty}\frac{1}{r^{2s}}\,ds=\frac{1}{2s-1}\Big(\frac{1}{R-1}\Big)^{2s-1}.$$

For $b=m+in\in I_1$ with $m\ge 1$, $n\ge 1$, and $|b|\ge R$, let

$$B(m,n)=\{(\xi,\eta): m\le\xi\le m+1, n\le\eta\le n+1\}.$$

Then for $(u,v)\in B(m,n)$,

$$\frac{1}{(u-1)^2+(v-1)^2}\ge\frac{1}{m^2+n^2}.$$

Also,

$$\begin{aligned}&(u-1)^2+(v-1)^2\ge(m-1)^2+(n-1)^2=m^2+n^2-2(m+n)+2\\ &\qquad\ge m^2+n^2-2\sqrt{2}\sqrt{m^2+n^2}+2=(\sqrt{m^2+n^2}-\sqrt{2})^2\ge(R-\sqrt{2})^2\equiv R_1^2.\end{aligned}$$

Hence,

$$\sum_{\substack{m\ge1,n\ge1\\ m^2+n^2>R^2}}\Big(\frac{1}{m^2+n^2}\Big)^s\le\sum_{\substack{m\ge1,n\ge1\\ m^2+n^2>R^2}}\iint_{B(m,n)}\Big(\frac{1}{(u-1)^2+(v-1)^2}\Big)^s du\,dv$$

$$\leq \iint_{\substack{u\geq 0, v\geq 0\\ u^2+v^2\geq R_1^2}} \Big(\frac{1}{u^2+v^2}\Big)^s \,du\,dv = \frac{\pi}{2}\int_{R_1}^\infty \frac{1}{r^{2s}} r\,dr = \frac{\pi}{2}\frac{r^{2-2s}}{2-2s}\Big|_{R_1}^\infty$$

$$= \frac{\pi}{2}\frac{1}{2s-2}\frac{1}{R_1^{2s-2}} = \frac{\pi}{4}\frac{1}{s-1}\left(\frac{1}{R-\sqrt{2}}\right)^{2s-2}.$$

A similar argument shows that

$$\sum_{\substack{m\geq 1, n\leq -1\\ m^2+n^2>R^2}} \Big(\frac{1}{m^2+n^2}\Big)^s \leq \frac{\pi}{4}\frac{1}{s-1}\Big(\frac{1}{R-\sqrt{2}}\Big)^{2s-2}. \tag{4.6}$$

Combining these estimates, we obtain

$$\sum_{b\in I_1, |b|>R} \frac{1}{|z+b|^{2s}} \exp(2s\theta_b(z)) \leq \exp\Big(\frac{2s}{\sqrt{R^2-R}}\Big)\Big(\frac{R}{R-1}\Big)^s$$

$$\cdot\left[\frac{1}{2s-1}\Big(\frac{1}{R-1}\Big)^{2s-1} + \frac{\pi}{2}\frac{1}{s-1}\Big(\frac{1}{R-\sqrt{2}}\Big)^{2s-2}\right] =: c_{R,s}.$$

and a similar estimate for the sum over I_2, where the factor $\pi/2$ is replaced by $\pi/4$, since we no longer include the bound in (4.6). The lemma follows immediately. □

For $s = 1.85$, evaluating the above expression gives 0.000796 for $R = 100$, 0.000236 for $R = 200$, and 0.000117 for $R = 300$. For $s = 1.60$, the corresponding expression for the set I_2 gives 0.005582 for $R = 100$, 0.002347 for $R = 200$, and 0.001427 for $R = 300$.

To use these results, we proceed for the finite sum analogously to Section 3 to get matrices A_s and B_s. To account for the terms where $|b| > R$, we note that for the lower matrix A_s, we can simply drop all terms where $|b| > R$, while for the upper matrix B_s, we add to the operator a term of the form $c_{R,s}v(0)$ so that we are now approximating the operator

$$(L_s^0 v)(z) = \sum_{b\in\mathcal{B}, |b|\leq R} \Big|\frac{d}{dz}\theta_b(z)\Big|^s v(\theta_b(z)) + c_{R,s}v(0) = \sum_{b\in\mathcal{B}, |b|\leq R} \frac{v(\theta_b(z))}{|z+b|^{2s}} + c_{R,s}v(0),$$

where $c_{R,s}$ is one of the constants in Lemma 4.3, depending on whether we are interested in I_1 or I_2.

5. Existence of C^m positive eigenvectors

In this section we shall describe some results concerning existence of C^m positive eigenvectors for a class of positive (in the sense of order-preserving) linear operators. We shall later indicate how one can often obtain explicit bounds on partial derivatives of the positive eigenvectors. As noted above, such estimates play a crucial role in our numerical method and therefore in obtaining rigorous estimates of Hausdorff dimension for invariant sets associated with iterated function systems.

The methods we shall describe can also be applied to the important case of graph directed iterated function systems, but for simplicity we shall restrict our attention

TABLE 4.1. Computation of Hausdorff dimension s for several values of h and R (rounded to 5 decimal places).

Set	h	R	lower s	upper s
I_1	.02	100	1.85459	1.85609
I_1	.01	100	1.85507	1.85595
I_1	.005	100	1.85518	1.85591
I_1	.02	200	1.85503	1.85604
I_1	.01	200	1.85550	1.85589
I_1	.02	300	1.85513	1.85603
I_2	.02	100	1.60240	1.60677
I_2	.01	100	1.60270	1.60668
I_2	.005	100	1.60277	1.60666
I_2	.02	200	1.60444	1.60654
I_2	.01	200	1.60474	1.60644
I_2	.02	300	1.60504	1.60650
I_3	.02		1.53705	1.53790
I_3	.01		1.53754	1.53774
I_3	.005		1.53765	1.53770

in this paper to a class of linear operators arising in the iterated function system case.

The starting point of our analysis is Theorem 5.5 in [46], which we now describe for a simple case. If H is a bounded open subset of $\mathbb{R}^n$ and m is a positive integer, $C^m(\bar{H})$ will denote the set of real-valued C^m maps $f : H \to \mathbb{R}$ such that all partial derivatives $D^\alpha f$ with $|\alpha| \le m$ extend continuously to $\bar{H}$. (Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with $\alpha_j \ge 0$ for all j, $D_j = \partial/\partial x_j$ for $1 \le j \le n$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$), $C^m(\bar{H})$ is a real Banach space with $\|f\| = \sup\{|D^\alpha f(x)| : x \in H, |\alpha| \le m\}$.

We say that H is *mildly regular* if there exist $\eta > 0$ and $M \ge 1$ such that whenever $x, y \in H$ and $\|x - y\| < \eta$, there exists a Lipschitz map $\psi : [0, 1] \to H$ with $\psi(0) = x$, $\psi(1) = y$ and

$$\int_0^1 \|\psi'(t)\| \, dt \le M\|x - y\|. \tag{5.1}$$

(Here $\|\cdot\|$ denotes any fixed norm on $\mathbb{R}^n$. If the norm is changed, (5.1) remains valid, but with a different constant M.)

Let $\mathcal{B}$ denote a finite index set with $|\mathcal{B}| = p$. For $\beta \in \mathcal{B}$, we assume

(H5.1) $b_\beta \in C^m(\bar{H})$ for all $\beta \in \mathcal{B}$ and $b_\beta > 0$ for all $x \in \bar{H}$ and all $\beta \in \mathcal{B}$.

(H5.2) $\theta_\beta : H \to H$ is a C^m map for all $\beta \in \mathcal{B}$, i.e., if $\theta_\beta(x) = (\theta_{\beta_1}(x), \dots \theta_{\beta_n}(x))$, then $\theta_{\beta_k} \in C^m(\bar{H})$ for all $\beta \in \mathcal{B}$ and for $1 \le k \le n$.

In (H5.1) and (H5.2), we always assume that $m \ge 1$.

We define $\Lambda : C^m(\bar{H}) \to C^m(\bar{H})$ by

$$(\Lambda(f))(x) = \sum_{\beta \in B} b_\beta(x) f(\theta_\beta(x)). \tag{5.2}$$

For integers $\mu \geq 1$, we define $\mathcal{B}_\mu := \{\omega = (j_1, \dots j_\mu) \mid j_k \in \mathcal{B} \text{ for } 1 \leq k \ \leq \mu\}$. For $\omega = (j_1, \dots j_\mu) \in \mathcal{B}_\mu$, we define $\omega_\mu = \omega$, $\omega_{\mu-1} = (j_1, \dots j_{\mu-1})$, $\omega_{\mu-2} = (j_1, \dots j_{\mu-2})$, $\cdots$, $\omega_1 = j_1$. We define

$$\theta_{\omega_{\mu-k}}(x) = (\theta_{j_{\mu-k}} \circ \theta_{j_{\mu-k-1}} \circ \cdots \circ \theta_{j_1})(x), \tag{5.3}$$

so

$$\theta_\omega(x) := \theta_{\omega_\mu}(x) = (\theta_{j_\mu} \circ \theta_{j_{\mu-1}} \circ \cdots \circ \theta_{j_1})(x). \tag{5.4}$$

For $\omega \in \mathcal{B}_\mu$, we define $b_\omega(x)$ inductively by $b_\omega(x) = b_{j_1}(x)$ if $\omega = (j_1) \in \mathcal{B} := \mathcal{B}_1$, $b_\omega(x) = b_{j_2}(\theta_{j_1}(x)) b_{j_1}(x)$ if $\omega = (j_1, j_2) \in \mathcal{B}_2$ and, for $\omega = (j_1, j_2, \dots j_\mu) \in \mathcal{B}_\mu$,

$$b_\omega(x) = b_{j_\mu}(\theta_{\omega_{j_{\mu-1}}}(x)) b_{\omega_{\mu-1}}(x). \tag{5.5}$$

If is not hard to show (see [42], [4], [46]) that

$$(\Lambda^\mu(f))(x) = \sum_{\omega \in \mathcal{B}_\mu} b_\omega(x) f(\theta_\omega(x)). \tag{5.6}$$

It is easy to prove (see [46]) that Λ defines a bounded linear map of $C^m(\bar{H}) \to C^m(\bar{H})$. We shall let $\hat{\Lambda}$ denote the complexification of Λ and let $\sigma(\hat{\Lambda})$ denote the spectrum of $\hat{\Lambda}$. We shall define $\sigma(\Lambda) = \sigma(\hat{\Lambda})$. If all the functions b_j and θ_j are C^N, then we can consider Λ as a bounded linear operator $\Lambda_m : C^m(\bar{H}) \to C^m(\bar{H})$ for $1 \leq m \leq N$, but one should note that in general $\sigma(\Lambda_m)$ will depend on m.

To obtain a useful theory for Λ, we need a further crucial assumption. For a given norm $\|\cdot\|$ on $\mathbb{R}^n$, we assume

(H5.3) There exists a positive integer μ and a constant $\kappa < 1$ such that for all $\omega \in \mathcal{B}_\mu$ and all $x, y \in H$,

$$\|\theta_\omega(x) - \theta_\omega(y)\| \leq \kappa \|x - y\|. \tag{5.7}$$

If we define $c = \kappa^{1/\mu} < 1$, it follows from (H5.3) that there exists a constant M such that for all $\omega \in B_\nu$ and all $\nu \geq 1$,

$$\|\theta_\omega(x) - \theta_\omega(y)\| \leq M c^\nu \|x - y\| \quad \forall x, y \in H. \tag{5.8}$$

If the norm $\|\cdot\|$ in (5.8) is replaced by a different norm $|\cdot|$, (5.8) remains valid, although with a different constant M. This in turn implies that (H5.3) will also be valid with the same constant κ, with $|\cdot|$ replacing $\|\cdot\|$ and with a possibly different integer μ.

The following theorem is a special case of Theorem 5.5 in [46].

Theorem 5.1. *Let H be a bounded open subset of $\mathbb{R}^n$ and assume that H is mildly regular. Let $X = C^m(\bar{H})$ and assume that (H5.1), (H5.2), and (H5.3) are satisfied (where $m \geq 1$ in (H5.1) and (H5.2)) and that $\Lambda : X \to X$ is given by* (5.2). *If $Y = C(\bar{H})$, the Banach space of real-valued continuous functions $f : \bar{H} \to \mathbb{R}$ and $L : Y \to Y$ is defined by* (5.2), *then $r(L) = r(\Lambda) > 0$, where $r(L)$ denotes the spectral radius of L and $r(\Lambda)$ denotes the spectral radius of Λ. If $\rho(\Lambda)$ denotes the essential spectral radius of Λ (see* [36],[42],[47], *and* [44]*), then $\rho(\Lambda) \leq c^m r(\Lambda)$ where $c = \kappa^{1/\mu}$ is as in* (5.8). *There exists $v \in X$ such that $v(x) > 0$ for all $x \in \bar{H}$ and*

$$\Lambda(v) = rv, \qquad r = r(\Lambda). \tag{5.9}$$

There exists $r_1 < r$ such that if $\xi \in \sigma(\Lambda) \setminus \{r\}$, then $|\xi| \leq r_1$; and $r = r(\Lambda)$ is an isolated point of $\sigma(\Lambda)$ and an eigenvalue of algebraic multiplicity 1. If $u \in X$ and $u(x) > 0 \, \forall x \in \bar{H}$, there exists a real number $s_u > 0$ such that

$$\lim_{k \to \infty} \left(\frac{1}{r} \Lambda \right)^k (u) = s_u v, \tag{5.10}$$

where the convergence in (5.10) *is in the C^m topology on X.*

Remark 5.1. *If α is a multi-index with $|\alpha| \leq m$, where $m \geq 1$ is as in (H5.1) and (H5.2), it follows from* (5.10) *that*

$$\lim_{k \to \infty} \left(\frac{1}{r} \right)^k D^\alpha \Lambda^k(u) = s_u D^\alpha v, \tag{5.11}$$

and

$$\lim_{k \to \infty} \left(\frac{1}{r} \right)^k \Lambda^k(u) = s_u v, \tag{5.12}$$

where the convergence in (5.11) *and* (5.12) *is in the topology of $C(\bar{H})$, the Banach space of continuous functions $f : \bar{H} \to \mathbb{R}$.*

It follows from (5.11) and (5.12) that for any multi-index α with $|\alpha| \leq m$,

$$\lim_{k \to \infty} \frac{(D^\alpha \Lambda^k(u))(x)}{\Lambda^k(u)(x)} = \frac{(D^\alpha(v))(x)}{v(x)}, \tag{5.13}$$

where the convergence in (5.13) is uniform in $x \in \bar{H}$. If we choose $u(x) = 1$ for all $x \in \bar{H}$, it follows from (5.6) that for all multi-indices α with $|\alpha| \leq m$, we have

$$\lim_{k \to \infty} \frac{D^\alpha (\sum_{\omega \in B_k} b_\omega(x))}{\sum_{\omega \in B_k} b_\omega(x)} = \frac{D^\alpha v(x)}{v(x)}, \tag{5.14}$$

where the convergence in (5.14) is uniform in $x \in \bar{H}$. We shall use (5.14) in our further work to obtain explicit bounds on $\sup \left\{ |D^\alpha v(x)|/v(x) : x \in \bar{H} \right\}$.

We shall also need information about positive eigenvectors when the index set $\mathcal{B}$ is countable, but not finite. Direct analogues of Theorem 5.5 in [46] exist when $\mathcal{B}$ is countable, but not finite, but such analogues were not stated or proved in [46]. Thus we shall make do with less precise theorems concerning strictly positive Lipschitz eigenvectors.

Given a metric space (S, d), a countable index set $\mathcal{B}$, and continuous maps $\theta_\beta : S \to S$ and $b_\beta : S \to \mathbb{R}$ for $\beta \in \mathcal{B}$, we shall say that the families $\{\theta_\beta : \beta \in \mathcal{B}\}$ and $\{b_\beta : \beta \in \mathcal{B}\}$ are *uniformly Lipschitz* if there exist constants M_1 and M_2, independent of $\beta \in \mathcal{B}$, such that

$$d(\theta_\beta(x), \theta_\beta(y)) \le M_1 d(x, y), \ \forall x, y \in S \text{ and } \forall \beta \in \mathcal{B}$$

and

$$|b_\beta(x) - b_\beta(y)| \le M_2 d(x, y), \ \forall x, y \in S \text{ and } \forall \beta \in \mathcal{B}.$$

If S is a subset of $\mathbb{R}^N$, we shall take the metric d to be given by some norm $\|\cdot\|$ on $\mathbb{R}^n$.

For (S, d) a compact metric space, $C(S)$ will denote the real Banach space of continuous functions $f : S \to \mathbb{R}$ with $\|f\| := \sup\{|f(x)| : x \in S\}$. If $b_\beta : S \to (0, \infty)$ is a positive, continuous function for all $\beta \in \mathcal{B}$, we shall assume that

$$\sum_{\beta \in \mathcal{B}} b_\beta(x) = b(x) < \infty \tag{5.15}$$

for all $x \in S$ and $x \mapsto b(x)$ is continuous on S. If D_k, $k \ge 1$ is any increasing sequence of finite subsets $D_k \subset \mathcal{B}$ with $\cup_{k \ge 1} D_k = \mathcal{B}$, Dini's theorem implies that

$$\lim_{k \to \infty} \sum_{\beta \in D_k} b_\beta(x) = b(x)$$

and that the convergence is uniform in $x \in S$. Using this fact, one can define for $f \in C(S)$, $L(f) \in C(S)$ by

$$(Lf)(x) = \sum_{\beta \in \mathcal{B}} b_\beta(x) f(\theta_\beta(x)). \tag{5.16}$$

Here, one is assuming that (5.15) holds with $x \mapsto b(x)$ continuous on S and that $\theta_\beta : S \to S$ is continuous for all $\beta \in S$, and under these assumptions, $L : C(S) \to C(S)$ is a bounded linear operator. Also, one can see that for integers $\mu \ge 1$ that

$$(L^\mu f)(x) = \sum_{\omega \in \mathcal{B}_\mu} b_\omega(x) f(\theta_\omega(x)), \tag{5.17}$$

where b_ω and θ_ω are as defined in equations (5.4) and (5.5).

If M is a fixed positive constant, we define a closed cone $K(M; S) \subset C(S)$ by

$$\begin{aligned} &K(M; S) \\ &\quad = \{f \in C(S) \,|\, f(x) \ge 0 \,\forall x \in S \text{ and } f(y) \le f(x) \exp(M d(x, y)) \,\forall x, y \in S\}. \end{aligned} \tag{5.18}$$

Our next theorem follows easily from Lemma 5.3 in Section 5 of [47] and Theorem 5.3 on page 86 of [42].

Theorem 5.2. *Let $H \subset \mathbb{R}^n$ be a bounded, open subset of $\mathbb{R}^n$ and let the metric on $\bar{H}$ be given by a fixed norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $\mathcal{B}$ be a countable (not finite) index set and assume that $\theta_\beta : \bar{H} \to \bar{H}$ and $b_\beta : \bar{H} \to (0, \infty)$, $\beta \in \mathcal{B}$, are continuous functions and that $\{\theta_\beta \,|\, \beta \in \mathcal{B}\}$ and $\{b_\beta \,|\, \beta \in \mathcal{B}\}$ are uniformly Lipschitz. Assume that for all $x \in \bar{H}$, $\sum_{\beta \in \mathcal{B}} b_\beta(x) := b(x) < \infty$ and that $x \mapsto b(x)$ is continuous. Assume that there exists an integer $\mu \ge 1$ and a constant $\kappa < 1$ such that for all $\omega \in \mathcal{B}_\mu$, $\mathrm{Lip}(\theta_\omega) \le \kappa$. Assume also that the family of maps $\{x \mapsto \log(b_\beta(x)) : \beta \in \mathcal{B}\}$ is* *uniformly Lipschitz. Then there exists a constant A such that for all integers $\nu \geq 1$ and for all $\omega \in \mathcal{B}_\nu$*

$$\text{Lip}(\theta_\omega) \leq Ac^\nu, \qquad c = \kappa^{1/\mu}. \tag{5.19}$$

Also, for each integer $\nu \geq 1$, the family of maps $\{x \mapsto \log(b_\omega(x)) : \omega \in \mathcal{B}_\nu\}$ is uniformly Lipschitz, so there exists $M_0 > 0$ such that $b_\omega \in K(M_0; \bar{H})$(see (5.17) *with $S := \bar{H}$) for all $\omega \in \mathcal{B}_\mu$. If $L : C(\bar{H}) \to C(\bar{H})$ is given by* (5.16) *with $\bar{H} := S$, L has a strictly positive eigenvector $v \in K(M_0/(1-\kappa); \bar{H})$ with eigenvalue $r = r(L) > 0$. The algebraic multiplicity of the eigenvalue r equals one, and r is the only eigenvalue of L of modulus r.*

Proof. In the following proof, we shall not distinguish in notation between L and its complexification $\tilde{L}$, but of course $\sigma(L)$ refers to the spectrum of $\tilde{L}$ and eigenvalues refer to (possibly complex) eigenvalues of $\tilde{L}$. We leave to the reader the proof of (5.17) and of the fact that for any $\nu \geq 1$, the set of maps $\{x \mapsto \log(b_\omega(x)) : \omega \in \mathcal{B}_\nu\}$ is uniformly Lipschitz. If we start with (5.17), rather than (5.16), Lemma 5.3 in [47] shows that L^μ has a strictly positive eigenvector $v \in K(M_0/(1-\kappa); \bar{H})$ with eigenvalue $r = r(L^\mu) = [r(L)]^\mu > 0$. If we apply Theorem 5.3, p. 86 in [42] to L^μ, we find that r^μ is the only eigenvalue of L^μ of modulus r^μ and r^μ has algebraic multiplicity one as an eigenvalue of L^μ. Since $[(1/r)L]^\mu v = v$ and $w = (1/r)L(v)$ is also a nonzero fixed point of $[(1/r)L]^\mu$, it must be that $[(1/r)L]v = \lambda v$ for some $\lambda \neq 0$. We must have $\lambda > 0$, because $(1/r)L(v)(x) > 0$ for all $x \in \bar{H}$ and $v(x) > 0$ for all $x \in \bar{H}$. This implies that $\lambda^\mu v = v$ and $\lambda > 0$; so $\lambda = 1$ and v is a strictly positive eigenvector of L with eigenvalue $r(L)$. If we now apply Theorem 5.3 of [42] to L, we find that $r(L)$ is an eigenvalue of L of algebraic multiplicity one and $r(L)$ is the only eigenvalue of L of modulus $r(L)$. Note, however, that $\sigma(L)$ may well contain elements of modulus $r(L)$. ☐

Corollary 5.3. *Let assumptions and notation be as in Theorem 5.2. Assume in addition that H is convex and that $b_\beta \in C^1(\bar{H})$ for all $\beta \in \mathcal{B}$. For each integer $\nu \geq 1$, define*

$$M_\nu = \sup\{\frac{\|\nabla b_\omega(x)\|}{b_\omega(x)} : \omega \in \mathcal{B}_\nu, x \in \bar{H}\},$$

where we use the Euclidean norm on $\mathbb{R}^n$. Define M_∞ by $M_\infty = \liminf_{\nu \to \infty} M_\nu$. If v is a strictly positive eigenvector of L in (5.15), *$v \in K(M_\infty, \bar{H})$.*

Proof. If $x, y \in H$, then because we assume that H is convex, $x^t := (1-t)x + ty \in H$ for $0 \leq t \leq 1$. (We use t as a superscript here.) If $\omega \in \mathcal{B}_\nu$, $\nu \geq 1$, it follows that

$$\begin{aligned}|\log(b_\omega(y) - \log(b_\omega(x)| &= \left|\int_0^1 \frac{d}{dt} \log b_\omega(x^t)\, dt\right| \\ &= \left|\int_0^1 \frac{\nabla b_\omega \cdot (y-x)}{b_\omega(x^t)}\, dt\right| \leq \int_0^1 \frac{\|\nabla b_\omega\| \|y-x\|}{b_\omega(x^t)}\, dt.\end{aligned}$$

This shows that $x \mapsto \log b_\omega(x)$ is Lipschitz on $\bar{H}$ with Lipschitz constant $\leq M_\nu$, so $b_\omega \in K(M_\nu; \bar{H})$ for $\omega \in \mathcal{B}_\nu$. If $Ac^\nu < 1$, the argument used in the proof of Lemma 5.3 in [47] now shows that $v \in K(M_\nu/(1-Ac^\nu); \bar{H})$. Since $\lim_{\nu \to \infty} Ac^\nu = 0$, we conclude that $v \in K(M_\infty; \bar{H})$. ☐

Remark 5.2. *Under slightly stronger assumptions, the bounded linear operator $L : C(S) \to C(S) := Y$ induces a bounded linear operator $\Lambda : X \to X$, where X denotes the Banach space of Lipschitz functions $f : S \to \mathbb{R}$. One can prove that $r(\Lambda) = r(L)$ and $\rho(\Lambda) < r(\Lambda)$, where $\rho(\Lambda)$ denotes the essential spectral radius of Λ. See* [47] *and Section 5 of* [42] *for details.*

In some applications, the domain H in Theorem 5.1 or Theorem 5.2 possesses some symmetry or symmetries, and this is often reflected in a corresponding symmetry of the unique, normalized positive eigenvector v in these theorems.

Corollary 5.4. *Let assumptions and notation be as in Theorem 5.1 or Theorem 5.2 and let v denote the unique normalized strictly positive eigenvector of L in Theorem 5.1 or Theorem 5.2. Assume that $\pi : \bar{H} \to \bar{H}$ is a C^m map, $m \geq 1$, such that $\pi(\pi(x)) = x$ for all $x \in \bar{H}$. Assume that there exists a one-one map $\beta \mapsto \bar{\beta}$ of $\mathcal{B}$ onto $\mathcal{B}$ such that $\pi(\theta_{\bar{\beta}}(x)) = \theta_\beta(\pi(x))$ and $b_\beta(\pi(x)) = b_{\bar{\beta}}(x)$ for all $\beta \in \mathcal{B}$ and all $x \in \bar{H}$. It then follows that $v(\pi(x)) = v(x)$ for all $x \in \bar{H}$.*

Proof. Define $w(x) = v(\pi(x))$, so $w(\theta_\beta(x)) = v(\pi(\theta_\beta(x)))$ for all $\beta \in \mathcal{B}$ and $x \in \bar{H}$. If $\lambda := r(\Lambda)$, it follow that

$$\lambda v(\pi(x)) = \lambda w(x) = \sum_{\beta \in \mathcal{B}} b_\beta(\pi(x)) v(\theta_\beta(\pi(x))) = \sum_{\beta \in \mathcal{B}} b_{\bar{\beta}}(\pi(x)) v(\theta_{\bar{\beta}}(\pi(x))).$$

Since $b_{\bar{\beta}}(\pi(x)) = b_\beta(x)$ and $v(\theta_{\bar{\beta}}(\pi(x))) = v(\pi(\theta_\beta(x))) = w(\theta_\beta(x))$, we find that

$$\lambda w(x) = \sum_{\beta \in \mathcal{B}} b_\beta(x) w(\theta_\beta(x)) = (\Lambda(w))(x),$$

so

$$w_1(x) = \frac{v(x) + w(x)}{2} = \frac{v(x) + v(\pi(x))}{2}$$

is a strictly positive eigenvector of Λ with eigenvalue λ and $w_1(\pi(x)) = w_1(x)$ for all $x \in \bar{H}$. By the uniqueness of the strictly positive eigenvector of Λ, there exists $\mu > 0$ such that $v(x) = \mu w(x)$ for all $x \in \bar{H}$, which implies that $v(\pi(x)) = v(x)$ for all $x \in \bar{H}$. ∎

Remark 5.3. *Suppose that H is a bounded, open mildly regular subset of $\mathbb{C} = \mathbb{R}^2$ and for all $z = x + iy$, $\bar{x} = x - iy \in H$. Define $\pi(z) = \bar{z}$ and assume that the hypotheses of Corollary 5.4 are satisfied, so $v(\bar{z}) = v(z)$ for all $z \in H$. Using this fact, the original eigenvalue problem can be reduced to an equivalent problem on the closure of H_+, where $H_+ = \{z \in H \,|\, \mathrm{Im}(z) > 0\}$.*

6. Estimates for derivatives of v_s: the one dimensional case

Throughout this section, we shall assume that $H \subset \mathbb{R}^1$ is a bounded, open set such that $H = \cup_{j=1}^n (c_j, d_j)$, where $[c_j, d_j] \cap [c_k, d_k] = \emptyset$ whenever $1 \leq j \leq n$, $1 \leq k \leq n$, and $j \neq k$. $\mathcal{B}$ will denote a finite index set. For $\beta \in \mathcal{B}$ and some integer $m \geq 1$, we assume

(H6.1:) For each $\beta \in \mathcal{B}$, $b_\beta \in C^m(\bar{H})$, $\theta_\beta \in C^m(\bar{H})$, $b_\beta(x) > 0$ for all $x \in \bar{H}$ and $\theta_\beta(H) \subset H$. There exist an integer $\mu \geq 1$ and a real number $\kappa < 1$ such that for all $\omega \in \mathcal{B}_\mu := \{(\beta_1, \beta_2, \cdots, \beta_\mu) \,|\, \beta_j \in \mathcal{B} \text{ for } 1 \le j \le \mu\}$ and for all $x, y \in \bar{H}$, $|\theta_\omega(x) - \theta_\omega(y)| \le \kappa |x-y|$, where $\theta_\omega := \theta_{\beta_\mu} \circ \theta_{\beta_{\mu-1}} \circ \cdots \circ \theta_{\beta_1}$ for $\omega = (\beta_1, \beta_2, \cdots, \beta_\mu) \in \mathcal{B}_\mu$.

As in Section 5, we define $Y = C(\bar{H})$ and $X_m = C^m(\bar{H})$. Assuming (H6.1), we define for $s \ge 0$, a bounded linear operator $L_s : Y \to Y$ by

$$(L_s f)(x) = \sum_{\beta \in \mathcal{B}} [b_\beta(x)]^s f(\theta_\beta(x)). \tag{6.1}$$

As in Section 5, $L_s(X_m) \subset X_m$ and $L_s|_{X_m}$ defines a bounded linear map of X_m to X_m which we denote by Λ_s. Theorem 5.1 is now directly applicable (replace $b_\beta(x)$ in Theorem 5.1 by $b_\beta(x)^s$) and yields information about $\sigma(\Lambda_s)$. In particular, $r(L_s) = r(\Lambda_s) > 0$ and there exists a unique (to with normalization) strictly positive, C^m eigenvector v_s of Λ_s with eigenvalue $r(\Lambda_s)$.

If $\omega = (\beta_1, \beta_2, \ldots, \beta_p) \in \mathcal{B}_p$, recall that we define $b_\omega(x)$ by

$$b_\omega(x) = b_{\beta_p}(\theta_{\beta_{p-1}} \circ \theta_{\beta_{p-2}} \circ \cdots \circ \theta_{\beta_1}(x)) \cdots b_{\beta_3}((\theta_{\beta_2} \circ \theta_{\beta_1})(x)) b_{\beta_2}((\theta_{\beta_1}(x)) b_{\beta_1}(x),$$

and

$$(L_s^p f)(x) = \sum_{\omega \in \mathcal{B}_p} [b_\omega(x)]^s f(\theta_\omega(x)). \tag{6.2}$$

Notice that L_s^p is of the same form as L_s and Theorem 5.1 is also directly applicable to L_s^p. Since v_s is also an eigenvector of L_s^p, we could also work with (6.2) instead of (6.1): $\mathcal{B}_p$ is an index set corresponding to $\mathcal{B}$, b_ω, $\omega \in \mathcal{B}_p$, corresponds to b_β, $\beta \in \mathcal{B}$, and θ_ω, $\omega \in \mathcal{B}_p$, corresponds to θ_β, $\beta \in \mathcal{B}$. In our subsequent work in this section, we shall start from (6.1), but the theorems we shall obtain translate directly to the case of using (6.2); and indeed it is sometimes desirable to start from (6.2) for some $p > 1$.

If m is as in (H6.1) and k is a positive integer with $k \le m$, we define $D = d/dx$, so $(Df)(x) = f'(x)$ and $(D^k f)(x) = f^{(k)}(x)$. We are interested in obtaining estimates for

$$\sup\{|D^k v_s(x)|/v_s(x) : x \in \bar{H}\}. \tag{6.3}$$

Hypothesis (H6.1) implies that there exist constants $M > 0$ and $c = \kappa^{1/\mu}$, (so $c < 1$), such that for all $x, y \in \bar{H}$, for all integers $\nu \ge 1$ and all $\omega \in \mathcal{B}_\nu$,

$$|\theta_\omega(x) - \theta_\omega(y)| \le M c^\nu |x - y|. \tag{6.4}$$

It follows that if we define $\epsilon_0 = 1$, and for $\nu \ge 1$,

$$\epsilon_\nu := \sup\Big\{|\theta_\omega(x) - \theta_\omega(y)|/|x-y| : \omega \in \mathcal{B}_\nu \text{ and } x, y \in H, x \ne y\Big\}, \tag{6.5}$$

we have that $\epsilon_\nu \le M c^\nu$ and $\sum_{\nu=1}^\infty \epsilon_\nu < \infty$.

We define constants C_1 and $C_1(s)$ for $s > 0$ by

$$C_1 = \sup\Big\{\frac{|D b_\beta(x)|}{b_\beta(x)} : \beta \in \mathcal{B}, x \in H\Big\} \tag{6.6}$$

and

$$C_1(s) = \sup\Big\{\frac{|Db_\beta(x)^s|}{b_\beta(x)^s} : \beta \in \mathcal{B}, x \in H\Big\}.$$

A calculation shows that for all $\omega \in \mathcal{B}_\nu$, $\nu \geq 1$,

$$\frac{|Db_\omega(x)^s|}{b_\omega(x)^s} = s\frac{Db_\omega(x)}{b_\omega(x)}, \tag{6.7}$$

so

$$C_1(s) = sC_1, \text{ for } s > 0. \tag{6.8}$$

We begin by considering (6.3) for the case $k = 1$. In our applications, we shall only need the case $s > 0$, so we shall restrict our attention to this case.

Theorem 6.1. *Assume that (H6.1) is satisfied. If C_1 and ϵ_ν, $\nu \geq 1$ are as in* (6.6) *and* (6.5) *respectively, then, for $s > 0$,*

$$\sup\Big\{\frac{|Dv_s(x)|}{v_s(x)} : x \in \bar{H}\Big\} \leq C_1 s\Big(\sum_{\nu=0}^{\infty}\epsilon_\nu\Big). \tag{6.9}$$

If $\delta \in \{0,1\}$ and $(-1)^\delta(Db_\omega)(x)/b_\omega(x)) \leq 0$ for all $\omega \in \mathcal{B}_\nu$, all $\nu \geq 1$ and all $x \in \bar{H}$, then $(-1)^\delta Dv_s(x) \leq 0$ for all $x \in \bar{H}$ and all $s > 0$.

Proof. For a fixed $\omega = (j_1, j_2, \dots, j_\nu) \in \mathcal{B}_\nu$, for notational convenience define $\xi_k(x) = (\theta_{j_k} \circ \theta_{j_{k-1}} \circ \cdots \circ \theta_{j_1})(x)$ and $\xi_0(x) = x$ for all $x \in \bar{H}$, so

$$b_\omega(x) = b_{j_\nu}(\xi_{\nu-1}(x))b_{j_{\nu-1}}(\xi_{\nu-2}(x))\cdots b_{j_1}(\xi_0(x)).$$

By the chain rule and product rule for differentiation, we find

$$Db_\omega(x) = b_\omega(x)\left[\sum_{k=0}^{\nu-1}\frac{b'_{j_{k+1}}(\xi_k(x))\xi'_k(x)}{b_{j_{k+1}}(\xi_k(x))}\right]. \tag{6.10}$$

It follows from (6.10) that

$$\frac{|Db_\omega(x)|}{b_\omega(x)} \leq \sum_{k=0}^{\nu-1} C_1\epsilon_k \leq C_1\sum_{k=0}^{\infty}\epsilon_k. \tag{6.11}$$

Using (6.7), we see that

$$\frac{|D(b_\omega(x)^s)|}{b_\omega(x)^s} \leq sC_1\sum_{k=0}^{\infty}\epsilon_k, \tag{6.12}$$

and it follows that

$$\frac{|D(\sum_{\omega\in\mathcal{B}_\nu}(b_\omega(x)^s)|}{\sum_{\omega\in\mathcal{B}_\nu}(b_\omega(x)^s)} \leq \frac{\sum_{\omega\in\mathcal{B}_\nu}|D(b_\omega(x)^s)|}{\sum_{\omega\in\mathcal{B}_\nu}b_\omega(x)^s} \leq sC_1\sum_{k=0}^{\infty}\epsilon_k.$$

Using Theorem 5.1 and (5.14), we conclude by letting $\nu \to \infty$ that

$$\frac{|Dv_s(x)|}{v_s(x)} \leq sC_1\sum_{k=0}^{\infty}\epsilon_k. \tag{6.13}$$

The final statement of the theorem follows by a similar argument, using (6.7) and (5.14). Details are left to the reader. $\square$

To facilitate the analysis of (6.3) when $k = 2$, we first prove a lemma.

Lemma 6.2. *Assume that (H6.1) is satisfied with $m \geq 2$ and define M_0 by*

$$M_0 = \sup\{|D^2\theta_\beta(x)| : \beta \in \mathcal{B}, x \in \bar{H}\}. \tag{6.14}$$

If $\omega \in \mathcal{B}_k$, then

$$|D^2\theta_\omega(x)| \leq M_0 \sum_{j=0}^{k-1} \epsilon_j^2 \epsilon_{k-j-1}. \tag{6.15}$$

Proof. Recall, for a fixed $\omega = (\beta_1, \beta_2, \cdots, \beta_k) \in \mathcal{B}_k$, we have defined $\xi_0(x) = x$ and $\xi_p(x)$, $1 \leq p \leq k$ by

$$\xi_p(x) = (\theta_{\beta_p} \circ \theta_{\beta_{p-1}} \circ \cdots \circ \theta_{\beta_1})(x).$$

Hence,

$$\xi_k'(x) = \prod_{j=1}^{k} \theta'_{\beta_j}(\xi_{j-1}(x)). \tag{6.16}$$

By differentiating (6.16), we find that

$$\xi_k''(x) = \sum_{j=0}^{k-1} (\theta''_{\beta_{j+1}}(\xi_j(x))\xi_j'(x)) \Big[\prod_{\substack{p=1 \\ p \neq j+1}}^{k} \theta'_{\beta_p}(\xi_{p-1}(x)) \Big]. \tag{6.17}$$

If there exist two distinct integers p and q with $1 \leq p \leq k$ and $1 \leq q \leq k$ such that $\theta'_{\beta_p}(\xi_{p-1}(x)) = 0$ and $\theta'_{\beta_q}(\xi_{q-1}(x)) = 0$, (6.17) implies that $\xi_k''(x) = 0$. If there exists exactly one integer q with $1 \leq q \leq k$ such that $\theta'_{\beta_q}(\xi_{q-1}(x)) = 0$, (6.17) implies that

$$\xi_k''(x) = \Big[\theta''_{\beta_q}(\xi_{q-1}(x))\xi_{q-1}'(x)\Big] \left[\prod_{p=1}^{q-1} \theta'_{\beta_p}(\xi_{p-1}(x))\right] \left[\prod_{p=q+1}^{k} \theta'_{\beta_p}(\xi_{p-1}(x))\right], \tag{6.18}$$

where we interpret $\prod_{p=q+1}^{k} \theta'_{\beta_p}(\xi_{p-1}(x)) = 1 = \epsilon_0$ if $q = k$. It follows from (6.18) that

$$|\xi_k''(x)| \leq M_0 \epsilon_{k-q} \epsilon_{q-1}^2. \tag{6.19}$$

If there does not exist p, $1 \leq p \leq k$, with $\theta'_{\beta_p}(\xi_{p-1}(x)) = 0$, (6.16) implies that $\xi_k'(x) \neq 0$, and we obtain from (6.17) that

$$\xi_k''(x) = \sum_{q=1}^{k} (\theta''_{\beta_q}(\xi_{q-1}(x))\xi_{q-1}'(x)) \left[\prod_{p=1}^{q-1} \theta'_{\beta_p}(\xi_{p-1}(x))\right] \left[\prod_{p=q+1}^{k} \theta'_{\beta_p}(\xi_{p-1}(x))\right]. \tag{6.20}$$

Then (6.20) implies that

$$|\xi_k''(x)| \leq \sum_{q=1}^{k} M_0 \epsilon_{k-q} \epsilon_{q-1}^2, \tag{6.21}$$

which completes the proof. □

If M and c, $0 < c < 1$, are chosen as in (6.4), Lemma 6.2 implies that for all $\omega \in \mathcal{B}_k$ and $k \geq 2$

$$
\begin{aligned}
(6.22) \quad |D^2\theta_\omega(x)| &\leq M_0 M c^{k-1} + \sum_{q=2}^{k-1} M_0 M^3 c^{k-2} c^q + M_0 M^2 c^{2k-2} \\
&= M_0 M c^{k-1}(1 + M c^{k-1}) + M_0 M^3 c^k \frac{1 - c^{k-2}}{1 - c}.
\end{aligned}
$$

Lemma 6.3. *Assume that (H6.1) is satisfied with $m \geq 2$ and define a constant C_2 by*

$$
C_2 = \sup\Big\{ \frac{|D^2 b_\beta(x)|}{b_\beta(x)} : \beta \in \mathcal{B}, x \in H \Big\}. \tag{6.23}
$$

Let C_1, C_2, and M_0 be as in (6.6), (6.23), *and* (6.14). *Then for $s > 0$, and for $\omega \in \mathcal{B}_\nu$, with $\nu \geq 1$, we have the estimates*

$$
\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \leq s^2 C_1^2 \Big(\sum_{k=0}^{\infty} \epsilon_k \Big)^2 + s \Big(\sum_{k=0}^{\infty} \epsilon_k^2 \Big) \Big[C_2 + C_1 M_0 \Big(\sum_{k=0}^{\infty} \epsilon_k \Big) \Big] \tag{6.24}
$$

and

$$
\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \geq -s \Big(\sum_{k=0}^{\infty} \epsilon_k^2 \Big) \Big[C_1^2 + C_2 + C_1 M_0 \Big(\sum_{k=0}^{\infty} \epsilon_k \Big) \Big]. \tag{6.25}
$$

Proof. For a fixed $\omega = (j_1, j_2, \ldots, j_\nu) \in \mathcal{B}_\nu$, let $\xi_k(x)$ be as defined in the proof of Theorem 6.1. A calculation gives

$$
\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} = s(s-1) \left(\frac{D(b_\omega(x))}{b_\omega(x)} \right)^2 + s \frac{D^2(b_\omega(x))}{b_\omega(x)}. \tag{6.26}
$$

Using (6.10) we see that

$$
D^2 b_\omega(x) = b_\omega(x) \left(\frac{D(b_\omega(x))}{b_\omega(x)} \right)^2 + b_\omega(x) D \left(\sum_{k=0}^{\nu-1} \frac{b'_{j_{k+1}}(\xi_k(x)) \xi'_k(x)}{b_{j_{k+1}}(\xi_k(x))} \right), \tag{6.27}
$$

which gives

$$
\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} = s^2 \left(\frac{D(b_\omega(x))}{b_\omega(x)} \right)^2 + s D \left(\sum_{k=0}^{\nu-1} \frac{b'_{j_{k+1}}(\xi_k(x)) \xi'_k(x)}{b_{j_{k+1}}(\xi_k(x))} \right) := T_1 + T_2. \tag{6.28}
$$

A calculation gives

$$
\begin{aligned}
(6.29) \quad & s D \left(\sum_{k=0}^{\nu-1} \frac{b'_{j_{k+1}}(\xi_k(x)) \xi'_k(x)}{b_{j_{k+1}}(\xi_k(x))} \right) \\
&= s \sum_{k=0}^{\nu-1} \frac{b''_{j_{k+1}}(\xi_k(x))(\xi'_k(x))^2 + b'_{j_{k+1}}(\xi_k(x)) \xi''_k(x)}{b_{j_{k+1}}(\xi_k(x))} - s \sum_{k=0}^{\nu-1} \frac{[b'_{j_{k+1}}(\xi_k(x)) \xi'_k(x)]^2}{[b_{j_{k+1}}(\xi_k(x))]^2}.
\end{aligned}
$$

It follows that

$$
T_2 \leq s \left(\sum_{k=0}^{\nu-1} C_2 \epsilon_k^2 + \sum_{k=0}^{\nu-1} C_1 |\xi''_k(x)| \right) \tag{6.30}
$$

and

$$T_2 \geq -s\left(\sum_{k=0}^{\nu-1} C_2\epsilon_k^2 + \sum_{k=0}^{\nu-1} C_1|\xi_k''(x)| + \sum_{k=0}^{\nu-1} C_1^2\epsilon_k^2\right). \tag{6.31}$$

Lemma 6.2 implies that

$$\sum_{k=0}^{\nu-1} |\xi_k''(x)| \leq M_0 \sum_{k=0}^{\infty}\Big(\sum_{q=1}^{k} \epsilon_{k-q}\epsilon_{q-1}^2\Big) = M_0\Big(\sum_{q=0}^{\infty}\epsilon_q^2\Big)\Big(\sum_{k=0}^{\infty}\epsilon_k\Big), \tag{6.32}$$

so we obtain from (6.30) and (6.31) that

$$T_2 \leq s\left(C_2\sum_{k=0}^{\infty}\epsilon_k^2 + C_1M_0\Big(\sum_{q=0}^{\infty}\epsilon_q^2\Big)\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right) \tag{6.33}$$

and

$$T_2 \geq -s\left((C_2 + C_1^2)\Big(\sum_{k=0}^{\infty}\epsilon_k^2\Big) + C_1M_0\Big(\sum_{q=0}^{\infty}\epsilon_q^2\Big)\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right). \tag{6.34}$$

Combining equations (6.28), (6.33), and (6.34) and using (6.11), we obtain for $s > 0$ and $\omega = \mathcal{B}_\nu$

$$\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \leq s^2C_1^2\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)^2 + s\Big(\sum_{k=0}^{\infty}\epsilon_k^2\Big)\left[C_2 + C_1M_0\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right] \tag{6.35}$$

and

$$\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \geq -s\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)^2\left[(C_2 + C_1^2) + C_1M_0\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right], \tag{6.36}$$

which completes the proof. ☐

Theorem 6.4. *Assume that (H6.1) is satisfied with $m \geq 2$, and for $s > 0$, let v_s denote the strictly positive, normalized C^m eigenvector of L_s in (6.1). For integers $\nu \geq 0$, define ϵ_ν by $\epsilon_0 = 1$ and by (6.5) for $\nu \geq 1$. In addition, let C_1, C_2, and M_0 be constants given by (6.6), (6.23), and (6.14), respectively. Then for all $x \in \bar{H}$, we have the following estimates.*

$$\frac{D^2v_s(x)}{v_s(x)} \leq s^2C_1^2\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)^2 + s\Big(\sum_{k=0}^{\infty}\epsilon_k^2\Big)\left[C_2 + C_1M_0\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right]$$

and

$$\frac{D^2v_s(x)}{v_s(x)} \geq -s\sum_{k=0}^{\infty}\epsilon_k^2\left[(C_2 + C_1^2) + C_1M_0\Big(\sum_{k=0}^{\infty}\epsilon_k\Big)\right].$$

Proof. Theorem 6.4 follows immediately from (5.14) and (6.35) and (6.36). ☐

The estimates given in Theorems 6.1 and 6.4 are somewhat crude. If one has more information about the coefficients $b_\beta(\cdot)$ and the maps $\theta_\beta(\cdot)$, $\beta \in \mathcal{B}$, one can obtain much sharper results. An example is provided by the following theorem.

Theorem 6.5. *Assume that (H6.1) is satisfied with $m \geq 2$. Assume, also, that $H = (a_1, a_2)$ is a bounded open interval in $\mathbb{R}$ and that $\theta'_\beta(u) \geq 0$, $\theta''_\beta(u) \geq 0$, $b'_\beta(u) \geq 0$, $b''_\beta(u) \geq 0$, and*

$$b''_\beta(u) b_\beta(u) - (1-s)[b'_\beta(u)]^2 \geq 0 \tag{6.37}$$

for all $\beta \in \mathcal{B}$, for all $u \in H$, and for a given real number s. If $s > 0$ and v_s is the strictly positive C^m eigenvector of Λ_s, it follows that for all $u \in \bar{H}$

$$Dv_s(u) \geq 0 \text{ and } D^2 v_s(u) \geq 0. \tag{6.38}$$

If, in addition, there exists a set $F \subset \bar{H}$ (possibly empty) such that for all $u \in \bar{H} \setminus F$ and all $\beta \in \mathcal{B}$, $b'_\beta(u) > 0$ and strict inequality holds in (6.37), then for all $u \in \bar{H} \setminus F$,

$$Dv_s(u) > 0 \text{ and } D^2 v_s(u) > 0. \tag{6.39}$$

Proof. For $\nu \geq 1$, let $\omega = (j_1, j_2, \cdots, , j_\nu)$ denote a fixed element of $\mathcal{B}_\nu$ and for $0 \leq k \leq \nu$, let $\xi_k(x)$ be as defined in the proof of Theorem 6.1. We leave to the reader the simple proof that $\xi'_k(x) \geq 0$ and $\xi''_k(x) \geq 0$ for all $x \in \bar{H}$ and $0 \leq k \leq \nu$. Using (6.7) and (6.10), it follows that

$$\frac{D(b_\omega(x)^s)}{b_\omega(x)^s} = s \frac{D b_\omega(x)}{b_\omega(x)} = s \sum_{k=0}^{\nu-1} \frac{b'_{j_{k+1}}(\xi_k(x)) \xi'_k(x)}{b_{j_{k+1}}(\xi_k(x))} \geq s \frac{b'_{j-1}(x)}{b_{j-1}(x)} \geq 0. \tag{6.40}$$

Using (5.14) and taking the limit as $\nu \to \infty$, we conclude that $Dv_s(x)/v_s(x) \geq 0$ for all $x \in \bar{H}$. If, in addition, there exists a set F as in the statement of Theorem 6.5 and if $x \notin F$, it follows that

$$\inf \left\{ s \frac{b'_\beta(x)}{b_\beta(x)} : \beta \in \mathcal{B} \right\} := s\delta_1(x) > 0,$$

so (6.40) then implies that

$$\frac{D(b_\omega(x)^s)}{b_\omega(x)^s} \geq s\delta_1(x).$$

Again using (5.14) and letting $\nu \to \infty$, we conclude that $Dv_s(x) \geq s\delta_1(x) > 0$ for all $x \in \bar{H} \setminus F$. Because all terms in the summation in (6.40) are nonnegative, we conclude that

$$s^2 \left(\frac{D b_\omega(x)}{b_\omega(x)} \right)^2 \geq s^2 \sum_{k=0}^{\nu-1} \frac{[b'_{j_{k+1}}(\xi_k(x))]^2 [\xi'_k(x)]^2}{[b_{j_{k+1}}(\xi_k(x))]^2}. \tag{6.41}$$

If one replaces $s^2[D b_\omega(x)/b_\omega(x)]^2$ in (6.28) by the lower bound in (6.41) and if one then uses (6.29) and simplifies, one obtains

(6.42)

$$\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \geq \sum_{k=0}^{\nu-1} \frac{[s b''_{j_{k+1}}(\xi_k(x)) b_{j_{k+1}}(\xi_k(x)) - (s - s^2) b'_{j_{k+1}}(\xi_k(x))^2][\xi'_k(x)]^2}{[b_{j_{k+1}}(\xi_k(x))]^2} + s \sum_{k=0}^{\nu-1} \frac{b'_{j_{k+1}}(\xi_k(x)) \xi''_k(x)}{b_{j_{k+1}}(\xi_k(x))}.$$

If (6.37) is satisfied, one deduces from (6.42) that $D^2(b_\omega(x)^s)/b_\omega(x)^s \geq 0$; and again using (5.14) and letting $\nu \to \infty$, one obtains that $D^2 v_s(x) \geq 0$ for all $x \in \bar{H}$. If a set F exists and if $x \notin F$ and one only takes the term $k = 0$ in the summation in (6.42), then because we assume that strict inequality holds in (6.37) for all $\beta \in \mathcal{B}$ and all $x \notin F$, we find that there is a number $\delta_2(x; s) > 0$ such that

$$\frac{D^2(b_\omega(x)^s)}{b_\omega(x)^s} \geq \delta_2(x; s).$$

Again, using (5.14) and letting $\nu \to \infty$, this implies that for $x \notin F$,

$$\frac{D^2 v_s(x)}{v_s(x)} \geq \delta_2(x; s) > 0,$$

which completes the proof. $\square$

Example 3.6: To illustrate the methods of this section, we consider a simple example which nevertheless has some interest because of a failure of smoothness which makes techniques in [26] inapplicable. For $0 \leq \lambda \leq 1$, define

$$\theta_1(x) = \frac{1}{3+2\lambda}(x + \lambda x^{7/2}), \qquad \theta_2(x) = \theta_1(x) + \frac{2+\lambda}{3+2\lambda},$$

so $\theta_j : [0,1] \to [0,1]$, $\theta_1(0) = 0$, and $\theta_2(1) = 1$. For simplicity we suppress the dependence of $\theta_j(x)$ on λ in our notation. If $\lambda = 0$, one obtains the iterated function system which gives the *middle thirds* Cantor set. If $\mathcal{B} = \{1, 2\}$ and $\lambda > 0$ and $\omega = (j_1, j_2, \dots, j_\nu) \in \mathcal{B}_\nu$, notice that $D^3\theta_\omega(x)$ is defined and Hölder continuous for all $x \in [0,1]$; but if $j_1 = 1$, $D^4\theta_\omega(x)$ is not defined. If $0 \leq \lambda \leq 1$, one can check that $0 < \theta_j'(x) < 1$ for $0 \leq x \leq 1$; and it follows that there exists a unique compact, nonempty set $J_\lambda \subset [0,1]$ such that

$$J_\lambda = \theta_1(J_\lambda) \cup \theta_2(J_\lambda).$$

Note that J_0 is the *middle thirds* Cantor set.

For $\lambda \in [0,1]$ fixed, and $0 < s \leq 1$, let $X = C^2[0,1]$ and $Y = C[0,1]$, and define

$$b_1(x) := b_2(x) := b(x) := D\theta_1(x) = \frac{1}{3+2\lambda}(1 + \tfrac{7}{2}\lambda x^{5/2}).$$

As in Section 1, define $\Lambda_s : X \to X$ and $L_s : Y \to Y$ by the same formula:

$$(\Lambda_s(f))(x) = b(x)^s[f(\theta_1(x)) + f(\theta_2(x))]. \tag{6.43}$$

Theorem 5.1 implies that $r(L_s) = r(\Lambda_s)$; and it follows, for example, from theorems in [47] that the Hausdorff dimension of J_λ is the unique value of s, $0 < s \leq 1$, for which $r(\Lambda_s) = 1$.

If $f \in Y$ is a nonnegative function, we have for $0 \leq \lambda \leq 1$ that

$$(L_s(f))(x) \geq \Big(\frac{1}{3+2\lambda}\Big)^s [f(\theta_1(x)) + f(\theta_2(x))] \geq \Big(\frac{1}{5}\Big)^s [f(\theta_1(x)) + f(\theta_2(x))].$$

If $u(x) = 1$ for $0 \leq x \leq 1$, it follows that

$$L_s(u) \geq \Big(\frac{1}{5}\Big)^s (2u),$$

which implies that $r(L_s) \geq 2(1/5)^s$. If log denotes the natural logarithm and $0 \leq s < \log(2)/\log(5)$, it follows that $r(L_s) > 1$. Thus we restrict attention to $0 \leq \lambda \leq 1$ and $s \geq \log(2)/\log(5) \approx .4307$. A calculation gives, for $0 < x \leq 1$ that

$$b''(x)b(x) - (1-s)[b'(x)]^2 = (\tfrac{7}{2})(\tfrac{5}{2})\lambda\{(\tfrac{3}{2})x^{1/2} - (\tfrac{7}{2})\lambda[(\tfrac{3}{2}) - (\tfrac{5}{2})(1-s)]x^3\}$$

$$\geq (\tfrac{7}{2})\lambda\{(\tfrac{3}{2}) - (\tfrac{7}{2})\lambda[(\tfrac{3}{2}) - (\tfrac{5}{2})(1-s)]x^3\} > 0.$$

It follows from Theorem 6.5 that $Dv_s(u) > 0$ and $D^2 v_s(u) > 0$ for $0 < u \leq 1$. If $\lambda = 0$, v_s is a constant and $r(L_s) = (2/3^s)$, and one obtains the well-known result that the Hausdorff dimension of the Cantor set is $\log(2)/\log(3)$.

It remains to apply Theorems 6.1 and 6.4 in our example. Because $D\theta_j(x) = b(x)$ and $0 < b(x) \leq \kappa(\lambda) = (2+7\lambda)/(6+\lambda)$, we can define $\epsilon_\nu(\lambda) := \epsilon_\nu = \kappa(\lambda)^\nu$, where ϵ_ν is defined as in (6.5). Because $b_1(x) = b_2(x) = b(x) = (3+2\lambda)^{-1}(1 + (\tfrac{7}{2})\lambda x^{5/2})$, to compute $C_1(\lambda) = C_1$ as in (6.6), we need to compute

$$\begin{aligned} C_1(\lambda) : = C_1 &= \sup\{[(\tfrac{7}{2})\lambda(\tfrac{5}{2})x^{3/2})][1 + (\tfrac{7}{2})\lambda x^{5/2})]^{-1} : 0 \leq x \leq 1\} \\ &= (\tfrac{5}{2}) \sup\{[(\tfrac{7}{2})\lambda u^3][1 + (\tfrac{7}{2})\lambda u^5]^{-1} : 0 \leq u \leq 1\}. \end{aligned}$$

An elementary but tedious calculus argument, which we leave to the reader, yields

$$C_1(\lambda) = \begin{cases} [(\tfrac{5}{2})(\tfrac{7}{2})\lambda][1 + (\tfrac{7}{2})\lambda]^{-1}, & 0 < \lambda \leq \tfrac{3}{7} \\ (\tfrac{7\lambda}{2})(\tfrac{3}{7\lambda})^{3/5}, & \tfrac{3}{7} \leq \lambda \leq 1. \end{cases} \tag{6.44}$$

It also follows from Theorems 6.1 and 6.5 that for $0 < x \leq 1$,

$$\begin{aligned} 0 < \frac{Dv_s(x)}{v_s(x)} &\leq sC_1(\lambda)\Big(\sum_{\nu=0}^{\infty} \epsilon_\nu\Big) \\ &= sC_1(\lambda)[1 - \kappa(\lambda)]^{-1} = sC_1(\lambda)(6+4\lambda)/(4-3\lambda). \end{aligned} \tag{6.45}$$

If $C_2 = C_2(\lambda)$ is as in (6.23), we have to compute

$$\begin{aligned} C_2 := C_2(\lambda) &= \sup\{[(\tfrac{7}{2})\lambda(\tfrac{5}{2})(\tfrac{3}{2})x^{1/2})][1 + (\tfrac{7}{2})\lambda x^{5/2})]^{-1} : 0 \leq x \leq 1\} \\ &= (\tfrac{5}{2})(\tfrac{3}{2}) \sup\{[(\tfrac{7}{2})\lambda u][1 + (\tfrac{7}{2})\lambda u^5]^{-1} : 0 \leq u \leq 1\}. \end{aligned}$$

A simple calculus exercise yields

$$C_2(\lambda) = \begin{cases} [(\tfrac{15}{4})(\tfrac{7}{2})\lambda][1 + (\tfrac{7}{2})\lambda]^{-1}, & 0 < \lambda \leq \tfrac{1}{14} \\ 3(\tfrac{1}{4})^{1/5}[(\tfrac{7}{2})\lambda]^{4/5}, & \tfrac{1}{14} \leq \lambda \leq 1. \end{cases} \tag{6.46}$$

If we recall the definition of M_0, we also obtain from Example 3.6 that

$$M_0 = M_0(\lambda) = \sup\Big\{\frac{1}{(3+2\lambda)}\frac{7}{2}\lambda\frac{5}{2}x^{3/2} : 0 \leq x \leq 1\Big\} = \frac{(35\lambda)}{4}\frac{1}{(3+2\lambda)}. \tag{6.47}$$

If we now refer to Theorem 6.4, we find for $0 < x \leq 1$, $.4307 \leq s$, and $0 < \lambda \leq 1$, that

$$\begin{aligned} 0 < \frac{D^2 v_s(x)}{v_s(x)} &\leq s^2[C_1(\lambda)]^2\Big(\frac{6+4\lambda}{4-3\lambda}\Big) \\ &+ s\frac{(6+4\lambda)^2}{(4-3\lambda)(8+11\lambda)}\left[C_2(\lambda) + C_1(\lambda)M_0(\lambda)\frac{(6+4\lambda)}{(4-3\lambda)}\right]. \end{aligned} \tag{6.48}$$

As was shown in Section 3 (see Theorem 3.3 and Table 3.4), with the aid of (6.48), we can obtain rigorous, high accuracy estimates (upper and lower bounds) for the Hausdorff dimension of J_λ for $0 < \lambda \leq 1$.

7. The Case of Möbius Transformations

By working with partial derivatives and using methods like those in Section 6, it is possible to obtain explicit estimates on partial derivatives of $v_s(x)$ in the generality of Theorem 5.1. However, for reasons of length and in view of the immediate applications in this paper, we shall not treat the general case here and shall now specialize to the case that the mappings $\theta_\beta(\cdot)$ are given by Möbius transformations which map a given bounded open subset H of $\mathbb{C} := \mathbb{R}^2$ into H. Specifically, throughout this section we shall usually assume:

(H7.1): $\gamma \geq 1$ is a given real number and $\mathcal{B}$ is a finite collection of complex numbers β such that $\mathrm{Re}(\beta) \geq \gamma$ for all $\beta \in \mathcal{B}$. For each $\beta \in \mathcal{B}$, $\theta_\beta(z) := 1/(z+\beta)$ for $z \in \mathbb{C} \setminus \{-\beta\}$.

As we note in Remark 7.6, the assumption in (H7.1) that $\gamma \geq 1$ is only a convenience; and the results of this section can be proved under the weaker assumption that $\gamma > 0$.

For $\gamma > 0$ we define $G_\gamma \in \mathbb{C}$ by

$$G_\gamma = \{z \in \mathbb{C} : |z - 1/(2\gamma)| < 1/(2\gamma)\}. \tag{7.1}$$

It is easy to check that if $w \in \mathbb{C}$ and $\mathrm{Re}(w) > \gamma$, then $(1/w) \in G_\gamma$. It follows that if $\mathrm{Re}(z) > 0$, $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq \gamma > 0$, then $\theta_\beta(z) \in \bar{G}_\gamma$. Let H be a bounded, open, mildly regular subset of $\mathbb{C} = \mathbb{R}^2$ such that $H \supset G_\gamma$ and $H \subset \{z \mid \mathrm{Re}(z) > 0\}$, and let $\mathcal{B}$ denote a finite set of complex numbers such that $\mathrm{Re}(\beta) \geq \gamma > 0$ for all $\beta \in \mathcal{B}$. We define a bounded linear map $\Lambda_s : C^m(\bar{H}) \to C^m(\bar{H})$, where m is a positive integer and $s \geq 0$, by

$$(\Lambda_s(f))(z) = \sum_{\beta \in \mathcal{B}} \left| \frac{d}{dz} \theta_\beta(z) \right|^s f(\theta_\beta(z)) := \sum_{\beta \in \mathcal{B}} \frac{1}{|z+\beta|^{2s}} f(\theta_\beta(z)). \tag{7.2}$$

As in Section 1, $L_s : C(\bar{H}) \to C(\bar{H})$ is defined by (7.2). We use different letters to emphasize that $\sigma(\Lambda_s) \neq \sigma(L_s)$, although $r(\Lambda_s) = r(L_s)$.

If all elements of $\mathcal{B}$ are real, we can restrict attention to the real line and, as we shall see, the analysis is much simpler. In this case we abuse notation and take $G_\gamma = (0, 1/\gamma) \subset \mathbb{R}^2$ and $H = (0, a)$, $a \geq 1/\gamma$. For $f \in C^m(\bar{H})$ and $x \in \bar{H}$, (7.2) takes the form

$$(\Lambda_s(f))(x) = \sum_{\beta \in \mathcal{B}} \frac{1}{(x+\beta)^{2s}} f(\theta_\beta(x)). \tag{7.3}$$

If, for $1 \leq j \leq n$, $M_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ is a 2×2 matrix with complex entries and $\det(M_j) = a_j d_j - b_j c_j$, define a Möbius transformation $\psi_j(z) = (a_j z + b_j)/(c_j z + d_j)$. It is well-known that

$$(\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n)(z) = (A_n z + B_n)/(C_n z + D_n), \tag{7.4}$$

where

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = M_1 M_2 \cdots M_n. \tag{7.5}$$

If $\mathcal{B}$ is a finite set of complex numbers β such that $\mathrm{Re}(\beta) \geq \gamma > 0$ for all $\beta \in \mathcal{B}$, we define $\mathcal{B}_\nu$ as before by

$$\mathcal{B}_\nu = \{\omega = (\beta_1, \beta_2, \dots, \beta_\nu) \,|\, \beta_j \in \mathcal{B} \text{ for } 1 \leq j \leq \nu\}$$

and $\theta_\omega = \theta_{\beta_n} \circ \theta_{\beta_{n-1}} \cdots \theta_{\beta_1}$. Given $\omega = (\beta_1, \beta_2, \dots, \beta_\nu) \in \mathcal{B}_\nu$, we define

$$\tilde{\omega} = (\beta_\nu, \beta_{\nu-1}, \dots, \beta_1) \tag{7.6}$$

so

$$\theta_{\tilde{\omega}} = \theta_{\beta_1} \circ \theta_{\beta_2} \cdots \theta_{\beta_n}. \tag{7.7}$$

For Λ_s as in (7.2) $\nu \geq 1$, and $f \in C^m(\bar{H})$, recall that

$$(\Lambda_s^\nu(f))(z) = \sum_{\omega \in \mathcal{B}_\nu} \Big|\frac{d\theta_\omega(z)}{dz}\Big|^s f(\theta_\omega(z)) = \sum_{\omega \in \mathcal{B}_\nu} \Big|\frac{d\theta_{\tilde{\omega}}(z)}{dz}\Big|^s f(\theta_{\tilde{\omega}}(z)). \tag{7.8}$$

The following lemma allows us to apply Theorem 5.1 to Λ_s in (7.2).

Lemma 7.1. *(Compare Remark 7.13.) Let β_1 and β_2 be complex numbers with $\mathrm{Re}(\beta_j) \geq \gamma \geq 1$ for $j = 1, 2$. If $\psi_j(z) = 1/(z + \beta_j)$ for $\mathrm{Re}(z) \geq 0$ and $\theta = \psi_1 \circ \psi_2$, then for all z, w with $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(w) \geq 0$,*

$$|\theta(z) - \theta(w)| \leq (\gamma^2 + 1)^{-2} |z - w|. \tag{7.9}$$

Proof. It suffices to prove that $|(d\theta/dz)(z)| \leq (\gamma^2+1)^{-2}$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Using (7.4) and (7.5) we see that

$$|(d\theta/dz)(z)| = |\beta_1|^{-2} |z + (1/\beta_1) + \beta_2|^{-2},$$

so it suffices to prove that $|\beta_1|^2 \, |z + (1/\beta_1) + \beta_2|^2 \geq (\gamma^2 + 1)^2$ for $\mathrm{Re}(z) \geq 0$. If we write $\beta_1 = u + iv$ with $u \geq \gamma$,

$$\mathrm{Re}(z + (1/\beta_1) + \beta_2) \geq u/(u^2 + v^2) + \gamma,$$

so

$$|z + (1/\beta_1) + \beta_2|^2 \geq [u/(u^2 + v^2) + \gamma]^2$$

and

$$\begin{aligned} |\beta_1|^2 \, |z + (1/\beta_1) + \beta_2|^2 &\geq (u^2 + v^2)\Big[\frac{u^2}{(u^2+v^2)^2} + \frac{2u\gamma}{(u^2+v^2)} + \gamma^2\Big] \\ &= \frac{u^2}{(u^2+v^2)} + 2u\gamma + \gamma^2(u^2+v^2) = g(u, v). \end{aligned}$$

Because $u \geq \gamma$, $g(u, 0) = 1 + 2\gamma^2 + \gamma^4 = (\gamma^2 + 1)^2$. Using the fact that $u \geq \gamma \geq 1$, we also see that for $v \geq 0$

$$\frac{\partial g(u,v)}{\partial v} = \frac{-u^2(2v)}{(u^2+v^2)^2} + 2\gamma^2 v \geq 0,$$

which implies that $g(u, v) \geq g(u, 0) = (\gamma^2 + 1)^2$ for $u \geq \gamma$ and $v \geq 0$. Since $g(u, -v) = g(u, v)$, $g(u, v) \geq (\gamma^2 + 1)^2$ for $v \leq 0$ and $u \geq \gamma$. □

With the aid of Lemma 7.1, the following theorem is an immediate corollary of Theorem 5.1.

Theorem 7.2. *Assume (H7.1) and let H be a bounded, open mildly regular subset of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ such that $H \supset G_\gamma$, where G_γ is defined by (7.1). For a given positive integer m and for $s > 0$, let $X = C^m(\bar{H})$ and $Y = C(\bar{H})$ and let $\Lambda_s : X \to X$ and $L_s : Y \to Y$ be given by (7.2). If $r(\Lambda_s)$ (respectively, $r(L_s)$) denote the spectral radius of Λ_s (respectively, L_s), we have $r(\Lambda_s) > 0$ and $r(\Lambda_s) = r(L_s)$. If $\rho(\Lambda_s)$ denotes the essential spectral radius of Λ_s,*

$$\rho(\Lambda_s) \leq (\gamma^2 + 1)^{-m} r(\Lambda_s). \tag{7.10}$$

For each $s > 0$, there exists $v_s \in X$ such that $v_s(z) > 0$ for all $z \in \bar{H}$ and $\Lambda_s(v_s) = r(\Lambda_s)v_s$. All the statements of Theorem 5.1 are true in this context whenever Λ and L in Theorem 5.1 are replaced by Λ_s and L_s respectively.

In the notation of Theorem 7.2, it follows from (5.14) that for any multi-index $\alpha = (\alpha_1, \alpha_2)$ and for $z = x + iy = (x, y)$

$$\lim_{\nu \to \infty} \frac{D^\alpha \left(\sum_{\omega \in \mathcal{B}_\nu} \left| \frac{d}{dz} \theta_\omega(z) \right|^s \right)}{\sum_{\omega \in \mathcal{B}_\nu} \left| \frac{d}{dz} \theta_\omega(z) \right|^s} = \frac{D^\alpha v_s(x, y)}{v_s(x, y)}, \tag{7.11}$$

where the convergence is uniform in $(x, y) := z \in \bar{H}$ and $D^\alpha = (\partial/\partial x)^{\alpha_1} (\partial/\partial y)^{\alpha_2}$.

Lemma 7.3. *Let β_j, $j \geq 1$ be a sequence of complex numbers with $\mathrm{Re}(\beta_j) \geq \gamma \geq 0$ for all j. For complex numbers z, define $\theta_{\beta_j}(z) = (z + \beta_j)^{-1}$ and define matrices $M_j = \left(\begin{smallmatrix} 0 & 1 \\ 1 & \beta_j \end{smallmatrix}\right)$. Then for $n \geq 1$,*

$$M_1 M_2 \cdots M_n = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}, \tag{7.12}$$

where $A_0 = 0$, $A_1 = 1$, $B_0 = 1$, $B_1 = \beta_1$ and for $n \geq 1$,

$$A_{n+1} = A_{n-1} + \beta_{n+1} A_n \text{ and } B_{n+1} = B_{n-1} + \beta_{n+1} B_n. \tag{7.13}$$

Also,

$$(\theta_{\beta_1} \circ \theta_{\beta_2} \cdots \theta_{\beta_n})(z) = (A_{n-1} z + A_n)/(B_{n-1} z + B_n),$$

and we have

$$\mathrm{Re}(B_n / B_{n-1}) \geq \gamma \tag{7.14}$$

and

$$\left| \frac{d}{dz} \left[\frac{A_{n-1} z + A_n}{B_{n-1} z + B_n} \right] \right|^s = |B_{n-1}|^{-2s} |z + B_n / B_{n-1}|^{-2s}. \tag{7.15}$$

Proof. Equation (7.12) follows by induction on n. It is obviously true for $n = 1$. If we assume that (7.12) is satisfied for some $n \geq 1$, then

$$M_1 M_2 \cdots M_n M_{n+1} = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \beta_{n+1} \end{pmatrix} = \begin{pmatrix} A_n & A_{n-1} + \beta_{n+1} A_n \\ B_n & B_{n-1} + \beta_{n+1} B_n \end{pmatrix},$$

which proves (7.12) with A_{n+1} and B_{n+1} defined by (7.13). Similarly, we prove (7.14) by induction on n. The case $n = 1$ is obvious, Assuming that (7.13) is satisfied for some $n \geq 1$, we obtain from (7.13) that

$$B_{n+1}/B_n = B_{n-1}/B_n + \beta_{n+1}.$$

Because $\mathrm{Re}(w) \ge \gamma$, where $w := B_n/B_{n-1}$, we see that $|1/w - 1/(2\gamma)| \le 1/(2\gamma)$ and $\mathrm{Re}(1/w) = \mathrm{Re}(B_{n-1}/B_n) \ge 0$, so

$$\mathrm{Re}(B_{n+1}/B_n) \ge \mathrm{Re}(B_{n-1}/B_n) + \mathrm{Re}(\beta_{n+1}) \ge \gamma.$$

Hence (7.13) is satisfied for all $n \ge 1$. Because $\det(M_j) = -1$ for all $j \ge 1$, we get that $\det\left(\begin{smallmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{smallmatrix}\right) = (-1)^n$, and (7.15) follows. $\square$

Before proceeding further, it will be convenient to establish some elementary calculus propositions. For $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $s > 0$, define

$$G(u,v;s) = (u^2+v^2)^{-s}.$$

Define $D_1 = (\partial/\partial u)$, so $D_1^m = (\partial/\partial u)^m$ for positive integers m; similarly, let $D_2 = (\partial/\partial v)$ and $D_2^m = (\partial/\partial v)^m$.

Lemma 7.4. *For positive integers m, there exist polynomials in u and v with coefficients depending on s, $P_m(u,v;s)$ and $Q_m(u,v;s)$, such that*

$$D_1^m G(u,v;s) = P_m(u,v;s)G(u,v;s+m), \; D_2^m G(u,v;s) = Q_m(u,v;s)G(u,v;s+m).$$

Furthermore, we have $P_1(u,v;s) = -2su$, $Q_1(u,v;s) = -2sv$, and for positive integers m,

$$P_{m+1}(u,v;s) = (u^2+v^2)(D_1P_m(u,v;s)) - 2(s+m)uP_m(u,v;s)$$

and

$$Q_{m+1}(u,v;s) = (u^2+v^2)(D_2Q_m(u,v;s)) - 2(s+m)vQ_m(u,v;s).$$

Proof. If $m = 1$,

$$D_1G(u,v;s) = (-2su)\,G(u,v;s+1), \qquad D_2G(u,v;s) = (-2sv)(u^2+v^2;s+1),$$

so $P_1(u,v;s) = -2su$ and $Q_1(u,v;s) = -2sv$.

We now argue by induction and assume we have proved the existence of $P_j(u,v;s)$ and $Q_j(u,v;s)$ for $1 \le j \le m$. It follows that

$$\begin{aligned} &D_1^{m+1}G(u,v;s) = D_1[P_m(u,v;s)G(u,v;s+m)] \\ &= [D_1P_m(u,v;s)]G(u,v;s+m)] + P_m(u,v;s)[-2(s+m)u]G(u,v;s+m+1) \\ &\qquad = [(u^2+v^2)(D_1P_m(u,v;s)) - 2(s+m)uP_m(u,v;s)]G(u,v;s+m+1). \end{aligned}$$

This proves the lemma with

$$P_{m+1}(u,v;s) := (u^2+v^2)(D_1P_m(u,v;s)) - 2(s+m)uP_m(u,v;s).$$

An exactly analogous argument, which we leave to the reader, shows that

$$Q_{m+1}(u,v;s) := (u^2+v^2)(D_2Q_m(u,v;s)) - 2(s+m)vQ_m(u,v;s).$$

$\square$

An advantage of working with Möbius transformations is that one can easily obtain tractable formulas for expressions like $(\theta_{\beta_1} \circ \theta_{\beta_2} \cdots \theta_{\beta_n})(z)$. Such formulas allow more precise estimates for the left hand side of (5.14) than we obtained in Section 6.

Lemma 7.5. *In the notation of Lemma 7.4, for all* $(u, v) \in \mathbb{R}^2 \setminus \{(0,0)\}$*, for all* $s > 0$*, and all positive integers* m*,* $P_m(u, v; s) = Q_m(v, u; s)$*.*

Proof. Fix $s > 0$. We have $P_1(u, v; s) = Q_1(v, u; s)$ for all $(u, v) \neq (0, 0)$. Arguing by mathematical induction, assume that for some positive integer m we have proved that $P_m(u, v; s) = Q_m(v, u; s)$ for all $(u, v) \neq (0, 0)$. For a fixed $(u, v) \neq (0, 0)$, we obtain, by virtue of the recursion formula in Lemma 7.4,

$$\begin{aligned} P_{m+1}(v, u; s) &= (u^2 + v^2) \lim_{\Delta v \to 0} \frac{P_m(v + \Delta v, u; s) - P_m(v, u; s)}{\Delta v} \\ &\quad - 2(s + m) v P_m(v, u; s) \\ &= (u^2 + v^2) \lim_{\Delta v \to 0} \frac{Q_m(u, v + \Delta v; s) - Q_m(u, v; s)}{\Delta v} \\ &\quad - 2(s + m) v Q_m(u, v; s) \\ &= Q_{m+1}(u, v; s). \end{aligned}$$

By mathematical induction, we conclude that $P_n(u, v; s) = Q_n(v, u; s)$ for all positive integers n. ∎

Remark 7.1. *By using the recursion formula in Lemma 7.4, one can easily compute* $P_j(u, v; s)$ *for* $1 \leq j \leq 4$*.*

$$\begin{aligned} &P_1(u, v; s) = -2su, \\ &P_2(u, v; s) = 2s(2s + 1)u^2 - 2sv^2, \\ &P_3(u, v; s) = -2s(2s + 1)(2s + 2)u^3 + (2s)(2s + 2)(3)uv^2, \\ &P_4(u, v; s) = (2s)(2s + 2)[(2s + 1)(2s + 3)u^4 - 6(2s + 3)u^2v^2 + 3v^4]. \end{aligned}$$

By virtue of Lemma 7.5, we also obtain formulas for $Q_j(v, u; s) = P_j(u, v; s)$. Also, Lemmas 7.4 and 7.5 imply that

$$\frac{D_1^j G(u, v; s)}{G(u, v; s)} = \frac{P_j(u, v; s)}{(u^2 + v^2)^j}, \qquad \frac{D_2^j G(u, v; s)}{G(u, v; s)} = \frac{P_j(v, u; s)}{(u^2 + v^2)^j}$$

and the latter formulas will play a useful role in this section. In particular, for a given constant $\gamma > 0$, we shall need good estimates for

$$\sup \left\{ \frac{D_k^j G(u, v; s)}{G(u, v; s)} : u \geq \gamma, v \in \mathbb{R} \right\} \text{ and } \inf \left\{ \frac{D_k^j G(u, v; s)}{G(u, v; s)} : u \geq \gamma, v \in \mathbb{R} \right\}$$

where $k = 1, 2$ and $1 \leq j \leq 4$. Although the arguments used to prove these estimates are elementary, these results will play a crucial role in our later work.

Lemma 7.6. *Let* $\gamma > 0$ *be a given constant and assume that* $u \geq \gamma$ *and* $v \in \mathbb{R}$*. Let* $D_1 = (\partial / \partial u)$ *and* $G(u, v; s) = (u^2 + v^2)^{-s}$*, where* $s > 0$*. For* $j \geq 1$ *we have*

$$\frac{D_1^j G(u, v; s)}{G(u, v; s)} = \frac{P_j(u, v; s)}{(u^2 + v^2)^j},$$

where $P_j(u, v; s)$ *is as defined in Remark 7.1; and the following estimates are satisfied.*

$$-\frac{2s}{\gamma} \leq \frac{D_1 G(u, v; s)}{G(u, v; s)} < 0,$$

$$-\frac{s}{4\gamma^2(s+1)} \le \frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)}{\gamma^2},$$
$$-\frac{2s(2s+1)(2s+2)}{\gamma^3} \le \frac{D_1^3 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+2)}{\gamma^3(s+2)^2},$$
$$-\frac{2s(s+1)(2s+2)(3)}{\gamma^4} \le \frac{D_1^4 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)(2s+2)(2s+3)}{\gamma^4}.$$

Proof. By Remark 7.1,

$$\frac{D_1^j G(u,v;s)}{G(u,v;s)} = \frac{P_j(u,v;s)}{(u^2+v^2)^j},$$

and Remark 7.1 provides formulas for $P_j(u,v;s)$. It follows that

$$\frac{D_1^j G(u,v;s)}{G(u,v;s)} = \frac{-2su}{u^2+v^2} < 0.$$

Since

$$\frac{2su}{u^2+v^2} \le \frac{2su}{u^2} \le \frac{2s}{\gamma},$$

we also see that

$$\frac{D_1 G(u,v;s)}{G(u,v;s)} \ge -\frac{2s}{\gamma}.$$

Using Remark 7.1, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1)u^2 - 2sv^2}{(u^2+v^2)^2},$$

so

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)u^2}{(u^2+v^2)^2}.$$

Since

$$\frac{u^2}{(u^2+v^2)^2} \le \frac{u^2}{u^4} \le \frac{1}{\gamma^2},$$

we find that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)}{\gamma^2},$$

If we write $v^2 = \rho u^2$, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1-\rho)}{u^2(1+\rho)^2},$$

and if $0 \le \rho \le 2s+1$, we obtain the upper bound given above and a lower bound of zero. If $\rho > 2s+1$, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \ge \frac{2s}{\gamma^2} \inf\left\{\frac{2s+1-\rho}{(1+\rho)^2} : \rho > 2s+1\right\}.$$

It is a simple calculus exercise to show that

$$\inf\left\{\frac{2s+1-\rho}{(1+\rho)^2} : \rho > 2s+1\right\} = -\frac{1}{8(s+1)},$$

achieved at $\rho = 4s+3$; and this gives the lower estimate $-s/[4\gamma^2(s+1)]$ of the lemma.

Using Remark 7.1 again, we see that

$$\frac{D_1^3 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+2)u[-(2s+1)u^2+3v^2]}{(u^2+v^2)^3}.$$

It follows that

$$\begin{aligned}\frac{D_1^3 G(u,v;s)}{G(u,v;s)} &\geq -2s(2s+1)(2s+2)\left[\frac{u}{(u^2+v^2)}\right]^3 \\ &\geq -2s(2s+1)(2s+2)\left[\frac{1}{u}\right]^3 \geq -2s(2s+1)(2s+2)\frac{1}{\gamma^3}.\end{aligned}$$

On the other hand, if we write $v^2 = \rho u^2$, then

$$\begin{aligned}\frac{D_1^3 G(u,v;s)}{G(u,v;s)} &= \frac{2s(2s+2)}{u^3}\frac{[3\rho-(2s+1)]}{(1+\rho)^3} \\ &\leq \frac{2s(2s+2)}{\gamma^3}\sup\left\{\frac{3\rho-(2s+1)}{(1+\rho)^3} : \rho \geq 0\right\}.\end{aligned}$$

Once again, a straightforward calculus argument shows that

$$\sup\left\{\frac{3\rho-(2s+1)}{(1+\rho)^3} : \rho \geq 0\right\} = \frac{1}{(s+2)^2}$$

and the supremum is achieved at $\rho = s+1$. Using this fact, we obtain the upper estimate of the lemma.

Finally, we obtain from Remark 7.1 that

$$\frac{D_1^4 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+2)[(2s+1)(2s+3)u^4 - 6(2s+3)u^2v^2 + 3v^4]}{(u^2+v^2)^4}.$$

Dropping the negative term in the numerator and observing that $3 \leq (2s+1)(2s+3)$ and $u^4+v^4 \leq (u^2+v^2)^2$, we see that

$$\begin{aligned}\frac{D_1^4 G(u,v;s)}{G(u,v;s)} &\leq \frac{(2s)(2s+1)(2s+2)(2s+3)(u^4+v^4)}{(u^2+v^2)^4} \\ &\leq \frac{(2s)(2s+1)(2s+2)(2s+3)}{(u^2+v^2)^2} \leq \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}.\end{aligned}$$

On the other hand, because $-u^4-v^4 \leq -2u^2v^2$, we obtain that

$$\begin{aligned}-\frac{D_1^4 G(u,v;s)}{G(u,v;s)} &\leq \frac{(2s)(2s+2)[-3u^4+6(2s+3)u^2v^2-3v^4]}{(u^2+v^2)^4} \\ &\leq \frac{3(2s)(2s+2)[-2u^2v^2+(4s+6)u^2v^2]}{(u^2+v^2)^4} \\ &\leq \frac{3(2s)(2s+2)[4(s+1)(u^2+v^2)^2/4]}{(u^2+v^2)^4} \\ &\leq \frac{3(2s)(2s+2)(s+1)}{(u^2+v^2)^2} \leq \frac{3(2s)(2s+2)(s+1)}{\gamma^4},\end{aligned}$$

which gives the lower estimate of Lemma 7.6. □

The following lemma gives analogous estimates for

$$\frac{D_2^j G(u,v;s)}{G(u,v;s)} = \frac{P_j(v,u;s)}{(u^2+v^2)^j}.$$

Lemma 7.7. *Let $\gamma > 0$ be a given real number, $D_2 = (\partial/\partial v)$ and for $s > 0$ and $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$, define $G(u,v;s) = (u^2+v^2)^{-s}$, If $u \geq \gamma$ and $v \in \mathbb{R}$, we have the following estimates.*

$$\frac{|D_2 G(u,v;s)|}{G(u,v;s)} \leq \frac{s}{\gamma},$$

$$-\frac{2s}{\gamma^2} \leq \frac{D_2^2 G(u,v;s)}{G(u,v;s)} \leq \frac{2s(2s+1)}{4\gamma^2},$$

$$\frac{|D_2^3 G(u,v;s)|}{G(u,v;s)} \leq \frac{2s(2s+2)}{\gamma^3} \max\left\{\frac{25\sqrt{5}}{72}, \frac{2s+1}{8}\right\}$$

$$-\frac{2s(s+1)(2s+2)(3)}{\gamma^4} \leq \frac{D_2^4 G(u,v;s)}{G(u,v;s)} \leq \frac{2s(2s+1)(2s+2)(2s+3)}{\gamma^4}.$$

Proof. By Remark 7.1, $P_1(v,u;s) = -2sv$, so

$$\frac{|D_2 G(u,v;s)|}{G(u,v;s)} = \frac{2s|v|}{u^2+v^2}.$$

The map $w \mapsto w/(u^2+w^2)$ has its maximum on $[0,\infty)$ at $w = u$, so $(2s|v|/(u^2+v^2) \leq s/u \leq s/\gamma$; and we obtain the first inequality in Lemma 7.7. Using Remark 7.1 again, we see that

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = \frac{2s[(2s+1)v^2 - u^2]}{(u^2+v^2)^2}.$$

It follows that

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = 2s(2s+1)\frac{|v|^2}{(u^2+v^2)^2}.$$

The map $v \mapsto |v|/(u^2+v^2)$ has its maximum at $|v| = u$, so $[|v|/(u^2+v^2)]^2 \leq 1/(4u^2) \leq 1/(4\gamma^2)$, and

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1)}{4\gamma^2}.$$

Similarly, one obtains

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} \geq -\frac{2su^2}{(u^2+v^2)^2} \geq -\frac{2s}{u^2} \geq -\frac{2s}{\gamma^2}.$$

With the aid of Remark 7.1 again, we see that

$$\frac{D_2^3 G(u,v;s)}{G(u,v;s)} = 2s(2s+2)v\frac{[-(2s+1)v^2 + 3u^2]}{(u^2+v^2)^3} := A(u,v).$$

For a fixed $u \geq \gamma$, $v \mapsto A(u,v)$ is an odd function of v, so if $\alpha(u) = \sup\{A(u,v) : v \in \mathbb{R}\}$, $-\alpha(u) = \inf\{A(u,v) : v \in \mathbb{R}\}$. If $v \leq 0$,

$$A(u,v) \leq (2s)(2s+1)(2s+2)\left[\frac{|v|}{u^2+v^2}\right]^3 \leq (2s)(2s+1)(2s+2)\left[\frac{u}{2u^2}\right]^3$$

$$\leq \frac{(2s)(2s+1)(2s+2)}{8\gamma^3}.$$

If $v > 0$,

$$A(u,v) \leq (2s)(2s+2)(3u^2)\frac{v}{(u^2+v^2)^3}.$$

A calculation shows that $v \mapsto v/(u^2+v^2)^3$ achieves its maximum for $v \geq 0$ at $v = u/\sqrt{5}$, so for $v > 0$,

$$A(u,v) \leq (2s)(2s+2)(3u^{-3})[\sqrt{5}(6/5)^3]^{-1} \leq (2s)(2s+2)\gamma^{-3}(25\sqrt{5}/72).$$

Note that $25\sqrt{5}/72 \approx .7764 < 1$. Using Remark 7.1 again, we see that

$$\frac{D_2^4 G(u,v;s)}{G(u,v;s)} = 2s(2s+2)\frac{[(2s+1)(2s+3)v^4 - 6(2s+3)u^2v^2 + 3u^4]}{(u^2+v^2)^4}.$$

Since $u^4 + v^4 \leq (u^2+v^2)^2$, it follows easily that

$$\frac{D_2^4 G(u,v;s)}{G(u,v;s)} \leq 2s(2s+2)(2s+1)(2s+3)\frac{u^4+v^4}{(u^2+v^2)^4} \leq 2s(2s+2)(2s+1)(2s+3)\gamma^{-4}.$$

Similarly, we see that

$$\begin{aligned}(2s+1)(2s+3)v^4 - 6(2s+3)u^2v^2 + 3u^4 &\geq 3(u^4+v^4) - 6(2s+3)[(u^2+v^2)/2]^2 \\ &\geq 3(u^2+v^2)^2 - 6[(u^2+v^2)/2]^2 - 6(2s+3)[(u^2+v^2)/2]^2.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{D_2^4 G(u,v;s)}{G(u,v;s)} &\geq 2s(2s+2)\frac{3 - 3/2 - 3/2(2s+3)}{(u^2+v^2)^2} \\ &\geq -(2s)(2s+2)3(s+1)(u^2+v^2)^{-2} \geq -(2s)(2s+2)(3s+3)\gamma^{-4},\end{aligned}$$

which completes the proof of Lemma 7.7. Note that $(2s)(2s+1)(2s+2)(2s+3) \geq 2s(2s+2)(3s+3)$. ☐

Remark 7.2. *Lemmas 7.6 and 7.7 show that whenever $u \geq \gamma > 0$, $s > 0$, $k = 1$ or $k = 2$, and $1 \leq j \leq 4$,*

$$\frac{|D_k^j G(u,v;s)|}{G(u,v;s)} \leq (2s)(2s+1)\cdots(2s+j-1)\gamma^{-j}.$$

We have not determined whether the above inequality holds for all $j \geq 1$.

Using Lemmas 7.6 and 7.7, we can give uniform estimates for the quantities $(\partial/\partial x)^j v_s(x,y)/v_s(x,y)$ and $(\partial/\partial y)^j v_s(x,y)/v_s(x,y)$, where $s > 0$, $1 \leq j \leq 4$, and $v_s(x,y)$ is the unique (to within normalization) strictly positive eigenvector of the linear operator $\Lambda_s : C^m(\bar{H}) \to C^m(\bar{H})$ in (7.2) for $m \geq 1$.

Theorem 7.8. *Let $\mathcal{B}$ be a finite set of complex numbers β such that* $\mathrm{Re}(\beta) \geq \gamma \geq 1$ *for all $\beta \in \mathcal{B}$. For $\beta \in \mathcal{B}$ and $s > 0$, define $\theta_\beta(z) = (z+\beta)^{-1}$. Let H be a bounded, mildly regular open subset of $\mathbb{C} := \mathbb{R}^2$ such that $H \supset G_\gamma = \{z \in \mathbb{C} : |z - 1/(2\gamma)| < 1/(2\gamma)\}$, and* $\mathrm{Re}(z) > 0$ *for all $z \in H$, so $\theta_\beta(H) \subset G_\gamma$ for all $\beta \in \mathcal{B}$. For a positive* *integer m, define a real Banach space $C^m(\bar{H}) = X$ and defined a bounded linear operator $\Lambda_s : X \to X$ by*

$$(\Lambda_s f)(z) = \sum_{\beta \in \mathcal{B}} \Big| \frac{d}{dz} \theta_\beta(z) \Big|^s f(\theta_\beta(z)).$$

Then Λ_s has a unique (to within normalization) positive eigenvector $v_s \in X$ and $v_s \in C^\infty$. Furthermore, we have the following estimates for $(x, y) \in \bar{H}$.

$$-\frac{2s}{\gamma} \le \frac{\partial v_s(x,y)}{\partial x} [v_s(x,y)]^{-1} \le 0, \tag{7.16}$$

$$-\frac{s}{4\gamma^2(s+1)} \le \frac{\partial^2 v_s(x,y)}{\partial x^2} [v_s(x,y)]^{-1} \le \frac{2s(2s+1)}{\gamma^2}, \tag{7.17}$$

$$-\frac{2s(2s+1)(2s+2)}{\gamma^3} \le \frac{\partial^3 v_s(x,y)}{\partial x^3} [v_s(x,y)]^{-1} \le \frac{(2s)(2s+2)}{\gamma^3(s+2)^2}, \tag{7.18}$$

$$-\frac{2s(2s+2)(3s+3)}{\gamma^4} \le \frac{\partial^4 v_s(x,y)}{\partial x^4} [v_s(x,y)]^{-1} \le \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}, \tag{7.19}$$

$$\Big| \frac{\partial v_s(x,y)}{\partial y} \Big| [v_s(x,y)]^{-1} \le \frac{s}{\gamma}, \tag{7.20}$$

$$-\frac{2s}{\gamma^2} \le \frac{\partial^2 v_s(x,y)}{\partial y^2} [v_s(x,y)]^{-1} \le \frac{2s(2s+1)}{4\gamma^2}, \tag{7.21}$$

$$\Big| \frac{\partial^3 v_s(x,y)}{\partial y^3} \Big| [v_s(x,y)]^{-1} \le \frac{(2s)(2s+2)}{\gamma^3} \max\{25\sqrt{5}/72, (2s+1)/8\}, \tag{7.22}$$

$$-\frac{2s(2s+2)(3s+3)}{\gamma^4} \le \frac{\partial^4 v_s(x,y)}{\partial y^4} [v_s(x,y)]^{-1} \le \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}. \tag{7.23}$$

Hence, if $D_1 = \partial/\partial x$ and $D_2 = \partial/\partial y$, we have for $k = 1, 2$ and $1 \le j \le 4$ that

$$\frac{|D_k^j v_s(x,y)|}{v_s(x,y)} \le \frac{(2s)(2s+1)\cdots(2s+j-1)}{\gamma^j}. \tag{7.24}$$

Proof. For any integer $m \ge 1$, we can view Λ_s as a bounded linear operator of $C^m(\bar{H})$ to $C^m(\bar{H})$. We know that Λ_s has a strictly positive eigenvector $v_s(x,y) \in C^m(\bar{H})$ such that $\sup\{v_s(x,y) \,|\, (x,y) \in \bar{H}\} = 1$. By the uniqueness of this eigenvector, $v_s(x,y)$ must actually be C^∞.

Using the notation of (7.6) and (7.7) and also using (7.15) in Lemma 7.3, we see that

$$\Big| \frac{d}{dz} \theta_{\tilde{\omega}}(z) \Big|^s = |B_{n-1}|^{-2s} |z + B_n/B_{n-1}|^{-2s}.$$

By Lemma 7.3, $\mathrm{Re}(B_n/B_{n-1}) \ge \gamma_\omega \ge \gamma$, so writing $\mathrm{Im}(B_n/B_{n-1}) = \delta_\omega$, we obtain that for $k = 1, 2$ and $1 \le j$,

$$\begin{aligned} D_k^j \left(\Big| \frac{d}{dz} \theta_{\tilde{\omega}}(z) \Big|^s \right) \Big| \frac{d}{dz} \theta_{\tilde{\omega}}(z) \Big|^s & \\ = D_k^j \Big[(x + \gamma_\omega)^2 + (y + \delta_\omega)^2 \Big]^{-s} & \Big[(x + \gamma_\omega)^2 + (y + \delta_\omega)^2 \Big]^s. \end{aligned} \tag{7.25}$$

However, if we write $(x+\gamma_\omega) = u \geq \gamma$ and $(y+\delta_\omega) = v$, we see that

$$\Big(\Big(\frac{\partial}{\partial x}\Big)^j \Big[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\Big]^{-s}\Big) \Big[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\Big]^{-s} = \Big[\Big(\frac{\partial}{\partial u}\Big)^j G(u,v;s)\Big] \left[G(u,v;s\right]^{-1}, \tag{7.26}$$

where the right hand side of the above equation is evaluated at $u = x+\gamma_\omega$ and $v = y+\delta_\omega$. If we combine (7.25) and (7.26) with the estimates in Lemma 7.6 and if we then use (7.11), we obtain the estimates on $(\partial/\partial x)^j v_s(x,y)$ given in (7.16) - (7.19).

Similarly, we have

$$\Big(\Big(\frac{\partial}{\partial y}\Big)^j \Big[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\Big]^{-s}\Big) \Big[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\Big]^{-s} = \Big[\Big(\frac{\partial}{\partial v}\Big)^j G(u,v;s)\Big] \left[G(u,v;s\right]^{-1}. \tag{7.27}$$

If we combine (7.25) and (7.27) with the estimates in Lemma 7.7 and if we then use (7.11), we obtain the estimates on $(\partial/\partial y)^j v_s(x,y)$ given in (7.20) - (7.23). ☐

Remark 7.3. *It turns out that exactly the same estimates given in Theorem 7.8 hold for a more general class of Perron-Frobenius operators which we shall need later. Let notation be as in Theorem 7.8, so* $\mathrm{Re}(\beta) \geq \gamma \geq 1$ *for* $\beta \in \mathcal{B}$ *and* $\theta_\beta(z) = 1/(z+\beta)$ *for* $\beta \in \mathcal{B}$*. Let* $\mathcal{A}$ *be a finite index set (possibly empty) of integers disjoint from* $\mathcal{B}$ *and for* $j \in \mathcal{A}$*, let* $z_j \in H$ *be a given point,* a_j *a positive real, and* $\theta_j : H \to G$ *the map defined by* $\theta_j(z) = z_j$ *for all* $z \in H$*, so* $\mathrm{Lip}(\theta_j) = 0$*. If* m *is a positive integer and* $s \geq 0$*, define a bounded linear map* $A_s : X := C^m(\bar{H}) \to X$ *by*

$$(A_s f)(z) = \sum_{\beta\in\mathcal{B}} \frac{1}{|z+\beta|^{2s}} f(\theta_\beta(z)) + \sum_{j\in\mathcal{A}} a_j^s f(\theta_j(z)). \tag{7.28}$$

Notice that A_s *satisfies all the hypotheses of Theorem 5.1, so all the conclusions of Theorem 5.1 hold. In particular,* A_s *has a unique (to within normalization) strictly positive eigenvector* $w_s \in C^m(\bar{H})$*. Because the eigenvector* w_s *is unique and* $m \geq 1$ *is arbitrary,* w_s *is* C^∞ *on* H*.*

Define an index set $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ *and for* $\delta \in \mathcal{D}$*, define* $b_\delta(z) = 1/|z+\beta|^{2s}$ *if* $\delta = \beta \in \mathcal{B}$ *and* $b_\delta(z) = a_j$ *if* $\delta = j \in \mathcal{A}$*. As usual, if* μ *is a positive integer, let*

$$\mathcal{D}_\mu = \{\omega = (\delta_1, \delta_2, \dots, \delta_\mu) \,|\, \delta_k \in \mathcal{D} \text{ for } 1 \leq k \leq \mu\}.$$

If $D_1 = \partial/\partial x$ *and* $D_2 = \partial/\partial y$*, for* $k \geq 1$ *and* $p = 1$ *or* 2*, we know that (writing* $z = x+iy := (x,y)$*)*

$$\frac{D_p^k w_s(x,y)}{w_s(x,y)} = \lim_{\mu\to\infty} \frac{D_p^k\Big(\sum_{\omega\in\mathcal{D}_\mu} b_\omega(x,y)\Big)}{\sum_{\omega\in\mathcal{D}_\mu} b_\omega(x,y)}.$$

If $\omega = (\delta_1, \delta_2, \dots, \delta_\mu) \in \mathcal{D}_\mu$ *and* $\delta_k \notin \mathcal{A}$ *for* $1 \leq k \leq \mu$*, we have seen in Lemmas 7.6 and 7.7 that* $D_p^k b_\omega(x,y)/b_\omega(x,y)$ *satisfies the same estimates given for* $D_p^k v_s(x,y)/v_s(x,y)$ *in equations* (7.16)- (7.24)*. Thus assume that* $\delta_t \in \mathcal{A}$ *for some* t*,* $1 \leq t \leq \mu$ *and* $\delta_{t'} \notin \mathcal{A}$ *for* $t' < t$*. A little thought shows that if* $t = 1$*,* $b_\omega(z)$

is a constant. If $t = 2$, $b_\omega(z) = c(\omega, z)b_{\delta_1}(z)$, where $c(\omega, z)$ is a positive constant. Generally, if $2 \le t \le \mu$, $b_\omega(z) = c(\omega, z)b_{\omega_{t-1}}(z)$, where $c(\omega, z)$ is a positive constant and $\omega_{t-1} = (\delta_{t-1}, \delta_{t-2}, \dots, \delta_1) \in \mathcal{D}_{t-1}$ and $\delta_1, \delta_2, \dots, \delta_{t-1} \in \mathcal{B}$. It follows that $D_p^k b_\omega(x, y)/b_\omega(x, y) = 0$ if $t = 1$ and otherwise

$$D_p^k b_\omega(x, y)/b_\omega(x, y) = D_p^k b_{\omega_{t-1}}(x, y)/b_{\omega_{t-1}}(x, y).$$

But using Lemmas 7.6 and 7.7 again, it follows that if $\delta_t \in \mathcal{A}$ for some t, $1 \le t \le \mu$, $D_p^k b_\omega(x, y)/b_\omega(x, y)$ is identically zero or $D_p^k b_\omega(x, y)/b_\omega(x, y)$ satisfies the same estimates given for $D_p^k v_s(x, y)/v_s(x, y)$. It follows that $D_p^k w_s(x, y)/w_s(x, y)$ satisfies the same estimates given for $D_p^k v_s(x, y)/v_s(x, y)$ in Theorem 7.8.

Corollary 7.9. *Let notation and hypotheses be as in Theorem 7.8. If $z_0 = (x_0, y_0) \in H$ and $z_1 = (x_1, y_1) \in H$,*

$$v_s(z_0) \le v_s(z_1) \exp[(\sqrt{5}s/\gamma)|z_1 - z_0|]. \tag{7.29}$$

Proof. Let $H_1 \supset H$ be a convex, bounded open set such that $\mathrm{Re}(z) > 0$ for all $z \in H_1$. (As usual, we identify $x + iy$ with (x, y).) For $z \in H_1$ and Λ_s given by (7.2) and viewed as a bounded linear operator $\Lambda_s : C^m(\bar{H}_1) \to C^m(\bar{H}_1)$, Λ_s has a strictly positive eigenvector $\hat{v}_s : \bar{H}_1 \to (0, \infty)$ in $C^m(\bar{H}_1)$. By uniqueness, $\hat{v}_s(z) = v_s(z)$ for all $z \in H$. Thus, by replacing H by H_1, we can assume that H is convex.

Define $z_t = (1 - t)z_0 + tz_1$ for $0 \le t \le 1$ and note that

$$\Big| \int_0^1 \frac{d}{dz} \log(v_s(z_t))\, dt \Big| = \Big| \log\Big(\frac{v_s(z_1)}{v_s(z_0)}\Big) \Big| \le \int_0^1 \Big| \frac{D_1 v_s(z_t)}{v_s(z_t)}(x_1 - x_0) + \frac{D_2 v_s(z_t)}{v_s(z_t)}(y_1 - y_0) \Big|\, dt.$$

Using (7.16) and (7.20), we obtain

$$\Big| \log\Big(\frac{v_s(z_1)}{v_s(z_0)}\Big) \Big| \le \int_0^1 \Big| \frac{2s}{\gamma}(x_1 - x_0) + \frac{s}{\gamma}(y_1 - y_0) \Big|\, dt \le \frac{\sqrt{5}s}{\gamma}\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

which gives the estimate in Corollary 7.9. ☐

Remark 7.4. *If $\mathcal{B} \subset \mathbb{C}$ is an infinite, countable index set with $\mathrm{Re}\,\beta \ge \gamma \ge 1$ and $\theta_\beta(z) = 1/(z + \beta)$, we can consider, in the notation of Corollary 7.9, $L_s : C(\bar{H}_1) \to C(\bar{H}_1)$ given by $(L_s f)(z) = \sum_{\beta \in \mathcal{B}} |\theta'_\beta(z)|^s f(\theta_\beta(z))$ for $s > \sigma(\mathcal{B})$, Here we assume that there exists $z_* \in G_\gamma$ and $s_* > 0$ such that $\sum_{\beta \in \mathcal{B}} [b_\beta(z_*)]^{s_*} < \infty$, where we have written $b_\beta(z) := |\theta'_\beta(x)|$. If then follows there exists a number $\sigma(\mathcal{B}) \ge 0$ such that for all $s > \sigma(\mathcal{B})$, the map L_s defines a bounded linear map of $C(\bar{H}_1)$ to $C(\bar{H}_1)$, while $\sum_{\beta \in \mathcal{B}} [b_\beta(z)]^s = \infty$ for all $z \in \bar{H}_1$ and all s with $0 \le s < \sigma(\mathcal{B})$. By using Lemmas 7.6 and 7.7 and the argument in Corollary 7.9, we see that for all $\omega \in \mathcal{B}_\mu$, and all $\mu \ge 1$, $b_\omega^s(x, y) \in K(\sqrt{5}s/\gamma; \bar{H}_1)$. With the aid of Lemma 5.3 in* [47] *and Theorem 5.3 on page 86 in* [42]*, we see that L_s has a unique strictly positive eigenvector v_s, and with the aid of Corollary 5.3 in Section 5, we conclude that $v_s \in K(\sqrt{5}s/\gamma; \bar{H}_1)$. In other words, the conclusion of Corollary 7.9 is also valid when $\mathcal{B}$ is countable but not finite.*

If (H7.1) is satisfied and all elements of $\mathcal{B}$ are real, we can, as already noted, restrict attention to the real line, take $G_\gamma := (0, \gamma)$ and $H := (0, a)$ to be open intervals with $a \geq \gamma$ and let Λ_s be given by (7.3) with $f \in C^m(\bar{H})$. Then (7.11) remains valid, but with z replaced by $x \in \bar{H} \subset \mathbb{R}$ and D^α replaced by D_1^ν, and $D_1 = d/dx$. Furthermore, for a fixed $\omega = (\beta_1, \beta_2, \ldots, \beta_\nu) \in \mathcal{B}_\nu$, Lemma 7.3 implies that there exists γ_ω, dependent on ω such that $\gamma_\omega \geq \gamma$, so that after using (7.15), we obtain

$$D_1^\nu(|D_1\theta_{\tilde{\omega}}(x)|^s)(|D_1\theta_{\tilde{\omega}}(x)|^s)^{-1} = D_1^\nu[(x+\gamma_\omega)^{-2s}](x+\gamma_\omega)^{2s}.$$

In this case, it is easy to carry out the calculation explicitly and obtain for all $\omega \in \mathcal{B}_\nu$ and $x \in \bar{H}$ that

$$0 < (-1)^\nu D_1^\nu(|D_1\theta_{\tilde{\omega}}(x)|^s)(|D_1\theta_{\tilde{\omega}}(x)|^s)^{-1} \leq \frac{(2s)(2s+1)\cdots(2s+\nu-1)}{\gamma^\nu}. \tag{7.30}$$

By using (7.11) and (7.30), we thus obtain the following theorem.

Theorem 7.10. *Let $\gamma \geq 1$ be a fixed real number and let $\mathcal{B}$ be a finite set of real numbers β with $\beta \geq \gamma$ for all $\beta \in \mathcal{B}$. Let $G_\gamma = (0, \gamma)$ and $H = (0, a) \supset G_\gamma$ be open intervals of real numbers, and for a positive integer m, let X_m denote the real Banach space $C^m(\bar{H})$. For $s > 0$ define a bounded linear operator $\Lambda_s : X_m \to X_m$ by*

$$(\Lambda_s(f))(x) = \sum_{\beta \in \mathcal{B}} (x+\beta)^{-2s} f(1/(x+\beta)).$$

Then Λ_s has a unique, normalized, strictly positive eigenvector $v_s(x)$, v_s is C^∞, and for all $\nu \geq 1$ and $x \in \bar{H}$,

$$0 \leq (-1)^\nu \frac{D_1^\nu v_s(x)}{v_s(x)} \leq \frac{(2s)(2s+1)\cdots(2s+\nu-1)}{\gamma^\nu}.$$

Remark 7.5. *One can prove that the eigenvector $v_s(x)$ in Theorem 7.10 extends to an analytic function $v_s(z)$ defined on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. In fact, much more general analyticity results of this type can be established. Since we shall not utilize such analyticity theorems in this paper, we omit the proofs.*

Remark 7.6. *Throughout this section we have assumed for convenience that $1 \leq \gamma \leq \mathrm{Re}(\beta)$ for all $\beta \in \mathcal{B}$, where $\mathcal{B}$ is a finite set of complex numbers. In fact, the main results of this section can be obtained under the assumption that $\mathrm{Re}(\beta) \geq \gamma > 0$ for all $\beta \in \mathcal{B}$. In the notation of this section, the key point is to prove that there exists an integer $\nu \geq 1$ and a constant $\kappa < 1$ such that for all $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ and for all $\omega \in \mathcal{B}_\nu$, one has*

$$|\theta_\omega(z) - \theta_\omega(w)| \leq \kappa^\nu |z - w|. \tag{7.31}$$

Inequality (7.31) can be established with the aid of the Carathéodory-Reiffen-Finsler metric (see [17] for the definition and basic results about the CRF metric) and the argument given in Section 6 of [47]. Once (7.31) has been established in the case $\mathrm{Re}(\beta) \geq \gamma > 0$, all the theorems of this section follow by the same arguments.

8. Computing the Spectral Radius of A_s and B_s

In previous sections, we have constructed matrices A_s and B_s such that $r(A_s) \leq r(L_s) \leq r(B_s)$. The $m \times m$ matrices A_s and B_s have nonnegative entries, so the Perron-Frobenius theory for such matrices implies that $r(B_s)$ is an eigenvalue of B_s with corresponding nonnegative eigenvector, with a similar statement for A_s. One might also hope that standard theory (see [41]) would imply that $r(B_s)$, respectively $r(A_s)$, is an eigenvalue of B_s with algebraic multiplicity one and that all other eigenvalues z of B_s (respectively, of A_s) satisfy $|z| < r(B_s)$ (respectively, $|z| < r(A_s)$). Indeed, this would be true if B_s were *primitive*, i.e., if B_s^k had all positive entries for some integer k. However, typically B_s has many zero columns and B_s is neither primitive nor *irreducible* (see [41]); and the same problem occurs for A_s. Nevertheless, the desirable spectral properties mentioned above are satisfied for both A_s and B_s. Furthermore B_s has an eigenvector w_s with all positive entries and with eigenvalue $r(B_s)$; and if x is any $m \times 1$ vector with all positive entries,

$$\lim_{k\to\infty} \frac{B_s^k(x)}{\|B_s^k(x)\|} = \frac{w_s}{\|w_s\|},$$

where the convergence rate is geometric. Of course, corresponding results hold for A_s. Such results justify standard numerical algorithms for approximating $r(B_s)$ and $r(A_s)$.

In this section, we shall prove these results in the one dimensional case. Similar theorems can be proved in the two dimensional case, but for reasons of length, we shall restrict our attention here to the one dimensional case and delay a more comprehensive discussion of the underlying issues to a later paper. The basic point, however, is simple: Although A_s and B_s both map the cone K of nonnegative vectors in $\mathbb{R}^m$ into itself, K is *not* the natural cone in which such matrices should be studied. We shall define below, for large positive real M, a cone $K_M \subset K$ such that $A_s(K_M) \subset K_M$ and $B_s(K_M) \subset K_M$. The cone K_M is the discrete analogue of a cone which has been used before in the infinite dimensional case (see [47], Section 5 of [42], Section 2 of [35] and [6]). Once we have proved that $A_s(K_M) \subset K_M$ and $B_s(K_M) \subset K_M$, we shall see that the desired spectral properties of A_s and B_s follow easily. In a later paper, we shall consider higher order piecewise polynomial approximations to the positive eigenvector v_s of L_s. We shall show that the corresponding matrices A_s and B_s no longer have all nonnegative entries, but still, under appropriate assumptions, map K_M into K_M.

Throughout this section, $[a, b]$ will denote a fixed, closed, bounded interval and s a fixed nonnegative real. For a given positive integer $n \geq 2$, and for integers j, $0 \leq j \leq n$, we shall write $h = (b-a)/n$ and $x_j = a + jh$. C will denote a fixed constant and we shall always assume at least that $h \leq 1$ and

$$|C|h/4 \leq 1. \tag{8.1}$$

In our applications C will depend on s, but we shall not indicate the dependence here. If $f : \{x_j \,|\, 0 \leq j \leq n\} \to \mathbb{R}$, one can extend f to a map $f^I : [a, b] \to \mathbb{R}$ by linear interpolation, so

$$f^I(x) = \frac{x - x_j}{h} f(x_{j+1}) + \frac{x_{j+1} - x}{h} f(x_j), \quad \text{for } x_j \leq x \leq x_{j+1}, \quad 0 \leq j < n. \tag{8.2}$$

We shall denote by X_n (or X, if n is obvious), the real vector space of maps $f : \{x_j \,|\, 0 \leq j \leq n\} \to \mathbb{R}$; obviously X_n is linearly isomorphic to $\mathbb{R}^{n+1}$. For a given positive real M, we shall denote by $K_M \subset X_n$ the closed cone given by

$$K_M = \{f \in X_n \,|\, f(x_{j+1}) \le f(x_j)\exp(Mh) \quad \text{and } f(x_j) \le f(x_{j+1})\exp(Mh), \quad 0 \le j < n\}. \tag{8.3}$$

The reader can verify that if $f \in K_M$, $f(x_j) \ge 0$ for $0 \le j \le n$, and either $f(x_j) > 0$ for all $0 \le j \le n$, or $f(x_j) = 0$ for all $0 \le j \le n$.

If x_j, $0 \le j \le n$, are as above, define a map $Q : [a,b] \to [0, h^2/4]$ by

$$Q(u) = (x_{j+1} - u)(u - x_j), \quad \text{for } x_j \le u \le x_{j+1}, \quad 0 \le j < n.$$

Lemma 8.1. *Assume that $\beta \in K_{M_0} \setminus \{0\}$ for some $M_0 > 0$, that $0 < h \le 1$ and that h and C satisfy* (8.1)*. Let $\theta : [a,b] \to [a,b]$ and define $\hat{\beta}_s \in X_n$ by*

$$\hat{\beta}_s(x_k) = [1 + \tfrac{1}{2}CQ(\theta(x_k))][\beta(x_k)]^s. \tag{8.4}$$

Then $\hat{\beta}_s \in K_{M_1}$, where $M_1 = sM_0 + (1+h)/2 \le M_0 + 1$.

Proof. Define $\psi \in X_n$ by

$$\psi(x_k) = 1 + \tfrac{1}{2}CQ(\theta(x_k))$$

and suppose we can prove that $\psi \in K_{(1+h)/2}$. For notational convenience define $b(x_k) = [\beta(x_k)]^s$. Then for $0 \le k < n$, we obtain

$$\psi(x_k)b(x_k) \le \psi(x_{k+1})\exp([1+h]h/2)b(x_{k+1})\exp(sM_0h) = \psi(x_{k+1})b(x_{k+1})\exp(M_1h),$$

and the same calculation gives

$$\psi(x_{k+1})b(x_{k+1}) \le \exp(M_1h)\psi(x_k)b(x_k),$$

which implies that $x_k \mapsto \psi(x_k)b(x_k)$ is an element of K_{M_1}.

Define $\delta = (1+h)/2$. Since $\psi(x_k) > 0$ for $0 \le k \le n$, one can check that $\psi(\cdot) \in K_\delta$ if and only if, for $0 \le k < n$,

$$|\log(\psi(x_{k+1})) - \log(\psi(x_k))| = \Big|\log\Big(\frac{\psi(x_{k+1})}{\psi(x_k)}\Big)\Big| \le \delta h.$$

Given x_k and x_{k+1} with $0 \le k < n$, write $\xi = \theta(x_k)$ and $\eta = \theta(x_{k+1})$. Define $u := \frac{1}{2}CQ(\theta(x_k))$ and $v = \frac{1}{2}CQ(\theta(x_{k+1}))$, so $\psi(x_k) = 1 + u$ and $\psi(x_{k+1}) = 1 + v$. Because u and v both lie in the interval $[0, Ch^2/8]$, (8.1) implies that $|u - v| \le h/2$, $|u| \le h/2$ and $|v| \le h/2$. It follows that

$$|\log(\psi(x_k)) - \log(\psi(x_{k+1}))| = |\log(1+u) - \log(1+v)| = \Big|\int_{1+v}^{1+u} (1/t)\,dt\Big|.$$

Because $0 \le 1/t \le 1/(1 - h/2) \le 1 + h$ for all $t \in [1+v, 1+u]$, we obtain

$$|\log(\psi(x_k)) - \log(\psi(x_{k+1}))| \le (1+h)|u - v| \le (1+h)h/2,$$

which proves the lemma. $\square$

Lemma 8.2. *Let assumptions and notation be as in Lemma 8.1. Let δ denote a fixed positive real and s a fixed nonnegative real. Assume, in addition that $\theta : [a,b] \to [a,b]$ is a Lipschitz map with $\mathrm{Lip}(\theta) \le c < 1$ and that, for $h = (b-a)/n$ and M_1 as in Lemma 8.1, $\exp(-[M_1+\delta]h) \ge (1+c)/2$ and $M > 0$ is such that $\exp(Mh) \ge 2$. Define a linear map $L_s : X_n \to X_n$ by $L_s(f) = g$, where*

$$g(x_k) := f^I(\theta(x_k))\hat{\beta}_s(x_k), \quad 0 \le k \le n.$$

Then, if $K_M \subset X_n$ is defined by (8.3), $L_s(K_M) \subset K_{M-\delta}$.

Proof. For a fixed k, $0 \le k < n$, define $\xi = \theta(x_k)$ and $\eta = \theta(x_{k+1})$. We must prove that if h and M satisfy the above constraints and $f \in K_M$, then

$$f^I(\xi)\hat{\beta}_s(x_k) \le \exp([M-\delta]h) f^I(\eta)\hat{\beta}_s(x_{k+1}),$$
$$f^I(\eta)\hat{\beta}_s(x_{k+1}) \le \exp([M-\delta]h) f^I(\xi)\hat{\beta}_s(x_k).$$

Using Lemma 8.1, we see that $x_k \mapsto \hat{\beta}_s(x_k)$ is an element of K_{M_1}, so the above inequalities will be satisfied if

$$f^I(\xi) \le \exp([M - M_1 - \delta]h) f^I(\eta), \tag{8.5}$$

$$f^I(\eta) \le \exp([M - M_1 - \delta]h) f^I(\xi). \tag{8.6}$$

For notational convenience, we write $M_2 = M_1 + \delta$. By interchanging the roles of ξ and η, we can assume that $\eta \le \xi$, and it suffices to prove that (8.5) and (8.6) are satisfied for M and h as in the statement of the Lemma. Define $j = n-1$ if $\xi \ge x_{n-1}$ and otherwise define j to be the unique integer, $0 \le j < n-1$, such that $x_j \le \xi < x_{j+1}$. Because $0 \le \xi - \eta \le ch < h$, there are only two cases to consider: either (i) $x_j \le \eta \le \xi$ or (ii) $x_{j-1} < \eta < x_j$ and $x_j \le \xi < x_{j+1}$.

We first assume that we are in case (i), so $\xi, \eta \in [x_j, x_{j+1}]$ and $0 \le \xi - \eta \le ch$, Using (8.2), we see that (8.5) is equivalent to proving

$$\begin{aligned}(x_{j+1} - \xi) f(x_j) &+ (\xi - x_j) f(x_{j+1}) \\ &\le \exp([M - M_2]h)[(x_{j+1} - \eta) f(x_j) + (\eta - x_j) f(x_{j+1})].\end{aligned} \tag{8.7}$$

Subtracting $(x_{j+1} - \eta) f(x_j) + (\eta - x_j) f(x_{j+1})$ from both sides of (8.7) shows that (8.7) will be satisfied if

$$\begin{aligned}(\xi - \eta)&[f(x_{j+1}) - f(x_j)] \\ &\le [\exp([M - M_2]h) - 1][(x_{j+1} - \eta) f(x_j) + (\eta - x_j) f(x_{j+1})].\end{aligned} \tag{8.8}$$

Equation (8.8) will certainly be satisfied if $f(x_{j+1}) \le f(x_j)$, so we can assume that $f(x_{j+1}) - f(x_j) > 0$ and $1 < f(x_{j+1})/f(x_j) \le \exp(Mh)$. If we divide both sides of (8.8) by $f(x_j)$ and recall that $\xi - \eta \le ch$, we see that the left hand side of (8.8) is dominated by $ch[\exp(Mh) - 1]$, while the right hand side of (8.8) is $\ge [\exp([M - M_2]h) - 1]h$, Thus, (8.8) will be satisfied if

$$c \le \frac{\exp([M - M_2]h) - 1}{\exp(Mh) - 1} = \exp(-M_2 h) + \frac{\exp(-M_2 h) - 1}{\exp(Mh) - 1}. \tag{8.9}$$

If $h > 0$ is chosen so that $\exp(-M_2 h) \ge (1+c)/2$, a calculation shows that (8.9) will be satisfied if

$$M \ge \log(2)/h, \tag{8.10}$$

where log denotes the natural logarithm. Thus, if $h > 0$ satisfies (8.1), $M \geq \log(2)/h$, and $\exp(-M_2 h) \geq (1+c)/2$, (8.5) is satisfied in case (i). Under the same conditions on h and M, an exactly analogous argument shows that (in case (i)), (8.6) is also satisfied.

We next consider case (ii), so $\xi \in [x_j, x_{j+1}]$, $\eta \in [x_{j-1}, x_j]$ and $0 \leq \xi - \eta \leq ch$. It follows that $\xi - x_j = c_1 h$ and $x_j - \eta = c_2 h$, where $c_1 \geq 0$, $c_2 \geq 0$, and $c_1 + c_2 \leq c < 1$. As before, we need to show that inequalities (8.5) and (8.6) are satisfied. Inequality (8.6) takes the form

$$\begin{aligned} f^I(\eta) &= \frac{\eta - x_{j-1}}{h} f(x_j) + \frac{x_j - \eta}{h} f(x_{j-1}) \\ &\leq \exp([M - M_2]h)\Big[\frac{\xi - x_j}{h} f(x_{j+1}) + \frac{x_{j+1} - \xi}{h} f(x_j)\Big], \end{aligned} \tag{8.11}$$

which is equivalent to

$$(\eta - x_{j-1}) + (x_j - \eta)\frac{f(x_{j-1})}{f(x_j)} \leq \exp([M - M_2]h)\Big[(\xi - x_j)\frac{f(x_{j+1})}{f(x_j)} + (x_{j+1} - \xi)\Big], \tag{8.12}$$

Since $f(x_{j-1})/f(x_j) \leq \exp(Mh)$, $f(x_{j+1})/f(x_j) \geq \exp(-Mh)$, $x_j - \eta = c_2 h$ and $\xi - x_j = c_1 h$, (8.12) will be satisfied if

$$(1 - c_2) + c_2 \exp(Mh) \leq \exp([M - M_2]h)[c_1 \exp(-Mh) + (1 - c_1)]. \tag{8.13}$$

Because $c_2 \leq c - c_1$, we have

$$(1 - c_2) + c_2 \exp(Mh) \leq (1 - c + c_1) + (c - c_1)\exp(Mh),$$

and inequality (8.13) will be satisfied if

$$(1 + c_1 - c) + (c - c_1)\exp(Mh) \leq \exp(-M_2 h)[c_1 + (1 - c_1)\exp(Mh)]. \tag{8.14}$$

A necessary condition that (8.14) be satisfied is that $\exp(-M_2 h) \geq (c - c_1)/(1 - c_1)$. Since $(c - c_1)/(1 - c_1) \leq c$ and $c < (1+c)/2$, we choose $h = (b-a)/n > 0$ sufficiently small that

$$\exp(-M_2 h) \geq (1 + c)/2. \tag{8.15}$$

For this choice of h, (8.14) will be satisfied if

$$(1 + c_1 - c) + (c - c_1)\exp(Mh) \leq \frac{1 + c}{2}[c_1 + (1 - c_1)\exp(Mh)],$$

which is equivalent to

$$(1 + c_1/2)(1 - c) \leq [(1 + c_1)(1 - c)/2]\exp(Mh). \tag{8.16}$$

Since $(2 + c_1)/(1 + c_1) \leq 2$, (8.16) will be satisfied if

$$2 \leq \exp(Mh). \tag{8.17}$$

Thus (8.11) will be satisfied if h satisfies (8.15) and, for this h, M satisfies (8.17).

Inequality (8.5) will be satisfied in case (ii) if

$$(\xi - x_j)\frac{f(x_{j+1})}{f(x_j)} + (x_{j+1} - \xi) \leq \exp([M - M_2]h)\Big[(\eta - x_{j-1}) + (x_j - \eta)\frac{f(x_{j-1})}{f(x_j)}\Big]. \tag{8.18}$$

The same reasoning as above shows that if $h > 0$ satisfies (8.15) and M then satisfies (8.17), (8.18) will be satisfied. Details are left to the reader. □

Theorem 8.3. *Let N denote a positive integer or $N = \infty$. For $1 \le j \le N$, assume that $\theta_j : [a,b] \to [a,b]$ is a Lipschitz map with $Lip(\theta_j) \le c < 1$, c independent of j. For $1 \le j \le N$, assume that $\beta_j \in K_{M_0} \setminus \{0\} \subset X_n$, where M_0 is independent of j. For $j \ge 1$, let C_j be a real number with $|C_j| \le C$, where C is independent of j; and for a fixed $s \ge 0$, define $\hat{\beta}_{j,s} \in X_n$ by*

$$\hat{\beta}_{j,s}(x_k) = [1 + \tfrac{1}{2} C_j Q(\theta_j(x_k))][\beta_j(x_k)]^s, \quad 0 \le k \le n.$$

Let $\delta > 0$ be a given real number and for $j \ge 1$ define a linear map $L_{j,s} : X_n \to X_n$ by

$$(L_{j,s} f)(x_k) = \hat{\beta}_{j,s}(x_k) f^I(\theta_j(x_k)).$$

If $N = \infty$, assume that there exists k_0, $0 \le k_0 \le n$, such that $\sum_{j=1}^{\infty} [\hat{\beta}_j(x_k)]^s < \infty$ and define a linear map $L_s : X_n \to X_n$ by $L_s = \sum_{j=1}^{N} L_{j,s}$. Assume that $h = (b-a)/n \le 1$ and $Ch/4 \le 1$ and define $M_2 = [sM_0 + (1+h)/2] + \delta$. Assume also that $\exp(-M_2 h) \ge (1+c)/2$ and that $M \in \mathbb{R}$ is such that $\exp(Mh) \ge 2$. Then we have that $L_s(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$.

Proof. Lemma 8.1 implies that $x_k \mapsto \hat{\beta}_{j,s}(x_k)$ is an element of K_{M_1}, where $M_1 = sM_0 + (1+h)/2$. It follows that if $N = \infty$ and $\sum_{j=1}^{N} \hat{\beta}_{j,s}(x_{k_0}) < \infty$, it must be true that $\sum_{j=1}^{N} \hat{\beta}_{j,s}(x_k) < \infty$ for all k, $0 \le k \le N$; and $L_s : X_n \to X_n$ is also a well-defined bounded linear map when $N = \infty$. Under our hypotheses, Lemma 8.2 implies that $L_{j,s}(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$, so $L_s(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$. ☐

At this point we need to recall some general results relating to u_0-*positive linear operators.* Recall that a closed subset C of a Banach space Y is called a closed cone if (i) $ax + by \in C$ whenever a and b are nonnegative reals and $x, y \in C$ and (ii) $C \cap (-C) = \{0\}$, where $-C = \{-x \,|\, x \in C\}$. A closed cone C in a real Banach space $(Y, \|\cdot\|)$ is called *reproducing* if $Y = \{f - g \,|\, f, g \in C\}$, and a closed cone C induces a partial ordering $\le_C$ on Y by $x \le_C y$ if and only if $y - x \in C$. If $x \in C$ and $y \in C$, we shall say that x and y are comparable (in the partial ordering $\le_C$) if there exist positive reals $\alpha > 0$ and $\beta > 0$ such that $\alpha x \le_C y$ and $y \le_C \beta x$. If x, y are comparable, we shall write

$$M(y/x; C) = \inf\{\beta > 0 \,|\, y \le_C \beta x\}, \qquad m(y/x; C) = \sup\{\alpha > 0 \,|\, \alpha x \le_C y\}.$$

The following proposition can be found in [30] and [31].

Proposition 8.4. *Let C be a closed, reproducing cone in a real Banach space Y, and let $A : Y \to Y$ be a bounded linear operator such that $A(C) \subset C$. Assume that there exists $v \in C \setminus \{0\}$ and $r > 0$ such that $Av = rv$. Assume (this is the u_0-positivity of A) that there exists $u_0 \in C \setminus \{0\}$ with the following property: For every $x \in C \setminus \{0\}$, there exists a positive integer $m(x)$ and positive reals $a(x)$ and $b(x)$ such that either (i) $a(x)u_0 \le_C A^{m(x)}(x) \le_C b(x)u_0$ or (ii) $A^{m(x)}(x) = 0$. If $\hat{A}$ denotes the complexification of A, r is an eigenvalue of $\hat{A}$ of algebraic multiplicity 1; and if $Aw = \lambda w$ for some $w \in C \setminus \{0\}$ and $\lambda > 0$, $\lambda = r$ and w is a positive scalar multiple of v. If $z \in \mathbb{C}$ is an eigenvalue of $\hat{A}$ and $z \ne r$, then $|z| < r$.*

Remark 8.1. *Note that Proposition 8.4 only gives information about eigenvalues of $\hat{A}$. If $\sigma(\hat{A})$ denotes the spectrum of $\hat{A}$, it is possible, under the assumptions of Proposition 8.4, that there exists $z \in \sigma(\hat{A})$ with $|z| = r$ and $z \ne r$.*

Remark 8.2. *Proposition 8.4 can be derived from the so-called Birkhoff-Hopf theorem, though we shall not do so here. We refer the reader to the papers* [2], [23], *and* [52] *for the original work by Birkhoff, Hopf, and Samelson. A general version of the Birkhoff-Hopf theorem, applications to spectral theory, and references to the literature can be found in* [12] *and* [11]*; see also Appendix A of* [33]. *Section 2.2 of* [35] *(particularly Lemma 2.12) is closely related to our work here.*

Theorem 8.5. *Let notation and assumptions be as in Theorem 8.3. Then L_s has an eigenvector $v \in K_{M-\delta} \setminus \{0\}$ with eigenvalue $r > 0$. If $\hat{L}_s$ denotes the complexification of L_s, r is an eigenvalue of $\hat{L}_s$ of algebraic multiplicity one; and if $L_s w = \lambda w$ for some $w \in K_M \setminus \{0\}$, $\lambda = r$, and w is a positive multiple of v. If z is an eigenvalue of $\hat{L}$ and $z \neq r$, then $|z| < r$.*

Proof. We shall need a very special case of Lemma 2.12 in [35]. Because $M - \delta < M$, Lemma 2.12 in [35] implies that all elements $x, y \in K_{M-\delta} \setminus \{0\}$ are comparable with respect to the partial ordering $\leq_{K_M}$ given by $K_M \supset K_{M-\delta}$. Furthermore, we have

$$\sup\{M(y/x; K_M)/m(y/x; K_M) : x, y \in K_{M-\delta} \setminus \{0\}\} < \infty.$$

Since Theorem 8.3 implies that $L_s(K_M \setminus \{0\}) \subset K_{M-\delta} \setminus \{0\}$, it follows that if $u \in K_{M-\delta} \setminus \{0\}$, $L_s u \in K_{M-\delta} \setminus \{0\}$ and u and $L_s u$ are comparable, so $L_s u \geq_{K_M} \alpha u$ for some $\alpha > 0$. This implies that $r(L_s) \geq \alpha > 0$. In our particular case, the cone K_M has nonempty interior in X_n, although in the generality of Lemma 2.12, this is not usually true. The Kreĭn-Rutman theorem implies that L_s has an eigenvector $v_s \in K_M$ with eigenvalue $r = r(L_s) > 0$; and since $r v_s = L_s(v_s)$, $v_s \in K_{M-\delta}$. If we define $u_0 := v_s$, Lemma 2.12 in [35] implies that $L_s(x)$ is comparable to v_s (with respect to the partial ordering $\leq_{K_M}$) for all $x \in K_M \setminus \{0\}$. Theorem 8.5 now follows directly from Proposition 8.4. □

Remark 8.3. *Since the linear maps A_s and B_s are both of the form of the map L_s in Theorem 8.3, Theorem 8.5 implies the desired spectral properties of A_s and B_s. With greater care it is possible to use results in* [11] *to estimate the so-called spectral clearance $q(L_s)$ of L_s, given by*

$$q(L_s) := \sup\{|z|/r : z \in \sigma(L_s) \text{ and } z \neq r(L_s)\} < 1.$$

Remark 8.4. *We claim that there is a constant E, which can be easily estimated, such that, for $h = (b-a)/n$ sufficiently small,*

$$r(B_s) \leq r(A_s)(1 + Eh^2).$$

(Of course we already know that $r(A_s) \leq r(B_s)$.) For a fixed $s \geq 0$, let $\beta_j(\cdot)$ and $\theta_j(\cdot)$ be as in Theorem 8.3. We know that A_s and B_s are of the form of L_s in Theorem 8.3, so we can write, for $0 \leq k \leq n$,

$$(A_s f)(x_k) = \sum_{j-1}^{N} [1 + (C_j/2) Q(\theta_j(x_k))][\beta_j(x_k)]^s f^I(\theta_j(x_k),$$

$$(B_s f)(x_k) = \sum_{j-1}^{N} [1 + (D_j/2) Q(\theta_j(x_k))][\beta_j(x_k)]^s f^I(\theta_j(x_k).$$

We assume that $h \leq 1$ and $Ch/4 \leq 1$, where C is a positive constant such that $\max(|C_j|, |D_j|) \leq C$ for $1 \leq j \leq N$. We assume also that for $1 \leq j \leq N$, $C_j \leq D_j$.

Let $K = \{f \in X_n \,|\, f(x_k) \geq 0$ *for* $0 \leq k \leq n\}$, *so* $A_s(K) \subset K$ *and* $B_s(K) \subset K$. *Define* $\mu \geq 1$ *by*

$$\mu = \sup\{[1 + \frac{D_j}{2} Q(\theta_j(x_k))][1 + \frac{C_j}{2} Q(\theta_j(x_k))]^{-1} : 1 \leq j \leq N, 0 \leq k \leq N\} \geq 1.$$

Then for all $f \in K$ *and* $0 \leq k \leq n$, $(B_s(f))(x_k) \leq \mu(A_s(f))(x_k)$, *which implies that* $r(B_s) \leq \mu r(A_s)$. *Since* $Q(u) \leq h^2/4$, *a little thought shows that* $\mu \leq (1 + Ch^2/8)(1 - Ch^2/8)^{-1} \leq 1 + Eh^2$, *which gives the desired estimate.*

9. Log convexity of the spectral radius of Λ_s

Throughout this section we shall assume that hypotheses (H5.1), (H5.2), and (H5.3) in Section 5 are satisfied and we shall also assume that H is a bounded, open, mildly regular subset of $\mathbb{R}^n$. As in Section 5, we shall write $X = C^m(\bar{H})$ and $Y = C(\bar{H})$. For $s \in \mathbb{R}$, we define $\Lambda_s : X \to X$ and $L_s : Y \to Y$ by

$$(\Lambda_s(f))(x) = \sum_{\beta \in \mathcal{B}} (b_\beta(x))^s f(\theta_\beta(x)) \tag{9.1}$$

and

$$(L_s(f))(x) = \sum_{\beta \in \mathcal{B}} (b_\beta(x))^s f(\theta_\beta(x)). \tag{9.2}$$

Theorem 5.1 implies that $r(\Lambda_s)$ is an algebraically simple eigenvalue of Λ_s for $s \in \mathbb{R}$ and that $\sup\{|z| : z \in \sigma(\Lambda_s), z \neq r(\Lambda_s)\} < r(\Lambda_s)$, where $\sigma(\Lambda_s)$ denotes the spectrum of Λ_x.

Let $\hat{X}$ denote of the complexification of X, so $\hat{X}$ is the Banach space of C^m maps $f : H \to \mathbb{C}$ such that $x \mapsto (D^\alpha f)(x)$ extends continuously to $\bar{H}$ for all multi-indices α with $|\alpha| \leq m$. For $s \in \mathbb{C}$ one can define $\hat{\Lambda}_s : \hat{X} \to \hat{X}$ by

$$(\hat{\Lambda}_s(f))(x) = \sum_{\beta \in \mathcal{B}} (b_\beta(x))^s f(\theta_\beta(x)) := \sum_{\beta \in \mathcal{B}} \exp(s \log b_{\beta(x)}) f(\theta_\beta(x)). \tag{9.3}$$

The reader can verify that $s \mapsto \hat{\Lambda}_s \in \mathcal{L}(\hat{X}, \hat{X})$ is an analytic map. Because $r(\hat{\Lambda}_s)$ is an algebraically simple eigenvalue of $\hat{\Lambda}_s$ for $s \in \mathbb{R}$ and $\sup\{|z| : z \in \sigma(\Lambda_s), z \neq r(\Lambda_s)\} < r(\Lambda_s)$, it follows from the kind of argument used on pages 227-228 of [43] that there is an open neighborhood U of $\mathbb{R} \in \mathbb{C}$ and the map $s \in U \mapsto r(\hat{\Lambda}_s)$ is analytic on U.

Theorem 9.1. *Assume that hypotheses (H5.1), (H5.2), and (H5.3) are satisfied with* $m \geq 1$ *and that* $H \subset \mathbb{R}^n$ *is a bounded, open mildly regular set. For* $s \in \mathbb{R}$, *let* Λ_s *and* L_s *be defined by* (9.1) *and* (9.2). *Then we have that* $s \mapsto r(\Lambda_s)$ *is log convex, i.e.,* $s \mapsto log(r(\Lambda_s))$ *is convex on* $[0, \infty)$.

Proof. Because Theorem 5.1 implies that $r(L_s) = r(\Lambda_s)$ for all real s, it suffices to take $s_0 < s_1$, and $0 < t < 1$ and prove that

$$r(L_{(1-t)s_0 + ts_1}) \leq r(L_{s_0})^{1-t} r(L_{s_1})^t.$$

We shall use an old trick (see [45] and the references therein). Let $v_{s_j}(x)$, $j = 0, 1$ denote the strictly positive eigenvector of L_{s_j} which is ensured by Theorem 5.1. Then

$$L_{s_j} v_{s_j} = r(L_{s_j}) v_{s_j}.$$

For a fixed t, $0 < t < 1$, define $s_t = (1-t)s_0 + ts_1$ and

$$w_t(x) = (v_{s_0}(x))^{1-t}(v_{s_1}(x))^t.$$

Then, using Hölder's inequality, we find that

$$\begin{aligned} (9.4)\quad (L_{s_t}(w_t))(x) &= \sum_{\beta\in\mathcal{B}} (b_\beta(x)^{s_0} v_{s_0}(x))^{1-t}(b_\beta(x)^{s_1} v_{s_1}(x))^t \\ &\le \Big(\sum_{\beta\in\mathcal{B}} (b_\beta(x)^{s_0} v_{s_0}(x)\Big)^{1-t}\Big(\sum_{\beta\in\mathcal{B}} (b_\beta(x)^{s_1} v_{s_1}(x)\Big)^t = [r(L_{s_0})^{1-t} r(L_{s_1})^t] w_t(x). \end{aligned}$$

Because $w_t(x) > 0$ for all $x \in \bar{H}$, a standard argument (see Lemma 5.9 in [47]) shows that

$$(9.5)\qquad r(L_{s_t}) = \lim_{k\to\infty} \|L_{s_t}^k\|^{1/k} = \lim_{k\to\infty} \|L_{s_t}^k(w_t)\|^{1/k}.$$

Using inequalities (9.4) and (9.5), we see that

$$r(L_{s_t}) \le r(L_{s_0})^{1-t} r(L_{s_1})^t.$$

□

In general, if V is a convex subset of a vector space X, we shall call a map $f : V \to [0, \infty)$ log convex if (i) $f(x) = 0$ for all $x \in V$ or (ii) $f(x) > 0$ for all $x \in V$ and $x \mapsto \log(f(x))$ is convex. Products of log convex functions are log convex, and Hölders inequality implies that sums of log convex functions are log convex.

Results related to Theorem 9.1 can be found in [45], [28], [29], [8], [15], and [14]. Note that the terminology *super convexity* is used to denote log convexity in [28] and [29], presumably because any log convex function is convex, but not conversely. Theorem 9.1, while adequate for our immediate purposes, can be greatly generalized by a different argument that does not require existence of strictly positive eigenvectors. This generalization (which we omit) contains Kingman's matrix log convexity result in [29] as a special case.

In our applications, the map $s \mapsto r(L_s)$ will usually be strictly decreasing on an interval $[s_1, s_2]$ with $r(L_{s_1}) > 1$ and $r(L_{s_2}) < 1$, and we wish to find the unique $s_* \in (s_1, s_2)$ such that $r(L_{s_*}) = 1$. The following hypothesis insures that $s \mapsto r(L_s)$ is strictly decreasing for all S.

(H9.1): Assume that $b_\beta(\cdot)$, $\beta \in \mathcal{B}$ satisfy the conditions of (H5.1). Assume also that there exists an integer $\mu \ge 1$ such that $b_\omega(x) < 1$ for all $\omega \in \mathcal{B}_\mu$ and all $x \in \bar{H}$.

Theorem 9.2. *Assume hypotheses (H5.1), (H5.2), (H5.3), and (H9.1) and let H be mildly regular. Then the map $s \mapsto r(\Lambda_s)$, $s \in \mathbb{R}$, is strictly decreasing and real analytic and* $\lim_{s\to\infty} r(\Lambda_s) = 0$.

Proof. If $L_s : C(\bar{H}) \to C(\bar{H})$ is given by (5.2), it is a standard result that $r(L_s^\nu) = (r(L_s))^\nu$ and $r(\Lambda_s^\nu) = (r(\Lambda_s))^\nu$ for all integers $\nu \geq 1$, and Theorem 5.1 implies that $r(L_s) = r(\Lambda_s)$. Thus it suffices to prove that for some positive integer ν, $s \mapsto r(L_s^\nu)$ is strictly decreasing and $\lim_{s\to\infty} r(L_s^\nu) = 0$.

Suppose that K denotes the set of nonnegative functions in $C(\bar{H})$ and $A : C(\bar{H}) \to C(\bar{H})$ is a bounded linear map such that $A(K) \subset K$. If there exists $w \in C(\bar{H})$ such that $w(x) > 0$ for all $x \in \bar{H}$ and if $(A(w))(x) \leq aw(x)$ for all $x \in \bar{H}$, it is well-known (and easy to verify) that $r(A) \leq a$, where $r(A)$ denotes the spectral radius of A. In our situation, we take $\nu = \mu$, where μ is as in (H9.1), and $A = (L_s)^\mu$. If $s < t$ and v_s is the strictly positive eigenvector for $(L_s)^\mu$, (H9.1) implies that there is a constant $c < 1$, $c = c(s,t)$, such that $cb_\omega(x)^s \geq b_\omega(x)^t$ for all $\omega \in \mathcal{B}_\mu$ and $x \in H$. Thus we find that

$$cr(L_s)^\mu v_s(x) = \sum_{\omega\in\mathcal{B}_\mu} cb_\omega(x)^s v_s(\theta_\omega(x)) \geq \sum_{\omega\in\mathcal{B}_\mu} b_\omega(x)^t v_s(\theta_\omega(x)) = (L_t^\mu(v_s))(x).$$

It follows that $r(L_t)^\mu \leq c(s,t) r(L_s)^\mu$, so $r(L_t) < r(L_s)$, for $s < t$. Because $0 < b_\omega(x) < 1$ for all $x \in \bar{H}$ and $\omega \in \mathcal{B}_\mu$, it is also easy to see that $\lim_{t\to\infty} \|(L_t)^\mu\| = 0$; and since $\|(L_t)^\mu\| \geq r(L_t^\mu)$, we see that $\lim_{t\to\infty} r(L_t^\mu) = 0$. ☐

Remark 9.1. *It is easy to construct examples for which (H9.1) is satisfied for some $\mu > 1$, but not satisfied for $\mu = 1$. The functions $\theta_1(x) := 9/(x+1)$ and $\theta_2(x) := 1/(x+2)$ both map the closed interval $\bar{H} = [1/11, 9]$ into itself. There is a unique nonempty compact set $J \subset \bar{H}$ such that*

$$J = \theta_1(J) \cup \theta_2(J).$$

For $s \in \mathbb{R}$, define $L_s : C(\bar{H}) \to C(\bar{H})$ by

$$(L_s f)(x) := \sum_{j=1}^{2} |D\theta_j(x)|^s f(\theta_j(x)) := \sum_{j=1}^{2} b_j(x)^s f(\theta_j(x)),$$

where $D := d/dx$. The Hausdorff dimension of J is the unique $s = s_*$, $0 < s_* < 1$, such that $r(L_s) = 1$. Our previous remarks show that

$$(L_s^2 f)(x) = \sum_{j=1}^{2}\sum_{k=1}^{2} |D(\theta_j \circ \theta_k)(x)|^s f(\theta_j \circ \theta_k)(x)).$$

One can check that (H9.1) is not satisfied for $\mu = 1$, but is satisfied for $\mu = 2$.

Remark 9.2. *Assume that the assumptions of Theorem 9.2 are satisfied and define $\psi(x) = \log(r(L_s)) = \log(r(\Lambda_s))$ (where $\log$ denotes the natural logarithm), so $s \mapsto \psi(s)$ is a convex, strictly decreasing function with $\psi(0) > 1$ (unless $|\mathcal{B}| = p = 1$) and $\lim_{s\to\infty} \psi(s) = -\infty$. We are interested in finding the unique value of s such that $\psi(s) = 0$. In general suppose that $\psi : [s_1, s_2] \to \mathbb{R}$ is a continuous, strictly decreasing, convex function such that $\psi(s_1) > 0$ and $\psi(s_2) < 0$, so there exists a unique $s = s_* \in (s_1, s_2)$ with $\psi(s_*) = 0$. If t_1 and t_2 are chosen so that $s_1 \leq t_1 < t_2 \leq s_*$ and t_{k+1} is obtained from t_{k-1} and t_k by the secant method, an elementary argument show that $\lim_{k\to\infty} t_k = s_*$. If $s_* \leq t_2 < t_1 < s_2$ and $s_1 \leq t_3$, a similar argument shows that $\lim_{k\to\infty} t_k = s_*$. If $\psi \in C^3$, elementary numerical analysis implies that the rate of convergence is faster than linear (= $(1+\sqrt{5})/2$). In our numerical work, we apply these observations, not directly to $\psi(s) = \log(r(\Lambda_s))$, but to convex decreasing functions which closely approximate $\log(r(\Lambda_s))$.*

One can also ask whether the maps $s \mapsto r(B_s)$ and $s \mapsto r(A_s)$ are log convex, where A_s and B_s are the previously described approximating matrices for L_s. An easier question is whether the map $s \mapsto r(M_s)$ is log convex, where A_s and B_s are obtained from M_s by adding error correction terms. We shall prove that $s \mapsto r(M_s)$ is log convex, at least in the one dimensional case. The proof in the two dimensional case is similar.

First, we need to recall a useful theorem of Kingman [29]. Let $M(s) = (a_{ij}(s))$ be an $m \times m$ matrix whose entries $a_{ij}(s)$ are either strictly positive for all s in a fixed interval J or are identically zero for all $s \in J$. Assume that $s \mapsto a_{ij}(s)$ is log convex on J for $1 \leq i, j \leq m$. Under these assumptions, Kingman [29] has proved that $s \mapsto r(M_s)$ is log convex.

Let $n \geq 2$ be a positive integer, and for $a < b$ given real numbers, define $x_k = a + kh$, $-1 \leq k \leq n+1$, $h = (b-a)/n$. Let X_n denote the vector space of real valued maps $f : \{x_k \,|\, 0 \leq k \leq n\} \to \mathbb{R}$, so X_n is a real vector space linearly isomorphic to $\mathbb{R}^{n+1}$. As usual, if $f \in X_n$, extend f to a map $f^I : [a,b] \to \mathbb{R}$ by linear interpolation, so

$$f^I(u) = \frac{u - x_k}{h} f(x_{k+1}) + \frac{x_{k+1} - u}{h} f(x_k), \qquad x_k \leq u \leq x_{k+1}, 0 \leq k \leq n.$$

For $1 \leq j \leq N$, assume that $\theta_j : [a,b] \to [a,b]$ are given maps and assume that $b_j : [a,b] \to (0,\infty)$ are given positive functions. For $s \in \mathbb{R}$, define a linear map $M_s : X_n \to X_n$ by $M_s(f) = g$, where

$$g(x_k) = \sum_{j=1}^{N} [b_j(x_k)]^s f^I(\theta_j(x_k)), \quad 0 \leq k \leq n,$$

so if $f(x_k) \geq 0$ for $0 \leq k \leq n$, $g(x_k) \geq 0$ for $0 \leq k \leq n$. We can write $g(x_k) = \sum_{m=0}^{n} a_{km}(x) f(x_m)$, where for $0 \leq k$, $m \leq n$,

$$a_{km}(x) = \sum_{j, x_{m-1} \leq \theta_j(x_k) \leq x_m} [b_j(x_k)]^s [\theta_j(x_k) - x_{m-1}]/h + \sum_{j, x_m \leq \theta_j(x_k) \leq x_{m+1}} [b_j(x_k)]^s [x_{m+1} - \theta_j(x_k)]/h.$$

If, for a given k and m, there is no j, $1 \leq j \leq N$, with $x_{m-1} \leq \theta_j(x_k) \leq x_{m+1}$, we define $a_{km} = 0$. Since the sum of log convex functions is log convex, $s \mapsto a_{km}(s)$ is log convex on $\mathbb{R}$. It follows from Kingman's theorem that $s \mapsto r(M_s)$ is log convex, where $r(M_s)$ denotes the spectral radius of M_s.

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