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A UNIFORMLY ACCURATE FINITE ELEMENT METHOD FOR THE REISSNER–MINDLIN PLATE*

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This paper is dedicated to Jim Douglas, Jr., on the occasion of his 60th birthday.

Abstract. A simple finite element method for the Reissner–Mindlin plate model in the primitive variables is presented and analyzed. The method uses nonconforming linear finite elements for the transverse displacement and conforming linear finite elements enriched by bubbles for the rotation, with the computation of the element stiffness matrix modified by the inclusion of a simple elementwise averaging. It is proved that the method converges with optimal order uniformly with respect to thickness.

Key words. Reissner–Mindlin plate, finite element, nonconforming

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1. Introduction.

The Reissner–Mindlin model describes the deformation of a plate subject to a transverse loading in terms of the transverse displacement of the midplane and the rotation of fibers normal to the midplane. This model, as well as its generalization to shells, is frequently used for plates and shells of small to moderate thickness. We present and analyze here a simple finite element method for the Reissner–Mindlin plate model. Our method uses linear finite elements for the transverse displacement and the rotation (the finite element space for the rotations are in fact slightly enriched by interior degrees of freedom) with the element stiffness matrix altered through the use of a simple elementwise average in the computation of the shear energy. We prove that the approximate values of the displacement and the rotation, together with their first derivatives, all converge at an optimal rate uniformly with respect to thickness. As far as we know, this is the only method for the Reissner–Mindlin problem in the primitive variables for which uniform optimal convergence results have been established.

Although the Reissner–Mindlin model is simple in appearance, its discretization is not straightforward. Most seemingly reasonable choices of finite element spaces lead to an approximate solution that is far more sensitive to the plate thickness than the true solution, and that grossly underestimates the displacement of thin plates. The root of this difficulty, referred to as *locking* of the numerical solution, is well understood. As the plate thickness tends to zero, the Reissner–Mindlin model enforces the Kirchhoff constraint so that the rotation of the normal fibers equals the gradient of the transverse displacement. On the continuous level this simply means that the solution of the Reissner–Mindlin model converges to the solution of a related biharmonic problem as the thickness tends to zero. On the discrete level, the standard finite element

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formulation similarly imposes the Kirchhoff constraint on the finite element subspaces in the limit. Therefore, if a finite element method based on the standard formulation is to approximate well uniformly with respect to plate thickness, then the subspace of trial functions for the transverse displacement whose gradients are in the rotation trial space must have good approximation properties. This simply does not occur in standard low-order finite element spaces.[1]

The most common approach to avoiding the locking problem is to modify the variational formulation when determining the element stiffness matrices so that only a weaker, discrete Kirchhoff hypothesis is enforced in the thin plate limit. One possibility, known as "reduced/selective integration," is to compute the terms of the stiffness matrix involving the difference between the transverse displacement gradient and the rotation, i.e., the terms arising from the shear strain energy, with a quadrature rule of low-order. Another possibility—which is closely related and in some cases equivalent—is to interpolate or project (as proposed here) the discrete transverse shear strain into a lower-order finite element space. The use of altered variational principles enlarges the class of possible methods, and the question then becomes how to modify the variational formulation and choose the finite element spaces in order to achieve good approximation for all values of the plate thickness. This problem can be posed mathematically in a simple way. We consider a family of problems, one for each value of the plate thickness, in which the loadings are all proportional but scaled so that the solution tends to a nonzero limit as the the plate thickness tends to zero. The goal is then to derive a simple finite element scheme which gives optimal order approximation of the exact solution uniformly in the thickness.

In the case of the corresponding beam model, the Timoshenko beam, the use of standard finite element spaces and an appropriate reduced integration scheme give methods that are uniformly optimal-order accurate. For example, we may use continuous piecewise linear elements for the displacement and rotation and one integration point per element. The uniform optimality of these methods is very clearly seen in computations, and has been rigorously established through mathematical analysis [1]. (See also [13] for an arch problem.) For the plate problem, however, simple analogous procedures do not work. For example, standard piecewise linear elements for the displacement and rotation suffer severely from locking—whether or not reduced integration is used. Many procedures, using more complex elements and/or a more involved modification of the variational form, have been proposed by engineers and applied successfully in certain regimes. (The literature is far too extensive to review here. The reader is referred to [7] and [10] and the papers referenced therein.) However, none of these has been justified in a rigorous way, and the claims made on empirical and heuristic grounds have not been totally satisfying. In addition, most of these procedures are quite complicated, more so than the one proposed here.

Brezzi and Fortin [4] derived a reformulation of the Reissner–Mindlin plate model and devised a finite element method for this formulation for which they were able to derive uniform error estimates. Their formulation involves the introduction of two scalar variables (the irrotational and solenoidal parts of the transverse shear strain)

[1]There is a second difficulty in the numerical discretization of the Reissner–Mindlin plate, namely, the existence of a boundary layer in the solution. This problem is quite separate from the locking phenomenon, as can be seen by considering the Timoshenko beam, which does not have a boundary layer, but for which standard numerical methods evidence locking. Although the boundary layer of the plate solution is rather weak and does not pose great problems for the numerical solution, it does complicate the numerical analysis, particularly for higher-order methods.

in addition to the primitive variables (the transverse displacement and the rotation vector). For their numerical method they approximate all these variables by independent finite elements. Although the reformulation is equivalent to the Reissner–Mindlin model on the continuous level, the finite element scheme they use for its approximation is not equivalent to any practical finite element scheme applied directly to the original Reissner–Mindlin model. This fact complicates the implementation and its generalization to nonlinear and shell problems.

Several other authors have proposed and analyzed finite element methods for the Reissner–Mindlin model in the primitive variables, although none has achieved optimal-order estimates uniformly in the thickness. (We do not consider a bound for the error to be uniform in the thickness if it depends on norms of the solution that grow unboundedly with decreasing thickness.) In particular see [3] and [11].

We conclude this introduction with a list of some basic notations used in the sequel. In § 2 we recall the Reissner–Mindlin model and its reformulation by Brezzi and Fortin and show that they are well posed. In § 3 we describe our finite element method. In § 4 we prove a discrete analogue of the Helmholtz Theorem, giving the decomposition of a vector field into an irrotational and a solenoidal field. This result is essential to the analysis of our method, which is given in §§ 5 and 6. Let us remark that the discrete Helmholtz Theorem appears to be a new result, which should be useful elsewhere in the analysis of finite element methods. We close the paper with an Appendix, in which we prove certain regularity results used in the paper.

We will use the usual L^2-based Sobolev spaces H^s. The space H^{-1} denotes the normed dual of $\mathring{H}^1$, the closure of C_0^∞ in H^1. We use a circumflex above a function space to denote the subspace of elements with mean value zero. An undertilde to a space denotes the 2-vector-valued analogue. The undertilde is also affixed to vector-valued functions and operators, and double undertildes are used for matrix-valued objects. Thus, for example, the notation $\underset{\sim}{f} \in \hat{\underset{\sim}{L}}{}^2(\Omega)$ means that $\underset{\sim}{f}$ is a square integrable function on a domain Ω taking values in $\mathbb{R}^2$ and $\int_\Omega \underset{\sim}{f} = 0$. The letter C denotes a generic constant, not necessarily the same in each occurrence. Finally, we use various standard differential operators:

$$\underset{\sim}{\operatorname{grad}}\, r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \quad \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\psi} = \begin{pmatrix} \partial \psi_1/\partial x & \partial \psi_1/\partial y \\ \partial \psi_2/\partial x & \partial \psi_2/\partial y \end{pmatrix}, \quad \underset{\sim}{\operatorname{curl}}\, p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix},$$

$$\operatorname{div} \underset{\sim}{\psi} = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}, \qquad \operatorname{rot} \underset{\sim}{\psi} = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}.$$

2. The Reissner–Mindlin plate model. Let Ω denote the region in $\mathbb{R}^2$ occupied by the midsection of the plate, and denote by ω and $\underset{\sim}{\phi}$ the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. The Reissner–Mindlin plate model determines ω and $\underset{\sim}{\phi}$ as the unique solution to the following variational problem.

Problem M[1]. Find $(\omega, \underset{\sim}{\phi}) \in \mathring{H}^1(\Omega) \times \mathring{\underset{\sim}{H}}{}^1(\Omega)$ such that

$$a(\underset{\sim}{\phi}, \underset{\sim}{\psi}) + \lambda t^{-2}(\underset{\sim}{\phi} - \underset{\sim}{\operatorname{grad}}\, \omega, \underset{\sim}{\psi} - \underset{\sim}{\operatorname{grad}}\, \mu) = (g, \mu) \quad \forall (\mu, \underset{\sim}{\psi}) \in \mathring{H}^1(\Omega) \times \mathring{\underset{\sim}{H}}{}^1(\Omega). \tag{2.1}$$

Here g is the scaled transverse loading function, t is the plate thickness, $\lambda = Ek/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, and k the shear correction factor, and the parentheses denote the usual L^2 inner product. The bilinear form a is defined

$$a(\underset{\sim}{\phi}, \underset{\sim}{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega \left[\left(\frac{\partial \phi_1}{\partial x} + \nu \frac{\partial \phi_2}{\partial y} \right) \frac{\partial \psi_1}{\partial x} + \left(\nu \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) \frac{\partial \psi_2}{\partial y} + \frac{1-\nu}{2} \left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) \left(\frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x} \right) \right].$$

By Korn's inequality, a is an inner product on $\underset{\sim}{\mathring{H}}^1(\Omega)$ equivalent to the usual one. Note that we have scaled the load by a constant multiple of the square of the thickness so that the solution tends to a nonzero limit as t tends to zero.

For simplicity of notation, we shall henceforth consider a simplified version of this model. Namely, we will consider the problem whose weak formulation is given by (2.1), with $\lambda = 1$, and

$$a(\underset{\sim}{\phi}, \underset{\sim}{\psi}) = (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}, \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\psi}).$$

It will be easy to check that all our results apply equally well to the true problem.

For our analysis we shall also make use of an equivalent formulation of the Reissner–Mindlin plate equations suggested by Brezzi and Fortin [4]. This formulation is derived from Problem M^1 by using the Helmholtz Theorem to decompose the shear strain vector

$$t^{-2}(\underset{\sim}{\operatorname{grad}}\, \omega - \underset{\sim}{\phi}) = \underset{\sim}{\operatorname{grad}}\, r + \underset{\sim}{\operatorname{curl}}\, p, \tag{2.2}$$

with $(r, p) \in \mathring{H}^1(\Omega) \times \hat{H}^1(\Omega)$.

Problem M^2. Find $(r, \underset{\sim}{\phi}, p, \omega) \in \mathring{H}^1(\Omega) \times \underset{\sim}{\mathring{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ such that

$$(\underset{\sim}{\operatorname{grad}}\, r, \underset{\sim}{\operatorname{grad}}\, \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{H}^1(\Omega), \tag{2.3}$$

$$(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}, \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\psi}) - (\underset{\sim}{\operatorname{curl}}\, p, \underset{\sim}{\psi}) = (\underset{\sim}{\operatorname{grad}}\, r, \underset{\sim}{\psi}) \quad \text{for all } \underset{\sim}{\psi} \in \underset{\sim}{\mathring{H}}^1(\Omega), \tag{2.4}$$

$$-(\underset{\sim}{\phi}, \underset{\sim}{\operatorname{curl}}\, q) - t^2(\underset{\sim}{\operatorname{curl}}\, p, \underset{\sim}{\operatorname{curl}}\, q) = 0 \quad \text{for all } q \in \hat{H}^1(\Omega), \tag{2.5}$$

$$(\underset{\sim}{\operatorname{grad}}\, \omega, \underset{\sim}{\operatorname{grad}}\, s) = (\underset{\sim}{\phi} + t^2 \underset{\sim}{\operatorname{grad}}\, r, \underset{\sim}{\operatorname{grad}}\, s) \quad \text{for all } s \in \mathring{H}^1(\Omega). \tag{2.6}$$

Note that (2.3) is a simple Poisson equation that decouples from the other three equations. Once r has been determined from (2.3), $\underset{\sim}{\phi}$ and p may be computed from (2.4) and (2.5) and then ω is defined by (2.6), which is again a simple Poisson equation. Thus all the difficulties of the problem have been concentrated in the system (2.4)–(2.5) for $\underset{\sim}{\phi}$ and p. When $t = 0$, this pair of equations gives the usual weak formulation of the Stokes equations, subject to the trivial change of variables $(\phi_1, \phi_2) \to (\phi_2, -\phi_1)$. For positive t, these two equations represent a singularly perturbed Stokes system.

If $(\omega, \underset{\sim}{\phi})$ solves Problem M^1 and $(r, p) \in \mathring{H}^1(\Omega) \times \hat{H}^1(\Omega)$ are defined (uniquely) by (2.2), then (2.1) implies (2.3) and (2.4). Also (2.2) implies (2.5) and (2.6), so $(r, \underset{\sim}{\phi}, p, \omega)$ solves Problem M^2. Conversely if $(r, \underset{\sim}{\phi}, p, \omega)$ solves Problem M^2 then, since $\underset{\sim}{\operatorname{grad}}\, \mathring{H}^1(\Omega) + \underset{\sim}{\operatorname{curl}}\, \hat{H}^1(\Omega) = \underset{\sim}{L}^2(\Omega)$, (2.5) and (2.6) together imply (2.2), and then (2.3) and (2.4) imply (2.1). This shows the equivalence of Problems M^1 and M^2.

The following theorem asserts that Problem M^2 (and consequently also Problem M^1) is well posed. A proof is provided in the Appendix. The analogous result for the usual formulation of the Reissner–Mindlin plate on a smooth domain can be found in [9].

THEOREM 2.1. *Let Ω be a convex polygon or smoothly bounded domain in the plane. For any $t \in (0,1]$ and any $g \in H^{-1}$, there exists a unique quadruple $(r, \underset{\sim}{\phi}, p, \omega) \in \mathring{H}^1(\Omega) \times \underset{\sim}{\mathring{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ solving Problem* M^2. *Moreover, $\underset{\sim}{\phi} \in H^2(\Omega)$ and there exists a constant C independent of t and g, such that*

$$\|r\|_1 + \|\underset{\sim}{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|\omega\|_1 \le C\|g\|_{-1}.$$

If $g \in L^2(\Omega)$, then r, $\omega \in H^2(\Omega)$ and

$$\|r\|_2 + \|\omega\|_2 \le C\|g\|_0.$$

Remarks. (1) Theorem 2.1 also provides a uniform bound in $L^2(\Omega)$ for the shear strain vector $t^{-2}(\underset{\sim}{\phi} - \underset{\sim}{\operatorname{grad}}\,\omega)$, due to (2.2).

(2) If Ω is a smoothly bounded domain and $g \in H^1(\Omega)$, then we can easily extend the proof in the Appendix to show that $\omega \in H^3(\Omega)$ and to estimate $\|\omega\|_3$ uniformly in t. Also it is clear that if $g \in H^s(\Omega)$, $s \ge 0$, then r can be bounded in $H^{s+2}(\Omega)$ uniformly in t. However, neither $\|\underset{\sim}{\phi}\|_3$ nor $\|p\|_2$ may be bounded independently of t, even when the boundary and g are smooth. This is because of the existence of a boundary layer in the solution for t small.

3. The finite element method. Our finite element method is based on the variational formulation in Problem M^1. As the trial space for $\underset{\sim}{\phi}$ we shall use standard continuous piecewise linear finite elements augmented by a bubble function on each triangle. As the trial space for ω we shall use nonconforming piecewise linear elements. Moreover, in the second term in (2.1), we shall project $\underset{\sim}{\phi}$ onto a piecewise constant function. The reasons for these choices will emerge when we analyze the method.

We assume henceforth that the domain Ω is a convex polygon, which is triangulated by a triangulation $\mathcal{T}_h$. As usual, the subscript h refers to the diameter of the largest triangle, and the constants in our error estimates will be independent of h assuming that a minimum angle condition is satisfied as $h \to 0$. Denoting by $\mathcal{P}_k(T)$ the set of functions on T which are the restrictions of polynomials of degree no greater than k, we define the following finite element spaces:

$$\begin{aligned}
M^k_{-1} &= \{\eta \in L^2(\Omega) : \eta|_T \in \mathcal{P}_k(T) \text{ for all } T \in \mathcal{T}_h\}, \\
M^k_0 &= M^k_{-1} \cap H^1(\Omega), \\
\mathring{M}^k_0 &= M^k_{-1} \cap \mathring{H}^1(\Omega), \\
\mathring{M}^1_* &= \{\eta \in L^2(\Omega) : \eta|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h, \text{ and } \eta \text{ is continuous at midpoints} \\
&\qquad\qquad \text{of element edges and vanishes at midpoints of boundary edges}\}, \\
B^k &= \left\{\eta \in M^k_0 : \eta \text{ vanishes on the boundary of every element}\right\}, \\
\mathring{N}^1_0 &= \mathring{M}^1_0 \oplus B^3.
\end{aligned}$$

Note that $\mathring{N}^1_0$ is the usual space of conforming $\mathcal{P}_1$ elements augmented by one bubble function for each triangle and $\mathring{M}^1_*$ is the usual nonconforming $\mathcal{P}_1$ approximation of $\mathring{H}^1(\Omega)$. For $\mu \in \mathring{M}^1_* + H^1(\Omega)$ we define $\underset{\sim}{\operatorname{grad}}_h\,\mu$ to be the $\underset{\sim}{L}^2(\Omega)$ function whose restriction to each triangle $T \in \mathcal{T}_h$ is given by $\underset{\sim}{\operatorname{grad}}\,\mu|_T$.

Our approximation scheme is given in the following problem.

Problem M_h^1. Find $(\omega_h, \underset{\sim}{\phi}_h) \in \mathring{M}_*^1 \times \underset{\sim}{\mathring{N}}{}_0^1$ such that

$$(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\psi}) + t^{-2}(\underset{\sim}{P}_0 \underset{\sim}{\phi}_h - \underset{\sim}{\operatorname{grad}}_h\, \omega_h, \underset{\sim}{\psi} - \underset{\sim}{\operatorname{grad}}_h\, \mu) = (g, \mu) \tag{3.1}$$
$$\text{for all } (\mu, \underset{\sim}{\psi}) \in \mathring{M}_*^1 \times \underset{\sim}{\mathring{N}}{}_0^1,$$

where $\underset{\sim}{P}_0 : \underset{\sim}{L}^2(\Omega) \to \underset{\sim}{M}{}_{-1}^0$ is the orthogonal projection. Note that since this projection can be computed independently on each element, the implementation of Problem M_h^1 causes no difficulty.

We check that Problem M_h^1 admits a unique solution for any $g \in L^2(\Omega)$ and any positive t. It suffices to show that if $(\omega_h, \underset{\sim}{\phi}_h) \in \mathring{M}_*^1 \times \underset{\sim}{\mathring{N}}{}_0^1$ satisfies (3.1) for $g = 0$, then $\omega_h = 0$ and $\underset{\sim}{\phi}_h = 0$. Taking $(\mu, \underset{\sim}{\psi}) = (\omega_h, \underset{\sim}{\phi}_h)$ in (3.1) we have that

$$\| \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h \|_0^2 + t^{-2} \| \underset{\sim}{P}_0 \underset{\sim}{\phi}_h - \underset{\sim}{\operatorname{grad}}\, \omega_h \|_0^2 = 0.$$

This implies first that $\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{\phi}_h = 0$, so $\underset{\sim}{\phi}_h = 0$; whence $\underset{\sim}{\operatorname{grad}}\, \omega_h = 0$ and $\omega_h = 0$.

4. A discrete version of the Helmholtz Theorem. The Helmholtz Theorem states that any L^2 vector field can be decomposed uniquely into the sum of the gradient of a function r in $\mathring{H}^1$ plus the curl of a function p in $\hat{H}^1$; moreover, the two summands are orthogonal in $\underset{\sim}{L}^2$. It is not true in general that if the vector field is piecewise constant, then r and p will be continuous piecewise linear functions. However, the following theorem gives an alternative orthogonal decomposition of a piecewise constant function into the curl of a continuous piecewise linear function and the *piecewise* gradient of a *nonconforming* piecewise linear function.

THEOREM 4.1.

$$\underset{\sim}{M}{}_{-1}^0 = \underset{\sim}{\operatorname{grad}}_h\, \mathring{M}_*^1 \oplus \underset{\sim}{\operatorname{curl}}\, \hat{M}_0^1. \tag{4.1}$$

This is an $\underset{\sim}{L}^2$ orthogonal decomposition.

Proof. It is obvious that the summands in (4.1) are piecewise constant functions, i.e., they are contained in $\underset{\sim}{M}{}_{-1}^0$. We verify that they are orthogonal in $\underset{\sim}{L}^2$. Let $r \in \mathring{M}_*^1$ and $p \in \hat{M}_0^1$. Then

$$(\underset{\sim}{\operatorname{grad}}_h\, r, \underset{\sim}{\operatorname{curl}}\, p) = \sum_T \int_T \underset{\sim}{\operatorname{grad}}_h\, r \cdot \underset{\sim}{\operatorname{curl}}\, p = -\sum_T \int_{\partial T} r \frac{\partial p}{\partial s_T}. \tag{4.2}$$

Here s_T is the counterclockwise tangent to ∂T. Now let e be any interior edge of the triangulation, say $e = T_+ \cap T_-$. Let $r_\pm = r|_{T_\pm}$ and $s_\pm = s|_{T_\pm}$. Since p is a piecewise linear function, the derivatives $\partial p_\pm / \partial s_{T_\pm}$ are constant on e, and since p is continuous, $\partial p_+ / \partial s_{T_+} = -\partial p_- / \partial s_{T_-}$. Since $r \in \mathring{M}_*^1$, $r_+ - r_-$ is a linear function on e that vanishes at the midpoint. It follows that

$$\int_e r_+ \frac{\partial p_+}{\partial s_{T_+}} + \int_e r_- \frac{\partial p_-}{\partial s_{T_-}} = 0.$$

For similar reasons, if e is a boundary edge contained in the triangle T, $\int_e r \partial p_+ / \partial s_T = 0$. Adding over all edges we conclude that the final sum in (4.2) is zero, so that $\underset{\sim}{\operatorname{grad}}_h\, r$ is orthogonal to $\underset{\sim}{\operatorname{curl}}\, p$.

It remains to show that any function $\underset{\sim}{\eta} \in \underset{\sim}{M}{}_{-1}^0$ may be decomposed as $\underset{\sim}{\operatorname{grad}}_h\, r + \underset{\sim}{\operatorname{curl}}\, p$ for some $r \in \mathring{M}_*^1$ and $p \in \hat{M}_0^1$. Define r as the approximate solution of the

Dirichlet problem for the equation $\Delta r = \operatorname{div} \underset{\sim}{\eta}$ using nonconforming piecewise linear elements. More precisely, $r \in \mathring{M}_*^1$ is defined by

$$(\underset{\sim}{\operatorname{grad}}_h r - \underset{\sim}{\eta}, \underset{\sim}{\operatorname{grad}}_h s) = 0 \quad \text{for all } s \in \mathring{M}_*^1. \tag{4.3}$$

Again let e denote an interior edge of the triangulation, and denote by $\underset{\sim}{n}_e$ one of the unit vectors normal to e. If we choose s in (4.3) to be the element of $\mathring{M}_*^1$, which is equal to 1 at the midpoint of e and which vanishes at all other edge midpoints, and integrate by parts over each triangle, we conclude (reasoning as before) that the constant values of $(\underset{\sim}{\eta} - \underset{\sim}{\operatorname{grad}}_h r) \cdot \underset{\sim}{n}_e$ from the two sides of e coincide. This implies that the distributional divergence of $\underset{\sim}{\eta} - \underset{\sim}{\operatorname{grad}}_h r$ belongs to $L^2(\Omega)$. Since this function is constant on every triangle, it follows that $\operatorname{div}(\underset{\sim}{\eta} - \underset{\sim}{\operatorname{grad}}_h r) = 0$ in the sense of distributions. Since Ω is simply connected (in fact, convex), there exists a function $p \in \hat{H}^1(\Omega)$ such that $\underset{\sim}{\operatorname{curl}} p = \underset{\sim}{\eta} - \underset{\sim}{\operatorname{grad}}_h r$. Finally, since $\underset{\sim}{\operatorname{curl}} p$ is constant on every triangle, p is necessarily linear on every triangle, i.e., $p \in \hat{M}_0^1$. $\square$

Remark. We have chosen to give a constructive proof of this theorem. An alternative approach is to check that

$$\dim \underset{\sim}{M}_{-1}^0 = \dim \underset{\sim}{\operatorname{grad}}_h \mathring{M}_*^1 + \dim \underset{\sim}{\operatorname{curl}} \hat{M}_0^1,$$

which, together with the orthogonality of the summands, establishes the theorem.

5. Error analysis. First we introduce a discrete version of Problem M^2 and show that it is equivalent to Problem M_h^1.

Problem M_h^2. Find $(r_h, \underset{\sim}{\phi}_h, p_h, \omega_h) \in \mathring{M}_*^1 \times \mathring{\underset{\sim}{N}}{}_0^1 \times \hat{M}_0^1 \times \mathring{M}_*^1$ such that

$$(\underset{\sim}{\operatorname{grad}}_h r_h, \underset{\sim}{\operatorname{grad}}_h \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{M}_*^1, \tag{5.1}$$

$$(\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) - (\underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\psi}) = (\underset{\sim}{\operatorname{grad}}_h r_h, \underset{\sim}{\psi}) \quad \text{for all } \underset{\sim}{\psi} \in \mathring{\underset{\sim}{N}}{}_0^1, \tag{5.2}$$

$$-(\underset{\sim}{\phi}_h, \underset{\sim}{\operatorname{curl}} q) - t^2(\underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\operatorname{curl}} q) = 0 \quad \text{for all } q \in \hat{M}_0^1, \tag{5.3}$$

$$(\underset{\sim}{\operatorname{grad}}_h \omega_h, \underset{\sim}{\operatorname{grad}}_h s) = (\underset{\sim}{\phi}_h + t^2 \underset{\sim}{\operatorname{grad}}_h r_h, \underset{\sim}{\operatorname{grad}}_h s) \quad \text{for all } s \in \mathring{M}_*^1. \tag{5.4}$$

LEMMA 5.1. *For any $g \in L^2(\Omega)$ and any $t \in (0, 1]$ there exists a unique solution $(r_h, \underset{\sim}{\phi}_h, p_h, \omega_h)$ to Problem M_h^2. Moreover, the pair $(\omega_h, \underset{\sim}{\phi}_h)$ is the unique solution of Problem M_h^1 and*

$$t^{-2}(\underset{\sim}{\operatorname{grad}} \omega_h - \underset{\sim}{P}_0 \underset{\sim}{\phi}_h) = \underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h. \tag{5.5}$$

Proof. Suppose $g = 0$. Choosing $\mu = r_h$ in (5.1) we see that $r_h = 0$. Next set $\underset{\sim}{\psi} = \underset{\sim}{\phi}_h$ in (5.2) and $q = p_h$ in (5.3) and subtract. This shows that $\underset{\sim}{\phi}_h = 0$ and $p_h = 0$. Finally, taking $s = \omega_h$, we conclude that $\omega_h = 0$. This establishes existence and uniqueness.

Since $\underset{\sim}{\operatorname{curl}} q$ and $\underset{\sim}{\operatorname{grad}}_h s$ are piecewise constant for $q \in \hat{M}_0^1$, $s \in \mathring{M}_*^1$, we may replace $\underset{\sim}{\phi}_h$ by $\underset{\sim}{P}_0 \underset{\sim}{\phi}_h$ in (5.3) and (5.4). Using the orthogonality proved in Theorem 4.1, we deduce that

$$(\underset{\sim}{\operatorname{grad}}_h \omega_h - \underset{\sim}{P}_0 \underset{\sim}{\phi}_h, \underset{\sim}{\operatorname{curl}} q) = t^2(\underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\operatorname{curl}} q) \quad \text{for all } q \in \hat{M}_0^1,$$

$$(\underset{\sim}{\operatorname{grad}}_h \omega_h - \underset{\sim}{P}_0 \underset{\sim}{\phi}_h, \underset{\sim}{\operatorname{grad}}_h s) = t^2(\underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\operatorname{grad}}_h s) \quad \text{for all } s \in \mathring{M}_*^1.$$

In light of Theorem 4.1, these two equations are equivalent to the single equation

$$(\underset{\sim}{\operatorname{grad}}_h \omega_h - \underset{\sim}{P}_0 \underset{\sim}{\phi}_h, \underset{\sim}{\eta}) = t^2(\underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\eta}) \quad \text{for all } \underset{\sim}{\eta} \in \underset{\sim}{M}^0_{-1},$$

from which (5.5) follows.

A similar application of orthogonality transforms (5.1) to

$$(\underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\operatorname{grad}}_h \mu) = (g, \mu) \quad \text{for all } \mu \in \mathring{M}^1_*.$$

Combining this equation with (5.2) gives

$$(\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) - (\underset{\sim}{\operatorname{grad}}_h r_h + \underset{\sim}{\operatorname{curl}} p_h, \underset{\sim}{\psi} - \underset{\sim}{\operatorname{grad}}_h \mu) = (g, \mu) \qquad \text{for all } (\underset{\sim}{\psi}, \mu) \in \underset{\sim}{\mathring{N}}^1_0 \times \mathring{M}^1_*,$$

or, in light of (5.5),

$$(\underset{\approx}{\operatorname{grad}} \underset{\sim}{\phi}_h, \underset{\approx}{\operatorname{grad}} \underset{\sim}{\psi}) + t^{-2}(\underset{\sim}{P}_0 \underset{\sim}{\phi}_h - \underset{\sim}{\operatorname{grad}} \omega_h, \underset{\sim}{\psi} - \underset{\sim}{\operatorname{grad}}_h \mu) = (g, \mu) \qquad \text{for all } (\underset{\sim}{\psi}, \mu) \in \underset{\sim}{\mathring{N}}^1_0 \times \mathring{M}^1_*,$$

i.e., $(\omega_h, \underset{\sim}{\phi}_h)$ solves Problem M^1_h. □

Problem M^2_h is similar to the discretization suggested by Brezzi and Fortin [4]. Like them, we use the space $\underset{\sim}{\mathring{N}}^1_0 \times \hat{M}^1_0$ (the MINI element of [2]) to discretize the Stokes-like system for $\underset{\sim}{\phi}_h$ and p_h. The MINI element is very appropriate in this context because it is stable for the Stokes system ($t = 0$) and the pressure elements are continuous so the penalty term in (5.3) makes sense. The only difference between Problem M^2_h and the method of [4] is that we use the nonconforming space $\mathring{M}^1_*$ for the variables r_h and ω_h rather than the conforming space $\mathring{M}^1_0$. While it might seem more natural to use a conforming space, the reduction to a method in the primitive variables, i.e., the equivalence to Problem M^1_h, depends on the choice of nonconforming elements.

In order to prove error estimates for our approximation scheme, we require some relatively standard results on nonconforming methods. The first lemma, which is of a sort basic to the analysis of nonconforming methods, can be found in essence in [8].

LEMMA 5.2. *There exists a constant C independent of h such that*

$$\left| \sum_{T \in \mathcal{T}_h} \int_{\partial T} p \underset{\sim}{\psi} \cdot \underset{\sim}{n}_T \right| \le Ch \|\underset{\sim}{\psi}\|_1 \inf_{q \in \mathring{H}^1(\Omega)} \|\underset{\sim}{\operatorname{grad}}_h (p - q)\|_0 \qquad \textit{for all } \underset{\sim}{\psi} \in \underset{\sim}{H}^1(\Omega),\ p \in \mathring{M}^1_* + \mathring{H}^1(\Omega).$$

LEMMA 5.3 (DISCRETE POINCARÉ INEQUALITY). *There exists a constant C independent of h such that*

$$\|p\|_0 \le C \|\underset{\sim}{\operatorname{grad}}_h p\|_0 \quad \textit{for all } p \in \mathring{M}^1_*.$$

Proof. For any $\underset{\sim}{\psi} \in \underset{\sim}{H}^1(\Omega)$, $p \in \mathring{M}^1_*$,

$$\int_\Omega p \operatorname{div} \underset{\sim}{\psi} = -\sum_{T \in \mathcal{T}_h} \left(\int_T \underset{\sim}{\operatorname{grad}}_h p \cdot \underset{\sim}{\psi} - \int_{\partial T} p \underset{\sim}{\psi} \cdot \underset{\sim}{n}_T \right) \le C \|\underset{\sim}{\operatorname{grad}}_h p\|_0 \|\underset{\sim}{\psi}\|_1,$$

by Lemma 5.2. Choosing $\underset{\sim}{\psi}$ so that $\operatorname{div} \underset{\sim}{\psi} = p$ and $\|\underset{\sim}{\psi}\|_1 \le C\|p\|_0$ gives the result. □

LEMMA 5.4. *Given $G \in L^2(\Omega)$ and $\underset{\sim}{F} \in \underset{\sim}{H}^1(\Omega)$, let u solve the boundary value problem*

$$-\Delta u = G - \operatorname{div} \underset{\sim}{F} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Let $u_h \in \mathring{M}^1_*$ be defined by

$$(\underset{\sim}{\operatorname{grad}}_h u_h, \underset{\sim}{\operatorname{grad}}_h v_h) = (G, v_h) + (\underset{\sim}{F}, \underset{\sim}{\operatorname{grad}}_h v_h) \quad \text{for all } v_h \in \mathring{M}^1_*.$$

Then

$$h\|\underset{\sim}{\operatorname{grad}}_h (u - u_h)\|_0 + \|u - u_h\|_0 \le Ch^2(\|G\|_0 + \|\underset{\sim}{F}\|_1),$$

where C is a constant independent of G, $\underset{\sim}{F}$, and h.

Proof. When $\underset{\sim}{F} = 0$ the convergence result for the gradient can be found in [14] along with remarks for deriving the $L^2(\Omega)$ estimate. The proof, which works equally well for nonzero $\underset{\sim}{F}$, may be obtained as an application of the basic abstract error estimates for nonconforming methods (see also [6, Thm. 4.2.2] for the gradient estimate and [6, Exercise 4.2.3] for the $L^2(\Omega)$ estimate), using Lemma 5.2 to bound the consistency errors. □

We now give the energy estimates for our method, the proof of which is similar to the proof in [4].

THEOREM 5.5. *There exists a constant C such that if $(r, \underset{\sim}{\phi}, p, \omega)$ and $(r_h, \underset{\sim}{\phi}_h, p_h, \omega_h)$ solve Problems M^2 and M^2_h, respectively, for some $g \in L^2(\Omega)$ and some $t \in (0, 1]$, then*

$$\|\underset{\sim}{\operatorname{grad}}_h (r - r_h)\|_0 + \|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_1 + \|p - p_h\|_0 + t\|p - p_h\|_1 + \|\underset{\sim}{\operatorname{grad}}_h (\omega - \omega_h)\|_0 \le Ch\|g\|_0.$$

The constant C is independent of g and t.

Proof. From (2.3), (5.1), and Lemma 5.4, it follows that

$$\|\underset{\sim}{\operatorname{grad}}_h (r - r_h)\|_0 \le Ch\|g\|_0. \tag{5.6}$$

Let $\underset{\sim}{\psi} \in \mathring{\underset{\sim}{N}}{}^1_0$ be arbitrary. From (2.4) and (5.2) it follows that

$$\begin{aligned}\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\|_0^2 &= \big(\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\psi}), \underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\big) + \big(\underset{\sim}{\operatorname{curl}}(p_h - p), \underset{\sim}{\phi}_h - \underset{\sim}{\psi}\big) \\ &\quad + \big(\underset{\sim}{\operatorname{grad}}_h (r_h - r), \underset{\sim}{\phi}_h - \underset{\sim}{\psi}\big).\end{aligned}$$

From (2.5) and (5.3) it follows that for any $\underset{\sim}{\psi} \in \mathring{\underset{\sim}{N}}{}^1_0$ and $q \in \hat{M}^1_0$,

$$\begin{aligned}t^2\|\underset{\sim}{\operatorname{curl}}(p_h - q)\|_0^2 &= t^2\big(\underset{\sim}{\operatorname{curl}}(p - q), \underset{\sim}{\operatorname{curl}}(p_h - q)\big) - \big(\underset{\sim}{\phi}_h - \underset{\sim}{\phi}, \underset{\sim}{\operatorname{curl}}(p_h - q)\big) \\ &= t^2\big(\underset{\sim}{\operatorname{curl}}(p - q), \underset{\sim}{\operatorname{curl}}(p_h - q)\big) \\ &\qquad - \big(\underset{\sim}{\phi}_h - \underset{\sim}{\psi}, \underset{\sim}{\operatorname{curl}}(p_h - q)\big) + \big(\underset{\sim}{\phi} - \underset{\sim}{\psi}, \underset{\sim}{\operatorname{curl}}(p_h - q)\big).\end{aligned}$$

Adding these two equations, we get

$$\begin{aligned}&\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\|_0^2 + t^2\|\underset{\sim}{\operatorname{curl}}(p_h - q)\|_0^2 \\ &\quad = \big(\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\psi}), \underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\big) \\ &\qquad + t^2\big(\underset{\sim}{\operatorname{curl}}(p - q), \underset{\sim}{\operatorname{curl}}(p_h - q)\big) - \big(\underset{\sim}{\operatorname{curl}}(p - q), \underset{\sim}{\phi}_h - \underset{\sim}{\psi}\big) \\ &\qquad + \big(\underset{\sim}{\phi} - \underset{\sim}{\psi}, \underset{\sim}{\operatorname{curl}}(p_h - q)\big) + \big(\underset{\sim}{\operatorname{grad}}_h (r_h - r), \underset{\sim}{\phi}_h - \underset{\sim}{\psi}\big).\end{aligned}$$

Integrating the third and fourth terms on the right-hand side by parts and applying the Schwarz inequality, the arithmetic-geometric mean inequality, and the Poincaré inequality, we get

(5.7)

$$\begin{aligned}\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\|_0^2 + t^2\|\underset{\sim}{\operatorname{curl}}(p_h - q)\|_0^2 \le C\big(&\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\psi})\|_0^2 + t^2\|\underset{\sim}{\operatorname{curl}}(p - q)\|_0^2 \\ &+ \|p - q\|_0^2 + \|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\psi})\|_0\|p_h - q\|_0 \\ &+ \|\underset{\sim}{\operatorname{grad}}_h\,(r - r_h)\|_0^2\big).\end{aligned}$$

In [2] it was shown that the MINI element is stable for the Stokes problem, i.e., that there exists $\gamma > 0$ independent of h such that for all $q \in \hat{M}_0^1$ there exists a nonzero $\underset{\sim}{\xi} \in \underset{\sim}{\mathring{N}}_0^1$ with

$$\gamma\|q\|_0\|\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\xi}\|_0 \le (\underset{\sim}{\operatorname{curl}}\,q, \underset{\sim}{\xi}).$$

Applying this result with q replaced by $p_h - q$, and again using (2.4) and (5.2), we have

$$\begin{aligned}&\gamma\|p_h - q\|_0\|\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\xi}\|_0 \\ &\quad\le \big(\underset{\sim}{\operatorname{curl}}(p_h - q), \underset{\sim}{\xi}\big) \\ &\quad= \big(\underset{\sim}{\operatorname{curl}}(p - q), \underset{\sim}{\xi}\big) + \big(\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\phi}), \underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\xi}\big) \\ &\qquad - \big(\underset{\sim}{\operatorname{grad}}_h\,(r_h - r), \underset{\sim}{\xi}\big) \\ &\quad\le C\big(\|p - q\|_0 + \|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\phi})\|_0 \\ &\qquad + \|\underset{\sim}{\operatorname{grad}}_h\,(r_h - r)\|_0\big)\|\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{\xi}\|_0,\end{aligned}$$

so for all $q \in \hat{M}_0^1$ and $\underset{\sim}{\psi} \in \underset{\sim}{\mathring{N}}_0^1$,

(5.8)

$$\begin{aligned}\|p_h - q\|_0 &\le C\big(\|p - q\|_0 + \|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\phi})\|_0 + \|\underset{\sim}{\operatorname{grad}}_h\,(r_h - r)\|_0\big) \\ &\le C\big(\|p - q\|_0 + \|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\|_0 + \|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\psi})\|_0 + \|\underset{\sim}{\operatorname{grad}}_h\,(r_h - r)\|_0\big).\end{aligned}$$

Substituting (5.7) in (5.8) and again applying the arithmetic-geometric mean inequality, we deduce that

$$\begin{aligned}&\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}_h - \underset{\sim}{\psi})\|_0 + t\|\underset{\sim}{\operatorname{curl}}(p_h - q)\|_0 \\ &\qquad\le C\big(\|\underset{\sim}{\phi} - \underset{\sim}{\psi}\|_1 + t\|p - q\|_1 + \|p - q\|_0 + \|\underset{\sim}{\operatorname{grad}}_h\,(r - r_h)\|_0\big).\end{aligned}$$

By the triangle inequality

$$\begin{aligned}&\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)\|_0 + t\|\underset{\sim}{\operatorname{curl}}(p - p_h)\|_0 \\ &\qquad\le C\big(\|\underset{\sim}{\phi} - \underset{\sim}{\psi}\|_1 + t\|p - q\|_1 + \|p - q\|_0 + \|\underset{\sim}{\operatorname{grad}}_h\,(r - r_h)\|_0\big),\end{aligned}$$

and with (5.8) we also have

$$\|p - p_h\|_0 \le C\big(\|\underset{\sim}{\phi} - \underset{\sim}{\psi}\|_1 + t\|p - q\|_1 + \|p - q\|_0 + \|\underset{\sim}{\operatorname{grad}}_h\,(r - r_h)\|_0\big).$$

Since $\underset{\sim}{\psi} \in \underset{\sim}{\mathring{N}}_0^1$ and $q \in \hat{M}_0^1$ are arbitrary, we may apply standard approximation theory, (5.6), and Theorem 2.1 to get

$$\|\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h)\|_0 + \|p - p_h\|_0 + t\|p - p_h\|_1 \le Ch(\|\underset{\sim}{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|g\|_0) \le Ch\|g\|_0.$$

It remains to bound $\|\omega - \omega_h\|_1$. Define $\bar{\omega}_h \in \mathring{M}^1_*$ by

$$(\underset{\sim}{\mathrm{grad}}_h \, \bar{\omega}_h, \underset{\sim}{\mathrm{grad}}_h \, s) = (\underset{\sim}{\phi} + t^2 \, \underset{\sim}{\mathrm{grad}} \, r, \underset{\sim}{\mathrm{grad}}_h \, s) \quad \text{for all } s \in \mathring{M}^1_*.$$

Then, by Lemma 5.4,

$$\|\underset{\sim}{\mathrm{grad}}_h \, (\omega - \bar{\omega}_h)\|_0 \le Ch\|\underset{\sim}{\phi} + t^2 \, \underset{\sim}{\mathrm{grad}} \, r\|_1 \le Ch\|g\|_0.$$

Thus it suffices to prove that

$$\|\underset{\sim}{\mathrm{grad}}_h \, (\omega_h - \bar{\omega}_h)\|_0 \le Ch\|g\|_0. \tag{5.9}$$

Now

$$\big(\underset{\sim}{\mathrm{grad}}_h \, (\omega_h - \bar{\omega}_h), \underset{\sim}{\mathrm{grad}}_h \, s\big) = \big(\underset{\sim}{\phi}_h - \underset{\sim}{\phi} + t^2 \underset{\sim}{\mathrm{grad}}_h \, (r_h - r), \underset{\sim}{\mathrm{grad}}_h \, s\big) \quad \text{for all } s \in \mathring{M}^1_*.$$

Taking $s = \omega_h - \bar{\omega}_h$ gives

$$\|\underset{\sim}{\mathrm{grad}}_h \, (\omega_h - \bar{\omega}_h)\|_0 \le C\big(\|\underset{\sim}{\phi}_h - \underset{\sim}{\phi}\|_0 + t^2\|\underset{\sim}{\mathrm{grad}}_h \, (r_h - r)\|_0\big) \le Ch\|g\|_0,$$

proving (5.9). $\square$

6. Error analysis continued: L^2 estimates. In this section we estimate $\underset{\sim}{\phi} - \underset{\sim}{\phi}_h$ and $\omega - \omega_h$ in L^2.

THEOREM 6.1. *Under the hypotheses of Theorem* 5.5,

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^2\|g\|_0.$$

Proof. First we note that Lemma 5.4 implies

$$\|r - r_h\|_0 \le Ch^2\|g\|_0. \tag{6.1}$$

To derive an L^2 estimate on the rotation vector, we will apply a variant of the usual Aubin–Nitsche duality argument. To do so, we introduce the following auxiliary problem.

Problem A. Find $(\underset{\sim}{\Phi}, P) \in \underset{\sim}{\mathring{H}}^1(\Omega) \times \hat{H}^1(\Omega)$ such that

$$\begin{aligned}&(\underset{\approx}{\mathrm{grad}} \, \underset{\sim}{\psi}, \underset{\approx}{\mathrm{grad}} \, \underset{\sim}{\Phi}) - (\underset{\sim}{\psi}, \underset{\sim}{\mathrm{curl}} \, P) = (\underset{\sim}{\phi} - \underset{\sim}{\phi}_h, \underset{\sim}{\psi}) \quad \text{for all } \underset{\sim}{\psi} \in \underset{\sim}{\mathring{H}}^1(\Omega),\\ &-(\underset{\sim}{\mathrm{curl}} \, q, \underset{\sim}{\Phi}) - t^2(\underset{\sim}{\mathrm{curl}} \, q, \underset{\sim}{\mathrm{curl}} \, P) = 0 \quad \text{for all } q \in \hat{H}^1(\Omega).\end{aligned}$$

We show in the Appendix that this problem admits a unique solution, and that

$$\|\underset{\sim}{\Phi}\|_2 + \|P\|_1 + t\|P\|_2 \le C\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_0. \tag{6.2}$$

Choosing $\underset{\sim}{\psi} = \underset{\sim}{\phi} - \underset{\sim}{\phi}_h$ and $q = p - p_h$ in Problem A gives

$$\begin{aligned}\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_0^2 =& (\underset{\approx}{\mathrm{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\mathrm{grad}} \, \underset{\sim}{\Phi}) - \big((\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\sim}{\mathrm{curl}} \, P\big) - \big(\underset{\sim}{\mathrm{curl}}(p - p_h), \underset{\sim}{\Phi}\big)\\ &- t^2(\underset{\sim}{\mathrm{curl}}(p - p_h), \underset{\sim}{\mathrm{curl}} \, P).\end{aligned}$$

Next use (2.4), (2.5), (5.2), and (5.3) to get, for any $(q, \underset{\sim}{\psi}) \in \mathring{M}^1_* \times \underset{\sim}{N}^1_0$,

$$\begin{aligned}\|\underset{\sim}{\phi} - \underset{\sim}{\phi}_h\|_0^2 &= \big(\underset{\approx}{\mathrm{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\mathrm{grad}}(\underset{\sim}{\Phi} - \underset{\sim}{\psi})\big) - \big(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h, \underset{\sim}{\mathrm{curl}}(P - q)\big)\\ &\quad - \big(\underset{\sim}{\mathrm{curl}}(p - p_h), \underset{\sim}{\Phi} - \underset{\sim}{\psi}\big) - t^2\big(\underset{\sim}{\mathrm{curl}}(p - p_h), \underset{\sim}{\mathrm{curl}}(P - q)\big)\\ &\quad + \big(\underset{\sim}{\mathrm{grad}}_h \, (r - r_h), \underset{\sim}{\psi}\big)\\ &= \big(\underset{\approx}{\mathrm{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), \underset{\approx}{\mathrm{grad}}(\underset{\sim}{\Phi} - \underset{\sim}{\psi})\big) - \big(\mathrm{rot}(\underset{\sim}{\phi} - \underset{\sim}{\phi}_h), P - q\big)\\ &\quad - \big(p - p_h, \mathrm{rot}(\underset{\sim}{\Phi} - \underset{\sim}{\psi})\big) - t^2\big(\underset{\sim}{\mathrm{curl}}(p - p_h), \underset{\sim}{\mathrm{curl}}(P - q)\big)\\ &\quad + \big(\underset{\sim}{\mathrm{grad}}_h \, (r - r_h), \underset{\sim}{\psi}\big).\end{aligned} \tag{6.3}$$

We now choose $\underset{\sim}{\psi}$ and q such that

$$\|\underset{\sim}{\Phi}-\underset{\sim}{\psi}\|_1 \le Ch\|\underset{\sim}{\Phi}\|_2, \qquad \|P-q\|_0 \le Ch\|P\|_1, \qquad \|P-q\|_1 \le Ch\|P\|_2.$$

Turning to the terms on the right-hand side of (6.3), we then have

$$\begin{aligned}
&\big(\underset{\approx}{\operatorname{grad}}(\underset{\sim}{\phi}-\underset{\sim}{\phi}_h), \underset{\approx}{\operatorname{grad}}(\underset{\sim}{\Phi}-\underset{\sim}{\psi})\big) - \big(\operatorname{rot}(\underset{\sim}{\phi}-\underset{\sim}{\phi}_h), P-q\big) - \big(p-p_h, \operatorname{rot}(\underset{\sim}{\Phi}-\underset{\sim}{\psi})\big)\\
&\qquad - t^2\big(\underset{\sim}{\operatorname{curl}}(p-p_h), \underset{\sim}{\operatorname{curl}}(P-q)\big)\\
(6.4)\qquad &\le Ch(\|\underset{\sim}{\phi}-\underset{\sim}{\phi}_h\|_1\|\underset{\sim}{\Phi}\|_2 + \|\underset{\sim}{\phi}-\underset{\sim}{\phi}_h\|_1\|P\|_1 + \|p-p_h\|_0\|\underset{\sim}{\Phi}\|_2\\
&\qquad + t^2\|p-p_h\|_1\|P\|_2)\\
&\le Ch^2\|g\|_0\|\underset{\sim}{\phi}-\underset{\sim}{\phi}_h\|_0
\end{aligned}$$

where we have used Theorem 5.5 and (6.2) in the last step. To treat the last term in (6.3) we integrate by parts, then apply Lemma 5.2, and finally invoke Theorem 5.5, (6.1), and (6.2) to obtain

$$\begin{aligned}
(\underset{\sim}{\operatorname{grad}}_h(r-r_h), \underset{\sim}{\psi}) &= -(r-r_h, \operatorname{div}\underset{\sim}{\psi}) + \sum_T \int_{\partial T}(r-r_h)\underset{\sim}{\psi}\cdot\underset{\sim}{n}_T\\
(6.5)\qquad &\le \|r-r_h\|_0\|\underset{\sim}{\psi}\|_1 + Ch\|\underset{\sim}{\operatorname{grad}}_h(r-r_h)\|_0\|\underset{\sim}{\psi}\|_1\\
&\le \big(\|r-r_h\|_0 + h\|\underset{\sim}{\operatorname{grad}}_h(r-r_h)\|_0\big)\|\underset{\sim}{\Phi}\|_2\\
&\le Ch^2\|g\|_0\|\underset{\sim}{\phi}-\underset{\sim}{\phi}_h\|_0.
\end{aligned}$$

Substituting (6.4) and (6.5) in (6.3) gives

$$\|\underset{\sim}{\phi}-\underset{\sim}{\phi}_h\|_0 \le Ch^2\|g\|_0$$

as desired.

To complete the proof of the theorem, we now bound $\|\omega-\omega_h\|_0$. Let $\theta=\omega-t^2r$ and $\theta_h=\omega_h-t^2r_h$. In light of (6.1), it suffices to show that

$$(6.6)\qquad \|\theta-\theta_h\|_0 \le Ch^2\|g\|_0.$$

Now $\theta\in \mathring{H}^1(\Omega)$ and, from (2.2), $-\Delta\theta = -\operatorname{div}\underset{\sim}{\phi}$. Therefore if we define $\bar\theta_h\in\mathring{M}^1_*$ by the equations

$$(6.7)\qquad (\underset{\sim}{\operatorname{grad}}_h\bar\theta_h, \underset{\sim}{\operatorname{grad}}_h s) = (\underset{\sim}{\phi}, \underset{\sim}{\operatorname{grad}}_h s)\quad \text{for all } s\in\mathring{M}^1_*,$$

it follows from Lemma 5.4 that

$$(6.8)\qquad \|\theta-\bar\theta_h\|_0 \le Ch^2\|\underset{\sim}{\phi}\|_1 \le Ch^2\|g\|_0.$$

From (5.4) and (6.7), we have

$$\big(\underset{\sim}{\operatorname{grad}}_h(\theta_h-\bar\theta_h), \underset{\sim}{\operatorname{grad}}_h s\big) = (\underset{\sim}{\phi}_h-\underset{\sim}{\phi}, \underset{\sim}{\operatorname{grad}}_h s)\quad \text{for all } s\in\mathring{M}^1_*.$$

Setting $s=\theta_h-\bar\theta_h$ gives

$$(6.9)\qquad \|\underset{\sim}{\operatorname{grad}}_h(\theta_h-\bar\theta_h)\|_0 \le C\|\underset{\sim}{\phi}_h-\underset{\sim}{\phi}\|_0 \le Ch^2\|g\|_0.$$

Now (6.6) follows from (6.8), (6.9), and Lemma 5.3. $\square$

7. Appendix. In this Appendix we prove that the solution to the Reissner–Mindlin system possesses the regularity we need for the foregoing analysis uniformly in the thickness. Similar results can be found in [4] and [9]. Note that we allow a forcing function in the second equation below (used in the derivation of the L^2 estimates) as well as a convex polygonal domain.

THEOREM 7.1. *Let Ω be a convex polygon or a smoothly bounded domain in the plane. For any $t \in (0,1]$, $G \in H^{-1}(\Omega)$, and $\underset{\sim}{F} \in \underset{\sim}{H}^{-1}(\Omega)$, there exists a unique quadruple $(r, \underset{\sim}{\phi}, p, \omega) \in \mathring{H}^1(\Omega) \times \mathring{\underset{\sim}{H}}^1(\Omega) \times \hat{H}^1(\Omega) \times \mathring{H}^1(\Omega)$ solving*

$$(7.1)\qquad (\underset{\sim}{\mathrm{grad}}\, r, \underset{\sim}{\mathrm{grad}}\, \mu) = (G, \mu) \quad \text{for all } \mu \in \mathring{H}^1(\Omega),$$

$$(7.2)\qquad (\underset{\approx}{\mathrm{grad}}\, \underset{\sim}{\phi}, \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{\psi}) - (\underset{\sim}{\mathrm{curl}}\, p, \underset{\sim}{\psi}) = (\underset{\sim}{\mathrm{grad}}\, r, \underset{\sim}{\psi}) + (\underset{\sim}{F}, \underset{\sim}{\psi}) \quad \text{for all } \underset{\sim}{\psi} \in \mathring{\underset{\sim}{H}}^1(\Omega),$$

$$(7.3)\qquad -(\underset{\sim}{\phi}, \underset{\sim}{\mathrm{curl}}\, q) - t^2(\underset{\sim}{\mathrm{curl}}\, p, \underset{\sim}{\mathrm{curl}}\, q) = 0 \quad \text{for all } q \in \hat{H}^1(\Omega),$$

$$(7.4)\qquad (\underset{\sim}{\mathrm{grad}}\, \omega, \underset{\sim}{\mathrm{grad}}\, s) = (\underset{\sim}{\phi} + t^2 \underset{\sim}{\mathrm{grad}}\, r, \underset{\sim}{\mathrm{grad}}\, s) \quad \text{for all } s \in \mathring{H}^1(\Omega).$$

Moreover, if $\underset{\sim}{F} \in \underset{\sim}{L}^2(\Omega)$, then $\underset{\sim}{\phi} \in H^2(\Omega)$ and there exists a constant C independent of t, G, and $\underset{\sim}{F}$ such that

$$(7.5)\qquad \|r\|_1 + \|\underset{\sim}{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|\omega\|_1 \le C(\|G\|_{-1} + \|\underset{\sim}{F}\|_0).$$

If also $G \in L^2(\Omega)$, then r, $\omega \in H^2(\Omega)$ and

$$(7.6)\qquad \|r\|_2 + \|\omega\|_2 \le C(\|G\|_0 + \|\underset{\sim}{F}\|_0).$$

Proof. Existence and uniqueness are easy. Clearly $r \in \mathring{H}^1(\Omega)$ is determined uniquely by (7.1). Then, subtracting (7.3) from (7.2) we can infer existence and uniqueness of $\underset{\sim}{\phi}$ and p by the Lax–Milgram theorem. Finally (7.4) determines w uniquely.

To prove (7.5), we note that the asserted bound on r is immediate, and the bound on ω follows from the bounds on $\underset{\sim}{\phi}$ and r. Thus it suffices to bound $\underset{\sim}{\phi}$ and p. Define $(\underset{\sim}{\phi}^0, p^0) \in \mathring{\underset{\sim}{H}}^1(\Omega) \times \hat{L}^2(\Omega)$ as the solution of (7.2), (7.3) with t set equal to zero. This is simply the Stokes system for $(\phi_2^0, -\phi_1^0, p^0)$, which admits a unique solution. Known regularity theory for the Stokes problem (see [5] in the case of a smooth domain and [12] in the polygonal case) gives

$$\|\underset{\sim}{\phi}^0\|_2 + \|p^0\|_1 \le C(\|\underset{\sim}{F}\|_0 + \|r\|_1) \le C(\|\underset{\sim}{F}\|_0 + \|G\|_{-1}).$$

From (7.2) and (7.3) and the definition of $\underset{\sim}{\phi}^0$ and p^0, we get

$$(\underset{\approx}{\mathrm{grad}}(\underset{\sim}{\phi} - \underset{\sim}{\phi}^0), \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{\psi}) - (\underset{\sim}{\mathrm{curl}}(p - p^0), \underset{\sim}{\psi}) + (\underset{\sim}{\phi} - \underset{\sim}{\phi}^0, \underset{\sim}{\mathrm{curl}}\, q) + t^2(\underset{\sim}{\mathrm{curl}}(p - p^0), \underset{\sim}{\mathrm{curl}}\, q)$$
$$= -t^2(\underset{\sim}{\mathrm{curl}}\, p^0, \underset{\sim}{\mathrm{curl}}\, q) \quad \text{for all } (\underset{\sim}{\psi}, q) \in \mathring{\underset{\sim}{H}}^1(\Omega) \times \hat{H}^1(\Omega).$$

Choosing $\underset{\sim}{\psi} = \underset{\sim}{\phi} - \underset{\sim}{\phi}^0$ and $q = p - p^0$, we obtain

$$\|\underset{\sim}{\phi} - \underset{\sim}{\phi}^0\|_1^2 + t^2\|p - p^0\|_1^2 \le Ct^2\|p^0\|_1\|p - p^0\|_1.$$

It easily follows that

$$(7.7)\qquad \|\underset{\sim}{\phi} - \underset{\sim}{\phi}^0\|_1 + t\|p - p^0\|_1 \le Ct\|p^0\|_1 \le Ct(\|\underset{\sim}{F}\|_0 + \|G\|_{-1}).$$

Hence also

$$\|p\|_1 \le C(\|\underset{\sim}{F}\|_0 + \|G\|_{-1}).$$

Applying standard estimates for second-order elliptic problems to (7.2), we further obtain

$$\|\underset{\sim}{\phi}\|_2 \le C(\|p\|_1 + \|r\|_1 + \|\underset{\sim}{F}\|_0) \le C(\|\underset{\sim}{F}\|_0 + \|G\|_{-1}).$$

Now from (7.3) and the definition of $\underset{\sim}{\phi}^0$ we get

$$t^2(\underset{\sim}{\operatorname{curl}}\, p, \underset{\sim}{\operatorname{curl}}\, q) = -(\underset{\sim}{\phi}, \underset{\sim}{\operatorname{curl}}\, q) = (\underset{\sim}{\phi}^0 - \underset{\sim}{\phi}, \underset{\sim}{\operatorname{curl}}\, q) \quad \text{for all } q \in \hat{H}^1(\Omega).$$

Thus p is the weak solution of the boundary value problem

$$-\Delta p = t^{-2} \operatorname{rot}(\underset{\sim}{\phi}^0 - \underset{\sim}{\phi}) \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial \underset{\sim}{n}} = 0 \quad \text{on } \partial\Omega,$$

and by standard a priori estimates

$$\|p\|_2 \le Ct^{-2}\|\underset{\sim}{\phi} - \underset{\sim}{\phi}^0\|_1 \le Ct^{-1}(\|\underset{\sim}{F}\|_0 + \|G\|_{-1}),$$

where we apply (7.7) at the last step. This completes the proof of (7.5).

The proof of (7.6) is straightforward. Since (7.1) is a weak form of Poisson's equation, the bound on r follows from standard elliptic regularity theory. The same reasoning applied to (7.4) then gives

$$\|\omega\|_2 \le C(\|\underset{\sim}{\phi}\|_1 + t^2\|r\|_2) \le C(\|G\|_0 + \|\underset{\sim}{F}\|_0). \qquad \square$$

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