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NONCONFORMING FINITE ELEMENT METHODS FOR THE EQUATIONS OF LINEAR ELASTICITY

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ABSTRACT. In the adaptation of nonconforming finite element methods to the equations of elasticity with traction boundary conditions, the main difficulty in the analysis is to prove that an appropriate discrete version of Korn's second inequality is valid. Such a result is shown to hold for nonconforming piecewise quadratic and cubic finite elements and to be false for nonconforming piecewise linears. Optimal-order error estimates, uniform for Poisson ratio $\nu \in [0, 1/2)$, are then derived for the corresponding P_2 and P_3 methods. This contrasts with the use of C^0 finite elements, where there is a deterioration in the convergence rate as $\nu \to 1/2$ for piecewise polynomials of degree ≤ 3. Modifications of the continuous methods and the nonconforming linear method which also give uniform optimal-order error estimates are discussed.

1. INTRODUCTION

The finite element approximation of the equations of linear isotropic elasticity may be accomplished in a variety of ways. The most straightforward approach is to use the pure displacement formulation and conforming finite elements. The analysis of this method is well understood. It works well if the elasticity tensor is positive definite, but suffers a deterioration in performance in some cases as the Poisson ratio approaches 1/2 (i.e., as the material becomes incompressible). Specifically, as discussed in [19], for piecewise linear elements, the method may not converge as the Poisson ratio approaches 1/2, and for piecewise polynomials of degree 2 and 3, the error in the method may be of order one less than the optimal approximation in the finite element subspace. For piecewise polynomials of degree ≥ 4, optimal-order error estimates are obtained for most meshes (cf. [18]).

A second approach is to use a mixed finite element method based on the Hellinger-Reissner variational principle. In this approach, both stresses and displacements are approximated and a stable combination of finite element spaces must be found to approximate these variables. While several pairs of stable spaces ([17 and 9]) are known for scalar second-order problems, the symmetry requirement on the stress tensor does not allow the direct use of these

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spaces for the system of linear elasticity. Several approaches to circumventing this difficulty have been analyzed and all of them have the important feature that the accuracy of the method does not deteriorate as the material becomes incompressible.

One of the mixed finite element approaches is to use macro elements. In this technique, the basic finite element mesh is subdivided and certain interior degrees of freedom are eliminated so that the resulting macro element will satisfy some additional constraint (in this case symmetry). In [15] a piecewise linear macro element is proposed and analyzed and in [3] a family of higher-order elements is developed.

Another approach, developed in [2], is to modify the Hellinger-Reissner variational principle by introducing a Lagrange multiplier to enforce the symmetry constraint. When this variational principle is discretized, the symmetry condition is partially relaxed and a stable triple of triangular finite elements is developed for the modified variational principle (which now includes an additional variable to approximate the multiplier). This idea has been extended in [16] to higher-order and rectangular elements and to elements for the three-dimensional equations of linear elasticity.

In [4], the problem of symmetric stress tensors is overcome by the development of a new mixed variational formulation of the elasticity equations in which the spaces no longer have any symmetry constraint. Thus, standard pairs of stable finite element spaces developed for the scalar problem may be directly applied. The method is quite simple in the case of displacement boundary conditions, but must be modified for pure traction or mixed boundary conditions due to the fact that the original stress variable does not appear in the new formulation.

One drawback in the use of mixed methods is the large number of variables involved, although this difficulty may be partially circumvented using techniques presented in [1]. The basic idea is to reformulate the discrete equations as a generalized displacement method in which the stress variable has been eliminated. In the simplest case of the approximation of Poisson's equation by the lowest-order Raviart-Thomas elements, it is shown that the method is equivalent to a slight modification of the approximation of Poisson's equation by nonconforming piecewise linear elements. Since this is the case, it is natural to ask whether nonconforming finite elements may be used directly in the approximation of the elasticity equations and whether the use of such methods would have any advantages over conforming or mixed finite element methods. For the case of scalar second-order equations, a detailed analysis of nonconforming methods is given in [17], and for the stationary Stokes problem, the use of such methods is analyzed in [12]. The case of nonconforming quadratics for both the scalar second-order problem and the stationary Stokes equations is considered in [13]. Since the stationary Stokes equations are closely related to the displacement-pressure formulation of elasticity, the extension of such methods to the equations of elasticity (involving displacements and the full stress tensor) would appear to be straightforward. In fact, the boundary conditions imposed play a crucial role, and it is only in the case of pure displacement boundary conditions, that an extension is obvious. The reason for this is that in the case of homogeneous displacement boundary conditions, the continuous problem may be transformed so that one works with a bilinear form involving the Dirichlet form $\int_\Omega \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{u} : \underset{\approx}{\mathrm{grad}}\, \underset{\sim}{v}$, instead of the more natural form $\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})$. The problem with this second form is that it is not at all clear whether the discrete analogue of Korn's second inequality, used to establish the coercivity of the form, holds for nonconforming finite elements. In fact, we show in §6 that it fails for nonconforming piecewise linear functions. The result of this failure is that the straightforward application of nonconforming piecewise linear elements to the approximation of the elasticity equations with pure traction boundary conditions leads to a discrete problem with a large space of solutions, while the solution of the continuous problem is unique up to addition of the three-dimensional space of rigid motions. This problem is completely avoided in the analysis of the Stokes problem in [12], since the basic problem is given in terms of the Dirichlet form, and only homogeneous Dirichlet boundary conditions are considered.

In this paper, we consider the approximation of the equations of elasticity with pure traction boundary conditions by nonconforming piecewise linear, quadratic, and cubic finite elements. For the piecewise quadratic and cubic cases, we use the straightforward extension of the nonconforming methods discussed in [12, 13, and 17]. For piecewise linears, we propose a slightly modified version in which a local projection is added. We then derive optimal-order error estimates for these methods in which the constant remains uniform as the material becomes incompressible. The keys to this analysis are the proof of appropriate discrete versions of Korn's second inequality and the equivalence of the displacement formulation of the elasticity equations with a Stokes-like formulation involving displacements and a single stress variable.

The nonconforming schemes we consider are equivalent to trivial mixed methods, where the stresses are discontinuous piecewise polynomials which are easily eliminated from the system. Since these methods share with other mixed methods the property that they do not deteriorate in accuracy as the material becomes incompressible, they raise the question whether the large number of variables present in the mixed methods mentioned previously contribute in any way to a better approximation. For one-dimensional problems, the results of Babuška and Osborn [7] prove that for rough coefficients, certain mixed formulations do perform better. In the case of two-dimensional problems, there are presently no general theoretical results of this type. Also relevant to the choice of methods for the numerical approximation of the elasticity equations is the remark made in the last section, that using ideas developed for the Stokes problem, the loss of accuracy near incompressiblity for conforming methods using piecewise polynomials of degree ≤ 3 is easily fixed. The number of unknowns for the modified methods constructed require less for linears, the same for quadratic, and more for cubics than the corresponding nonconforming methods. As mentioned previously, for piecewise polynomials of degree greater than three, the standard displacement method using conforming elements suffers no loss of accuracy. Since it uses fewer unknowns than other methods, it thus appears preferable.

An outline of the paper is as follows. In the next section, we include the notation to be used along with some preliminary results useful in the paper. In particular, a statement of a continuous version of Korn's second inequality is given along with a proof which allows generalization to nonconforming finite elements. Section 3 describes the approximate problems and §4 contains the statement and proofs of the discrete versions of Korn's second inequality needed for the analysis of these methods. An error analysis of the methods is presented in §5. In §6, we examine the case of nonconforming linears, showing why Korn's second inequality fails and proposing a modified method to deal with this difficulty. This method produces a nonsymmetric approximation to the stress tensor $\underset{\approx}{\sigma}$ and is shown to be equivalent to a mixed formulation (similar to that in [2]) in which the symmetry of the stress tensor is relaxed through the use of a Lagrange multiplier. In §7, modified forms of the standard finite element method for conforming piecewise polynomials of degree ≤ 3, which alleviate the problem of deterioration of accuracy for nearly incompressible materials, are discussed.

2. Notations and preliminaries

We will use the usual L^2-based Sobolev spaces H^s. An undertilde to a space denotes the 2-vector-valued analogue. The undertilde is also used to denote vector-valued functions and operators, and double undertildes are used for matrix-valued objects. The letter C denotes a generic constant, not necessarily the same in each occurrence. We will use various standard differential operators defined as follows:

$$\underset{\sim}{\operatorname{grad}} p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}, \qquad \underset{\sim}{\operatorname{curl}} p = \begin{pmatrix} \partial p/\partial y \\ -\partial p/\partial x \end{pmatrix},$$

$$\operatorname{div} \underset{\sim}{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \qquad \operatorname{rot} \underset{\sim}{v} = -\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x},$$

$$\underset{\approx}{\operatorname{grad}} \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix}, \qquad \underset{\approx}{\operatorname{curl}} \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial y & -\partial v_1/\partial x \\ \partial v_2/\partial y & -\partial v_2/\partial x \end{pmatrix},$$

$$\underset{\approx}{\operatorname{div}} \underset{\approx}{\tau} = \begin{pmatrix} \partial \tau_{11}/\partial x + \partial \tau_{12}/\partial y \\ \partial \tau_{21}/\partial x + \partial \tau_{22}/\partial y \end{pmatrix}, \qquad \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \frac{1}{2}[\underset{\approx}{\operatorname{grad}} \underset{\sim}{v} + (\underset{\approx}{\operatorname{grad}} \underset{\sim}{v})^t].$$

We also define two constant tensors

$$\underset{\approx}{\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \underset{\approx}{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and for any tensor $\underset{\approx}{\tau}$

$$\operatorname{tr}(\underset{\approx}{\tau}) = \underset{\approx}{\tau} : \underset{\approx}{\delta}, \qquad \operatorname{as}(\underset{\approx}{\tau}) = \underset{\approx}{\tau} : \underset{\approx}{\chi},$$

where

$$\underset{\approx}{\sigma} : \underset{\approx}{\tau} = \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}\tau_{ij}.$$

An easy calculation shows that

$$\underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \underset{\approx}{\operatorname{grad}}\,\underset{\sim}{v} - \frac{1}{2}\operatorname{rot}\underset{\sim}{v}\,\underset{\approx}{\chi}. \tag{2.1}$$

The traction boundary value problem for the equations of plane strain linear isotropic elasticity may be written in the form

$$\underset{\approx}{\sigma} = \mu\left[\underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \frac{\nu}{1-2\nu}\operatorname{div}\underset{\sim}{u}\,\underset{\approx}{\delta}\right] \quad \text{in } \Omega, \tag{2.2}$$

$$-\underset{\approx}{\operatorname{div}}\,\underset{\approx}{\sigma} = \underset{\sim}{f} \qquad \text{in } \Omega, \tag{2.3}$$

$$\underset{\approx}{\sigma}\underset{\sim}{n} = \underset{\sim}{g} \qquad \text{on } \partial\Omega, \tag{2.4}$$

where $\underset{\approx}{\sigma}$ denotes the stresses, $\underset{\sim}{u}$ the displacements, $\underset{\sim}{f}$ the body forces, $\underset{\sim}{g}$ the boundary tractions, E is Young's modulus, ν the Poisson ratio, and we have set $\mu = E/(1+\nu)$.

In order for a solution to exist, the data $\underset{\sim}{f}$ and $\underset{\sim}{g}$ must satisfy the compatibility condition

$$\int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx + \int_{\partial\Omega}\underset{\sim}{g}\cdot\underset{\sim}{v}\,ds = 0 \qquad \text{for all} \quad \underset{\sim}{v} \in \underset{\sim}{\mathrm{RM}},$$

where $\underset{\sim}{\mathrm{RM}}$, the space of rigid motions, is defined by

$$\underset{\sim}{\mathrm{RM}} = \left\{\underset{\sim}{v} : \underset{\sim}{v} = (a+by, c-bx), a, b, c \in \mathbb{R}\right\}.$$

When this compatibility condition is satisfied, the solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ will be unique in $\underset{\approx}{L}^2 \times \underset{\sim}{\widehat{V}}$, where

$$\underset{\sim}{\widehat{V}} = \left\{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \int_\Omega \underset{\sim}{v}\,dx = 0, \int_\Omega \operatorname{rot}\underset{\sim}{v}\,dx = 0\right\}.$$

A weak mixed formulation of the elasticity equations is

Problem E. Find $\underset{\approx}{\sigma} \in \underset{\approx}{H}_S$ and $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$ such that

$$\int_\Omega \mathbf{A}\underset{\approx}{\sigma} : \underset{\approx}{\tau}\,dx - \int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\tau}\,dx = 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{H}_S,$$

$$\int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\,dx = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx + \int_{\partial\Omega}\underset{\sim}{g}\cdot\underset{\sim}{v}\,ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}},$$

where

$$\mathbf{A}\underset{\approx}{\sigma} = \frac{1}{\mu}\left[\underset{\approx}{\sigma} - \nu\,\mathrm{tr}(\underset{\approx}{\sigma})\underset{\approx}{\delta}\right]$$

and

$$\underset{\approx}{H}_S = \{\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) : \tau_{12} = \tau_{21}\}.$$

For $0 \le \nu < 1/2$, $\underset{\approx}{\sigma}$ may be easily eliminated from the elasticity system (2.2)–(2.4). The resulting pure displacement problem has the following well-known weak formulation:

Problem P. Find $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$ such that

$$B(\underset{\sim}{u}, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \qquad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}},$$

where

$$B(\underset{\sim}{u}, \underset{\sim}{v}) = \mu\left(\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\, dx + \frac{\nu}{1-2\nu}\int_\Omega \mathrm{div}\,\underset{\sim}{u}\,\mathrm{div}\,\underset{\sim}{v}\, dx\right).$$

Using the identity (2.1), we may also write

$$B(\underset{\sim}{u}, \underset{\sim}{v}) = \mu\Big(\int_\Omega \underset{\approx}{\mathrm{grad}}\,\underset{\sim}{u} : \underset{\approx}{\mathrm{grad}}\,\underset{\sim}{v}\, dx - \frac{1}{2}\int_\Omega \mathrm{rot}\,\underset{\sim}{u}\,\mathrm{rot}\,\underset{\sim}{v}\, dx + \frac{\nu}{1-2\nu}\int_\Omega \mathrm{div}\,\underset{\sim}{u}\,\mathrm{div}\,\underset{\sim}{v}\, dx\Big).$$

If we define

$$p = -\,\mathrm{tr}\,\underset{\approx}{\sigma} = -\frac{\mu}{1-2\nu}\mathrm{div}\,\underset{\sim}{u},$$

then

$$\underset{\approx}{\sigma} = \mu\underset{\approx}{\varepsilon}(\underset{\sim}{u}) - \nu p\underset{\approx}{\delta}, \tag{2.5}$$

and the equations of elasticity may also be written in the form:

Problem S. Find $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$, $p \in L^2(\Omega)$ such that

$$\begin{aligned}\mu\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\, dx - \nu\int_\Omega p\,\mathrm{div}\,\underset{\sim}{v}\, dx \\ = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}},\end{aligned} \tag{2.6}$$

$$\int_\Omega \mathrm{div}\,\underset{\sim}{u} q\, dx = -\mu^{-1}(1-2\nu)\int_\Omega pq\, dx \quad \text{for all } q \in L^2(\Omega). \tag{2.7}$$

This formulation is valid even in the incompressible limit $\nu = 1/2$ (the stationary Stokes equations).

The proof of existence and uniqueness of the weak solution to Problem E, P, or S depends on the use of Korn's second inequality, which insures the coerciveness of the bilinear form $\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\, dx$ for $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$. One version of this result may be stated as follows:

Theorem 2.1 (Korn's second inequality). *For all* $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$, *there exists a constant* K *independent of* $\underset{\sim}{u}$ *such that*

$$\|\underset{\approx}{\varepsilon}(\underset{\sim}{u})\|_0 \geq K \|\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}\|_0.$$

Unlike the proof of Korn's first inequality, which establishes the above result for $\underset{\sim}{u} \in \underset{\sim}{H}{}^1_0(\Omega)$, the proof of Theorem 2.1 is not elementary, and many proofs have been given in the literature. Since a discrete version of this inequality will be the essential ingredient in the analysis of the nonconforming finite element approximations to the elasticity equations given in the next section, we now present a proof of Theorem 2.1 which may be generalized to the case of nonconforming finite elements.

The key fact used in the proof is the following lemma (cf. [14] for smoothly bounded domains and [6] in the case of a polygon).

Lemma 2.2. *Given* $p \in L^2(\Omega)$, *with* $\int_\Omega p = 0$, *there exists* $\underset{\sim}{v} \in \underset{\sim}{H}{}^1_0(\Omega)$ *such that*

$$\operatorname{div} \underset{\sim}{v} = p \quad \text{in } \Omega, \qquad \|\underset{\sim}{v}\|_1 \leq C\|p\|_0,$$

with C *independent of* $\underset{\sim}{v}$ *and* p.

Proof of Theorem 2.1. Using (2.1), we have for all $\underset{\approx}{\tau} \in \underset{\approx}{L}{}^2$,

$$\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx = \int_\Omega (\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u} - \frac{1}{2} \operatorname{rot} \underset{\sim}{u} \underset{\approx}{\chi}) : \underset{\approx}{\tau}\, dx.$$

Using Lemma 2.2, we may choose $\underset{\approx}{\tau} = \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u} - \underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}$, where $\underset{\sim}{z} \in \underset{\sim}{H}{}^1_0(\Omega)$ satisfies

$$\operatorname{div} \underset{\sim}{z} = \operatorname{rot} \underset{\sim}{u} \quad \text{in } \Omega, \qquad \|\underset{\sim}{z}\|_1 \leq C\|\operatorname{rot} \underset{\sim}{u}\|_0.$$

Then

$$\|\underset{\approx}{\tau}\|_0 \leq \|\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}\|_0 + \|\underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}\|_0 \leq C\|\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}\|_0.$$

Now using the $\underset{\approx}{L}{}^2$ orthogonality of $\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}$ and $\underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}$, we obtain

$$\begin{aligned}\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx &= \int_\Omega \left(\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u} : \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u} - \frac{1}{2} \operatorname{rot} \underset{\sim}{u}[\operatorname{rot} \underset{\sim}{u} - \operatorname{div} \underset{\sim}{z}]\right) dx \\ &= \|\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}\|_0^2.\end{aligned}$$

Hence,

$$\|\underset{\approx}{\varepsilon}(\underset{\sim}{u})\|_0 \geq \frac{\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx}{\|\underset{\approx}{\tau}\|_0} \geq K\|\underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u}\|_0. \quad \square$$

3. Approximation scheme

In this section we consider nonconforming finite element methods based on the variational formulation of Problem P. In the case of cubics, we use the straightforward generalization of the method analyzed in [17] for scalar second-order problems, and for quadratics, we use the ideas in [13]. Unfortunately, this straightforward generalization does not work for nonconforming piecewise linear elements. The reason for this, to be made more precise later, is that a needed discrete Korn's inequality fails for this space, and thus the form $B(\underset{\sim}{u}, \underset{\sim}{v})$ is not coercive. Some modifications of the basic method which get around this problem are discussed in §6.

We assume henceforth that the domain Ω is a polygon, which is triangulated by a triangulation $\mathcal{T}_h$. As usual, the subscript h refers to the diameter of the largest triangle in $\mathcal{T}_h$, and the constants in our error estimates will be independent of h, assuming that a minimum angle condition is satisfied as $h \to 0$.

Denoting by λ_i the barycentric coordinates of a triangle T and by $\mathcal{P}_k(T)$ the set of functions on T which are the restrictions of polynomials of degree no greater than k, we define the following finite element spaces with respect to the triangulation $\mathcal{T}_h$:

$$
\begin{aligned}
M_{-1}^k &= \left\{ \eta \in L^2(\Omega) : \eta|_T \in \mathcal{P}_k(T) \text{ for all } T \in \mathcal{T}_h \right\}, \\
M_0^k &= M_{-1}^k \cap H^1(\Omega), \\
\mathring{M}_0^k &= M_{-1}^k \cap H_0^1(\Omega), \\
M_*^k &= \{\eta \in M_{-1}^k : \eta \text{ is continuous at the } k \text{ Gauss points} \\
&\qquad\qquad \text{on each edge of } \mathcal{T}_h\}, \\
B^k &= \{\eta \in M_0^k : \eta|_T \in \lambda_1\lambda_2\lambda_3 \operatorname{span}\{x_1^i x_2^{k-3-i}, 0 \le i \le k-3\}\}, \\
B_*^2 &= \{\eta \in M_{-1}^2 : \eta \text{ equals zero at the two Gauss points} \\
&\qquad\qquad \text{on each edge of } \mathcal{T}_h\}.
\end{aligned}
$$

Note that M_*^k are the usual nonconforming approximations of $H^1(\Omega)$. For $\mu \in M_*^k + H^1(\Omega)$, we define $\underset{\sim}{\operatorname{grad}}_h \mu$ to be the $\underset{\sim}{L}^2(\Omega)$ function whose restriction to each triangle $T \in \mathcal{T}_h$ is given by $\underset{\sim}{\operatorname{grad}} \mu|_T$. Analogous definitions hold for rot_h, $\underset{\approx}{\operatorname{grad}}_h$, div_h, $\underset{\approx}{\operatorname{curl}}_h$, and $\underset{\approx}{\varepsilon}_h$. Note that

$$
\underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) = \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} - \frac{1}{2} \operatorname{rot}_h \underset{\sim}{u} \underset{\approx}{\chi}. \tag{3.1}
$$

Finally, define

$$
\underset{\sim}{\widehat{V}}{}_h^k = \left\{ \underset{\sim}{v} \in \underset{\sim}{M}_*^k : \int_\Omega \underset{\sim}{v}\, dx = \int_\Omega \operatorname{rot}_h \underset{\sim}{v}\, dx = 0 \right\}.
$$

The nonconforming finite element approximation schemes for Problem P are then given for $k = 2$ and $k = 3$ as follows:

Problem P_h^k. Find $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{V}}_h^k$ such that

$$B_h(\underset{\sim}{u}_h, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}}_h^k,$$

where

$$\begin{aligned} B_h(\underset{\sim}{u}, \underset{\sim}{v}) &= \mu\left(\int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}_h(\underset{\sim}{v})\, dx + \frac{\nu}{1-2\nu}\int_\Omega \operatorname{div}_h \underset{\sim}{u} \operatorname{div}_h \underset{\sim}{v}\, dx\right) \\ &= \mu\left(\int_\Omega \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} : \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{v}\, dx - \frac{1}{2}\int_\Omega \operatorname{rot}_h \underset{\sim}{u} \operatorname{rot}_h \underset{\sim}{v}\, dx \right. \\ &\qquad \left. + \frac{\nu}{1-2\nu}\int_\Omega \operatorname{div}_h \underset{\sim}{u} \operatorname{div}_h \underset{\sim}{v}\, dx\right). \end{aligned}$$

Note that since $\operatorname{div} \underset{\sim}{\widehat{V}}_h^k \subseteq M_{-1}^{k-1}$, it it easy to see that Problem P_h^k is equivalent to the following discretization of Problem S:

Problem S_h^k. Find $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{V}}_h^k$, $p_h \in M_{-1}^{k-1}$ such that

$$\begin{aligned} \mu \int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}_h) : \underset{\approx}{\varepsilon}_h(\underset{\sim}{v})\, dx - \nu \int_\Omega p_h \operatorname{div}_h \underset{\sim}{v}\, dx \\ = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}}_h^k, \end{aligned} \tag{3.2}$$

$$\int_\Omega \operatorname{div}_h \underset{\sim}{u}_h q\, dx = -\mu^{-1}(1-2\nu)\int_\Omega p_h q\, dx \quad \text{for all } q \in M_{-1}^{k-1}. \tag{3.3}$$

If we define

$$\underset{\approx}{H}_{S,h}^k = \{\tau \in \underset{\approx}{H}_S : \tau_{ij}|_T \in \mathscr{P}_k(T)\},$$

then, since $\underset{\approx}{\varepsilon}(\underset{\sim}{\widehat{V}}_h^k) \subseteq \underset{\approx}{H}_{S,h}^{k-1}$, it is also easy to see that Problem P_h^k is equivalent to the following discretization of Problem E:

Problem E_h^k. Find $\underset{\approx}{\sigma}_h \in \underset{\approx}{H}_{S,h}^{k-1}$ and $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{V}}_h^k$ such that

$$\int_\Omega \mathrm{A}\underset{\approx}{\sigma}_h : \underset{\approx}{\tau}\, dx - \int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}_h) : \underset{\approx}{\tau}\, dx = 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{H}_{S,h}^{k-1},$$

$$\int_\Omega \underset{\approx}{\sigma}_h : \underset{\approx}{\varepsilon}_h(\underset{\sim}{v})\, dx = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}}_h^k.$$

Note that the above approximations also make sense in the incompressible limit $\nu = 1/2$.

Once an approximation $\underset{\sim}{u}_h$ to $\underset{\sim}{u}$ has been computed, an approximation $\underset{\approx}{\sigma}_h$ to

$$\underset{\approx}{\sigma} = \mu\left[\underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \frac{\nu}{1-2\nu}\operatorname{div} \underset{\sim}{u}\underset{\approx}{\delta}\right] = \mu\underset{\approx}{\varepsilon}(\underset{\sim}{u}) - \nu p \underset{\approx}{\delta}$$

may be computed from the formula

$$\underset{\approx}{\sigma}_h = \mu\left[\underset{\approx}{\varepsilon}_h(\underset{\sim}{u}_h) + \frac{\nu}{1-2\nu}\operatorname{div}_h \underset{\sim}{u}_h\underset{\approx}{\delta}\right] = \mu\underset{\approx}{\varepsilon}_h(\underset{\sim}{u}_h) - \nu p_h \underset{\approx}{\delta}. \tag{3.4}$$

4. Discrete Korn's inequality

The essential difference between the analysis of nonconforming finite element methods for the system of elasticity and the analysis of the scalar second-order problem studied in [17] is the need in the elasticity equations for a version of Korn's second inequality to insure the coerciveness of the bilinear form. Since the nonconforming spaces are not in $H^1(\Omega)$, this fact does not follow from the continuous case. In this section, we address this problem by giving a proof of a discrete version of Korn's second inequality.

Theorem 4.1. *For all* $\underset{\sim}{v} \in \underset{\sim}{\widehat{V}}_h^k$, $k = 2, 3$, *there exists a constant* K *independent of* $\underset{\sim}{v}$ *such that*

$$\|\underset{\approx}{\varepsilon}_h(\underset{\sim}{v})\|_0 \geq K \|\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{v}\|_0. \tag{4.1}$$

To prove Theorem 4.1, we use a discrete version of Lemma 2.2., which states a result about two well-known stable pairs of conforming finite elements for the Stokes problem, i.e., $(\underset{\sim}{W}_h^2, R_h^2) = (\underset{\sim}{\overset{\circ}{M}}{}_0^2 \cup \underset{\sim}{B}^3, M_{-1}^1)$ and $(\underset{\sim}{W}_h^3, R_h^3) = (\underset{\sim}{\overset{\circ}{M}}{}_0^3 \cup \underset{\sim}{B}^4, M_{-1}^2)$. We include a proof for future reference.

Lemma 4.2. *Given* $p \in R_h^k$ $(k = 2$ *or* $3)$ *with* $\int_\Omega p\, dx = 0$, *there exists* $\underset{\sim}{v}_h \in \underset{\sim}{W}_h^k$ *such that*

$$\int_\Omega \operatorname{div} \underset{\sim}{v}_h q\, dx = \int_\Omega pq\, dx \quad \text{for all } q \in R_h^k, \qquad \|\underset{\sim}{v}_h\|_1 \leq C\|p\|_0,$$

with C *independent of* h *and* p.

Proof. For $\underset{\sim}{v} \in [H^1(T)]^2$, define an interpolant $\Pi_1 \underset{\sim}{v} \in [\mathscr{P}_2(T)]^2$ by the following:

$$\Pi_1 v_i(a) = I_h v_i(a) \quad \text{for each vertex } a \text{ of } T,$$
$$\int_e (\Pi_1 v_i - v_i)\, ds = 0 \quad \text{for each edge } e \text{ of } T,$$

where $I_h v_i$ denotes the Clément interpolant of v_i (cf. [14, pp. 110]). Then

$$\int_T \operatorname{div}(\underset{\sim}{v} - \Pi_1 \underset{\sim}{v})\, dx = \sum_{j=1}^{3} \int_{e_j} (\Pi_1 \underset{\sim}{v} - \underset{\sim}{v}) \cdot \underset{\sim}{n}\, ds = 0,$$

and it is well known that

$$\|\Pi_1 \underset{\sim}{v}\|_1 \leq C\|\underset{\sim}{v}\|_1.$$

Using the ideas in [12] and [10], we next define for $\underset{\sim}{w} \in [H^1(T)]^2$, with $\int_T \operatorname{div} \underset{\sim}{w}\, dx = 0$, an interpolant $\Pi_2 \underset{\sim}{w}$ with $\Pi_2 w_i$ in the space of bubble functions of degree $k+1$ (i.e., $\in \lambda_1\lambda_2\lambda_3 \mathscr{P}_{k-2}(T)$), and defined by the following:

$$\int_T \operatorname{div}(\Pi_2 \underset{\sim}{w} - \underset{\sim}{w}) q\, dx = 0 \quad \text{for all } q \in \mathscr{P}_{k-1}(T)$$

and for $k = 3$ by the additional condition

$$\int_T [x_1(\Pi_2 \underset{\sim}{w})_2 - x_2(\Pi_2 \underset{\sim}{w})_1]\, dx = \int_T [x_1(\underset{\sim}{w})_2 - x_2(\underset{\sim}{w})_1]\, dx.$$

Note that since $\Pi_2 \underset{\sim}{w}$ vanishes on the boundary of T,

$$\int_T \Pi_2 \underset{\sim}{w} \cdot \nabla q\, dx = -\int_T \operatorname{div} \underset{\sim}{w} q\, dx \quad \text{for all } q \in \mathscr{P}_{k-1}(T).$$

It can then be shown that $\Pi_2 \underset{\sim}{w}$ is well defined and satisfies $\|\Pi_2 \underset{\sim}{w}\|_1 \le C\|\underset{\sim}{w}\|_1$. Choosing $\underset{\sim}{v}_h = \Pi_1 \underset{\sim}{v} + \Pi_2(\underset{\sim}{v} - \Pi_1 \underset{\sim}{v})$, where $\underset{\sim}{v}$ is given by Lemma 2.2, we get that

$$\int_T \operatorname{div} \underset{\sim}{v}_h q\, dx = \int_T \operatorname{div} \underset{\sim}{v} q\, dx = \int_T pq\, dx \quad \text{for all } q \in \mathscr{P}_{k-1}(T),$$
$$\|\underset{\sim}{v}_h\|_1 \le C\|\underset{\sim}{v}\|_1 \le C\|p\|_0,$$

which establishes the lemma. □

Proof of Theorem 4.1. Using (3.1), we have for all $\underset{\approx}{\tau} \in \underset{\approx}{L}^2$,

$$\int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx = \int_\Omega \left(\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} - \frac{1}{2} \operatorname{rot}_h \underset{\sim}{u} \underset{\approx}{\chi} \right) : \underset{\approx}{\tau}\, dx.$$

Using Lemma 4.2, we may choose $\underset{\approx}{\tau} = \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} - \underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}$, where $\underset{\sim}{z} \in \underset{\sim}{W}_h^k$ satisfies

(4.2)
$$\int_\Omega \operatorname{div} \underset{\sim}{z} q\, dx = \int_\Omega \operatorname{rot}_h \underset{\sim}{u} q\, dx \quad \text{for all } q \in R_h^k,$$
$$\|\underset{\sim}{z}\|_1 \le C\|\operatorname{rot}_h \underset{\sim}{u}\|_0.$$

Then

$$\|\underset{\approx}{\tau}\|_0 \le \|\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}\|_0 + \|\underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}\|_0 \le C\|\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}\|_0.$$

Now observe that

$$\int_\Omega \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} : \underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}\, dx = \sum_T \int_{\partial T} \underset{\sim}{u} \cdot \frac{\partial \underset{\sim}{z}}{\partial s}\, ds = 0,$$

since on boundary edges $\underset{\sim}{z} = 0$ and on interior edges, contributions from adjoining triangles cancel. The cancellation occurs since the integrand along the edges involves only tangential derivatives of $\underset{\sim}{z}$ which are polynomials of degree $\le k-1$ (occurring with opposite signs) and moments of $\underset{\sim}{u}$ of order $\le k-1$ on each edge which are continuous across edges. Using this $\underset{\approx}{L}^2$ orthogonality of $\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}$ and $\underset{\approx}{\operatorname{curl}}\, \underset{\sim}{z}$ and (4.2), we obtain

$$\int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx = \int_\Omega \left(\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} : \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u} - \frac{1}{2} \operatorname{rot}_h \underset{\sim}{u} [\operatorname{rot}_h \underset{\sim}{u} - \operatorname{div} \underset{\sim}{z}] \right) dx$$
$$= \|\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}\|_0^2.$$

Hence,

$$\|\underset{\approx}{\varepsilon}_h(\underset{\sim}{u})\|_0 \geq \frac{\int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx}{\|\underset{\approx}{\tau}\|_0} \geq K\|\underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}\|_0. \quad \square$$

5. Error estimates

In this section we give estimates for the errors

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{1,h} \equiv \|\underset{\approx}{\operatorname{grad}}_h (\underset{\sim}{u} - \underset{\sim}{u}_h)\|_0 \quad \text{and} \quad \|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_0.$$

Note that the estimates obtained do not deteriorate as the material becomes incompressible (i.e., $\nu \to 1/2$). The techniques of the proof use the ideas developed in [12, 17, and 13], and the saddle point analysis developed by Babuška and Brezzi. The discrete Korn's inequalities derived in the previous section are used to establish the coercivity of the bilinear form. Although the general approach to deriving error estimates for mixed finite element approximations is now fairly standard, the analysis of nonconforming finite elements is not as widely known. Hence, we provide a derivation of the error estimates. For more background on this subject, the interested reader is advised to consult the general treatment of error estimates for mixed finite element approximations given in the recent book of Brezzi and Fortin [10].

Theorem 5.1. *Let* $(\underset{\sim}{u}, p)$ *and* $(\underset{\sim}{u}_h, p_h)$ *be the solutions to Problems* S *and* S_h^k, *respectively* $(k = 2$ *or* $3)$. *Then there exists a constant* C, *independent of* $\underset{\sim}{u}$ *and* h, *and uniform for* $0 \leq \nu \leq 1/2$, *such that*

$$\begin{aligned} &\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{1,h} + \nu\|p - p_h\|_0 \\ &\leq C \inf \left(\|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{1,h} + \|p - q_h\|_0 \right. \\ &\qquad \left. + \sup \frac{\sum_T \int_{\partial T} \underset{\approx}{\sigma}\underset{\sim}{n} \cdot \underset{\sim}{w}_h\, ds - \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{w}_h\, ds}{\|\underset{\sim}{w}_h\|_{1,h}} \right), \end{aligned}$$

where the inf *is taken over all* $\underset{\sim}{v}_h \in \underset{\sim}{\widehat{V}}_h^k$ *and* $q_h \in M_{-1}^{k-1}$, *and the* sup *is taken over all* $\underset{\sim}{w}_h \in \underset{\sim}{\widehat{V}}_h^k$.

Proof. The key ingredient in the proof (e.g., see [8]) is the stability condition

$$\inf_{0 \neq q_h \in M_{-1}^{k-1}} \sup_{0 \neq \underset{\sim}{v}_h \in \underset{\sim}{\widehat{V}}_h^k} \frac{\int_\Omega \operatorname{div}_h \underset{\sim}{v}_h q_h\, dx}{\|\underset{\sim}{v}_h\|_{1,h}\|q_h\|_0} \geq \gamma. \tag{5.1}$$

For the case $k = 3$, such a condition has been established (for most commonly used meshes) in [11] in the stronger case when $\underset{\sim}{v}_h \in M_*^3$ vanishes at the Gauss points on $\partial\Omega$ and p is replaced by $p - \overline{p}$, where $\overline{p}$ denotes the mean value of

p on Ω. Using that result, we can find $\underset{\sim}{z}_h \in M_*^3$ and vanishing at the Gauss points on $\partial\Omega$ satisfying

$$\operatorname{div}_h \underset{\sim}{z}_h = p - \bar{p}, \qquad \|\underset{\sim}{z}_h\|_{1,h} \le C\|p - \bar{p}\|_0.$$

Setting $\underset{\sim}{v}_h = \underset{\sim}{z}_h - \bar{\underset{\sim}{z}}_h + \frac{1}{2}(x - \bar{x}, y - \bar{y})\bar{p}$, it is easy to check that $\underset{\sim}{v}_h \in \hat{\underset{\sim}{V}}_h^3$ and satisfies

$$\operatorname{div}_h \underset{\sim}{v}_h = p, \qquad \|\underset{\sim}{v}_h\|_{1,h} \le C\|p\|_0,$$

from which (5.1) follows. In the case $k = 2$, (5.1) is established by first noting the result of [13] that the space of nonconforming piecewise quadratics consists of conforming piecewise quadratics plus the functions $c_T B_T^2$, where B_T^2 is the piecewise quadratic vanishing at the two Gauss points on each of the edges of the triangle T and is zero outside of T. The proof of (5.1) is now almost identical to that given in Lemma 4.2 for the choice of conforming quadratics plus cubic bubble functions for velocities and discontinuous piecewise linear elements for pressure. We need only replace the cubic bubble function $\lambda_1\lambda_2\lambda_3$ by the function B_T^2. To see that the nonconforming version of the Π_2 interpolant is well defined, note that $\int_T B_T^2\,dx \ne 0$ and

$$\int_\Omega \operatorname{div}_h (\underset{\sim}{c}_T B_T^2) q\,dx = -\int_T \underset{\sim}{c}_T B_T^2 \nabla q\,dx,$$

for all $q \in M_{-1}^1$, which follows from the facts that the two-point Gauss integration formula is exact for polynomials of degree ≤ 3 on each edge and B_T^2 vanishes at these points. The modification given above in the cubic case to produce orthogonality to rigid motions can also be applied in the quadratic case. We remark that the result obtained in [13] does not directly establish (5.1) since the interpolant constructed uses point values, although it is sufficient for the optimal-order error estimates given in Theorem 5.3 below.

To simplify the exposition of the remainder of the proof, we define

$$a_h(u, v) = \mu \int_\Omega \underset{\approx}{\varepsilon}_h(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}_h(\underset{\sim}{v})\,dx.$$

Multiplying (2.3) by $\underset{\sim}{v}_h \in \hat{\underset{\sim}{V}}_h^k$, integrating by parts, and using (2.5) and (2.4), we obtain

$$a_h(\underset{\sim}{u}, \underset{\sim}{v}_h) - \nu \int_\Omega p \operatorname{div}_h \underset{\sim}{v}_h\,dx = \int_\Omega \underset{\sim}{f}\underset{\sim}{v}_h\,dx + \int_\Omega \underset{\sim}{g}\underset{\sim}{v}_h\,ds + G_h(\underset{\sim}{v}_h),$$

where

$$G_h(\underset{\sim}{v}_h) = \sum_T \int_{\partial T} \underset{\approx}{\sigma}\underset{\sim}{n} \cdot \underset{\sim}{v}_h\,ds - \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}_h\,ds$$

is the error due to the use of nonconforming elements. Hence, for any $\underset{\sim}{u}_I \in \hat{\underset{\sim}{V}}_h^k$

and any $p_I \in M_{-1}^{k-1}$, we have

$$\begin{aligned}
a_h(\underset{\sim}{u}_I, \underset{\sim}{v}_h) - \nu \int_\Omega p_I \operatorname{div}_h \underset{\sim}{v}_h \, dx
&= a_h(\underset{\sim}{u}_I - \underset{\sim}{u}, \underset{\sim}{v}_h) - \nu \int_\Omega (p_I - p) \operatorname{div}_h \underset{\sim}{v}_h \, dx \\
&\quad + \int_\Omega \underset{\sim}{f}\underset{\sim}{v}_h \, dx + \int_\Omega \underset{\sim}{g}\underset{\sim}{v}_h \, ds + G_h(\underset{\sim}{v}_h).
\end{aligned}$$

Subtracting (3.2), we then obtain

$$\begin{aligned}
&a_h(\underset{\sim}{u}_I - \underset{\sim}{u}_h, \underset{\sim}{v}_h) - \nu \int_\Omega (p_I - p_h) \operatorname{div}_h \underset{\sim}{v}_h \, dx \\
&\quad = a_h(\underset{\sim}{u}_I - \underset{\sim}{u}, \underset{\sim}{v}_h) - \nu \int_\Omega (p_I - p) \operatorname{div}_h \underset{\sim}{v}_h \, dx + G_h(\underset{\sim}{v}_h).
\end{aligned} \tag{5.2}$$

Using (2.7) and (3.3), we easily obtain for all $q \in M_{-1}^{k-1}$ that

$$\begin{aligned}
&\int_\Omega \operatorname{div}_h (\underset{\sim}{u}_I - \underset{\sim}{u}_h) q \, dx \\
&\quad = -\mu^{-1}(1 - 2\nu) \int_\Omega (p_I - p_h) q \, dx + \int_\Omega \operatorname{div}_h (\underset{\sim}{u}_I - \underset{\sim}{u}) q \, dx \\
&\qquad + \mu^{-1}(1 - 2\nu) \int_\Omega (p_I - p) q \, dx.
\end{aligned} \tag{5.3}$$

Choosing $\underset{\sim}{v}_h = \underset{\sim}{u}_I - \underset{\sim}{u}_h$ in (5.2) and $q = p_I - p_h$ in (5.3), and combining these results, we obtain

$$\begin{aligned}
&a_h(\underset{\sim}{u}_I - \underset{\sim}{u}_h, \underset{\sim}{u}_I - \underset{\sim}{u}_h) + \nu\mu^{-1}(1 - 2\nu) \int_\Omega (p_I - p_h)^2 \, dx \\
&\quad = a_h(\underset{\sim}{u}_I - \underset{\sim}{u}, \underset{\sim}{u}_I - \underset{\sim}{u}_h) - \nu \int_\Omega (p_I - p) \operatorname{div}_h (\underset{\sim}{u}_I - \underset{\sim}{u}_h) \, dx \\
&\qquad + G_h(\underset{\sim}{u}_I - \underset{\sim}{u}_h) + \nu \int_\Omega (p_I - p_h) \operatorname{div}_h (\underset{\sim}{u}_I - \underset{\sim}{u}) \, dx \\
&\qquad + \nu\mu^{-1}(1 - 2\nu) \int_\Omega (p_I - p)(p_I - p_h) \, dx.
\end{aligned}$$

It then follows from the discrete Korn's inequality and the Schwarz inequality that

$$\begin{aligned}
&\mu K \|\underset{\sim}{u}_I - \underset{\sim}{u}_h\|_{1,h}^2 + \nu\mu^{-1}(1 - 2\nu) \int_\Omega (p_I - p_h)^2 \, dx \\
&\quad \le G_h(\underset{\sim}{u}_I - \underset{\sim}{u}_h) + \mu \|\underset{\sim}{u}_I - \underset{\sim}{u}_h\|_{1,h} \|\underset{\sim}{u}_I - \underset{\sim}{u}\|_{1,h} + 2\nu \|p_I - p\|_0 \|\underset{\sim}{u}_I - \underset{\sim}{u}_h\|_{1,h} \\
&\qquad + 2\nu \|p_I - p_h\|_0 \|\underset{\sim}{u}_I - \underset{\sim}{u}\|_{1,h} + \nu\mu^{-1}(1 - 2\nu) \|p_I - p_h\|_0 \|p_I - p\|_0.
\end{aligned}$$

Next, applying the stability condition (5.1), and using (5.2) and the Schwarz inequality, we get

$$\begin{aligned}\nu\|p_I - p_h\|_0 &\le \nu\gamma^{-1} \sup_{0\neq \underset{\sim}{v}_h\in\widehat{\underset{\sim}{V}}_h} \frac{\int_\Omega \operatorname{div}_h \underset{\sim}{v}_h (p_I - p_h)\,dx}{\|\underset{\sim}{v}_h\|_{1,h}} \\ &\le \gamma^{-1}\Bigg[\mu\|\underset{\sim}{u}_I - \underset{\sim}{u}_h\|_{1,h} + \mu\|\underset{\sim}{u}_I - \underset{\sim}{u}\|_{1,h} + 2\nu\|p_I - p\|_0 \\ &\qquad + \sup_{\underset{\sim}{w}_h\in\widehat{\underset{\sim}{V}}_h} \frac{G_h(\underset{\sim}{w}_h)}{\|\underset{\sim}{w}_h\|_{1,h}}\Bigg].\end{aligned}$$

Combining these results and using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}&\|\underset{\sim}{u}_I - \underset{\sim}{u}_h\|_{1,h} + \nu\|p_I - p_h\|_0 \\ &\qquad \le C\left[\|\underset{\sim}{u}_I - \underset{\sim}{u}\|_{1,h} + \|p_I - p\|_0 + \sup_{\underset{\sim}{w}_h\in\widehat{\underset{\sim}{V}}_h} \frac{G_h(\underset{\sim}{w}_h)}{\|\underset{\sim}{w}_h\|_{1,h}}\right].\end{aligned}$$

Theorem (5.1) now follows directly from the triangle inequality. □

Corollary 5.2. *We have*

$$\begin{aligned}\|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_{0,h} \le C \inf\Bigg(&\|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{1,h} + \|p - q_h\|_0 \\ &+ \sup \frac{\sum_T \int_{\partial T} \underset{\approx}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{w}_h\,ds - \int_{\partial\Omega} \underset{\sim}{g}\cdot\underset{\sim}{w}_h\,ds}{\|\underset{\sim}{w}_h\|_{1,h}}\Bigg).\end{aligned}$$

Proof. This follows immediately from (2.5) and (3.4). □

Using again the results in [12] and [13], we then obtain the following optimal-order error estimates.

Theorem 5.3. *Let* $\underset{\sim}{u}$ *and* $\underset{\sim}{u}_h$ *be the solutions to Problems* P *and* P_h^k, *respectively* ($k = 2$ *or* 3) *and* $\underset{\approx}{\sigma}$ *and* $\underset{\approx}{\sigma}_h$ *defined by* (2.2) *and* (3.4). *If* $\underset{\sim}{u} \in \underset{\sim}{H}^{k+1}(\Omega)$ *and* $\underset{\approx}{\sigma} \in \underset{\approx}{H}^k(\Omega)$, *then*

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{1,h} + \|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_0 \le Ch^k(\|\underset{\sim}{u}\|_{k+1} + \|\underset{\approx}{\sigma}\|_k),$$

where C *is independent of* $\underset{\sim}{u}$ *and* h, *and uniform for* $0 \le \nu < 1/2$.

6. Nonconforming piecewise linear elements

As mentioned previously, inequality (4.1) does not hold for the space $\widehat{\underset{\sim}{V}}{}_h^1$ of nonconforming piecewise linear elements. To establish this fact, we use a dimension-counting argument. First observe that the subspace of $\widehat{V}_h^1$ with $\varepsilon_h(u) = 0$ has dimension $\geq 2e - 3T - 3$, where e and T denote the number of edges and triangles, respectively, in the triangulation $\mathscr{T}_h$. This follows from the facts that the dimension of $\widehat{V}_h^1$ is $2e - 3$, and since $\varepsilon_h(u)$ is constant on every triangle, the constraint $\varepsilon_h(u) = 0$ imposes at most $3T$ independent constraints. But $2e - 3T - 3 = e_B - 3$, where e_B denotes the number of edges lying on $\partial\Omega$. As soon as $\mathscr{T}_h$ consists of more than one triangle, this dimension will be positive. On the other hand, the dimension of the subspace of $\widehat{V}_h^1$ with $\operatorname{grad}_h u = 0$ is clearly zero. Hence, there must exist functions in $\widehat{V}_h^1$ for which (4.1) fails.

We now consider a possible remedy for this problem, in which we make a slight modification of the basic piecewise linear nonconforming method by introducing a local projection in one of the terms. To describe this projection, we assume that the domain Ω has been first triangulated by a triangulation $\mathscr{T}_{h'}$. The triangulation $\mathscr{T}_h$ is then created by adding three interior edges per triangle formed by connecting the midpoints of the sides of each triangle $T' \in \mathscr{T}_{h'}$. We then define with respect to the coarse triangulation $\mathscr{T}_{h'}$ the finite element space

$$G_{h'} = \left\{ \beta \in L^2(\Omega) : \beta|_{T'} \in \mathscr{P}_0(T') \text{ for all } T' \in \mathscr{T}_{h'} \right\}$$

and let P_0 denote the L^2 projection into $G_{h'}$.

In order to establish a discrete Korn's inequality, we next replace the operator ε_h satisfying (3.1) by an operator ε_h^* defined by

$$\varepsilon_h^*(u) = \operatorname{grad}_h u - \frac{1}{2} P_0 \operatorname{rot}_h u \chi. \tag{6.1}$$

The approximation scheme is then given by:

Problem P_h^1. Find $u_h \in \widehat{V}_h^1$ such that

$$B_h^*(u_h, v) = \int_\Omega f \cdot v \, dx + \int_{\partial\Omega} g \cdot v \, ds \quad \text{for all } v \in \widehat{V}_h^1,$$

where

$$\begin{aligned} B_h^*(u, v) &= \mu \left(\int_\Omega \varepsilon_h^*(u) : \varepsilon_h^*(v) \, dx + \frac{\nu}{1 - 2\nu} \int_\Omega \operatorname{div}_h u \operatorname{div}_h v \, dx \right) \\ &= \mu \left(\int_\Omega \operatorname{grad}_h u : \operatorname{grad}_h v \, dx - \frac{1}{2} \int_\Omega P_0 \operatorname{rot}_h u \operatorname{rot}_h v \, dx \right. \\ &\qquad \left. + \frac{\nu}{1 - 2\nu} \int_\Omega \operatorname{div}_h u \operatorname{div}_h v \, dx \right). \end{aligned}$$

Note that since $P_0 \operatorname{div} \hat{V}_h^1 \subseteq G_{h'}$, it is easy to see that Problem P_h^1 is equivalent to:

Problem S_h^1. Find $u_h \in \hat{V}_h^1$, $p_h \in G_{h'}$ such that

$$\mu \int_\Omega \varepsilon_h^*(u_h) : \varepsilon_h^*(v)\, dx - \nu \int_\Omega p_h \operatorname{div}_h v\, dx = \int_\Omega f \cdot v\, dx + \int_{\partial\Omega} g \cdot v\, ds \text{ for all } v \in \hat{V}_h^1,$$

$$\int_\Omega \operatorname{div}_h u_h q\, dx = -\mu^{-1}(1 - 2\nu) \int_\Omega p_h q\, dx \quad \text{for all } q \in G_{h'}.$$

Note that the above approximation also makes sense in the incompressible limit $\nu = 1/2$. The approximate stress σ_h is then defined by

$$\text{(6.2)} \quad \begin{aligned} \sigma_h &= \mu \left[\operatorname{grad}_h u_h - \frac{1}{2} P_0 \operatorname{rot}_h u_h \chi + \frac{\nu}{1 - 2\nu} \operatorname{div}_h u_h \delta \right] \\ &= \mu \left[\operatorname{grad}_h u_h - \frac{1}{2} P_0 \operatorname{rot}_h u_h \chi \right] - \nu p_h \delta. \end{aligned}$$

The result of this change will be that the analysis will now depend on a modified form of the discrete Korn's inequality given in Lemma 4.1 in which the operator ε_h is replaced by the operator ε_h^*. Specifically, we shall prove:

Theorem 6.1. *For all* $v \in \hat{V}_h^1$, *there exists a constant* K *independent of* v *such that*

$$\text{(6.3)} \qquad \|\varepsilon_h^*(v)\|_0 \geq K \|\operatorname{grad}_h v\|_0.$$

To do so, we again need a discrete version of Lemma 2.2, giving a pressure space which, together with continuous piecewise linear finite elements, forms a stable pair of spaces for approximating the Stokes problem. A proof of the following lemma may be found in [14].

Lemma 6.2. *Given* $p \in G_{h'}$ *with* $\int_\Omega p = 0$, *there exists* $v \in \mathring{M}_0^1$ *such that*

$$\int_\Omega \operatorname{div} v q\, dx = \int_\Omega pq\, dx \quad \text{for all } q \in G_{h'}, \qquad \|v\|_1 \leq C\|p\|_0,$$

with C *independent of* v *and* p.

We now prove Theorem 6.1, using an argument similar to the one used in the proof of Theorem 4.1.

Proof of Theorem 6.1. Using (6.1), we have for all $\tau \in L^2$,

$$\int_\Omega \varepsilon_h^*(u) : \tau\, dx = \int_\Omega \left(\operatorname{grad}_h u - \frac{1}{2} P_0 \operatorname{rot}_h u \chi \right) : \tau\, dx.$$

Using Lemma 6.2, we may choose $\underset{\approx}{\tau} = \underset{\approx}{\text{grad}}_h \underset{\sim}{u} - \underset{\approx}{\text{curl}}\, \underset{\sim}{z}$, where $\underset{\sim}{z} \in \overset{\circ}{\underset{\sim}{M}}{}_0^1$ satisfies

$$(6.4) \quad \int_\Omega \operatorname{div} \underset{\sim}{z} q\, dx = \int_\Omega \operatorname{rot}_h \underset{\sim}{u} q\, dx \quad \text{for all } q \in G_{h'}, \qquad \|\underset{\sim}{z}\|_1 \le C \|\operatorname{rot}_h \underset{\sim}{u}\|_0.$$

Then

$$\|\underset{\approx}{\tau}\|_0 \le \|\underset{\approx}{\text{grad}}_h \underset{\sim}{u}\|_0 + \|\underset{\approx}{\text{curl}}\, \underset{\sim}{z}\|_0 \le C \|\underset{\approx}{\text{grad}}_h \underset{\sim}{u}\|_0.$$

Now observe that

$$\int_\Omega \underset{\approx}{\text{grad}}_h \underset{\sim}{u} : \underset{\approx}{\text{curl}}\, \underset{\sim}{z}\, dx = \sum_T \int_{\partial T} \underset{\sim}{u} \cdot \frac{\partial \underset{\sim}{z}}{\partial s}\, ds = 0,$$

since on boundary edges $\underset{\sim}{z} = 0$ and on interior edges, contributions from adjoining triangles cancel. The cancellation occurs since the integrand along the edges involves only tangential derivatives of $\underset{\sim}{z}$ which are polynomials of degree 0 (occurring with opposite signs) and average values of $\underset{\sim}{u}$ on each edge which are continuous across edges. (This argument is given in more detail in [5].) Using the $\underset{\approx}{L}^2$ orthogonality of $\underset{\approx}{\text{grad}}_h \underset{\sim}{u}$ and $\underset{\approx}{\text{curl}}\, \underset{\sim}{z}$ and (6.4), we obtain

$$\begin{aligned}\int_\Omega & \underset{\approx}{\varepsilon}_h^*(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx \\ &= \int_\Omega \left(\underset{\approx}{\text{grad}}_h \underset{\sim}{u} : \underset{\approx}{\text{grad}}_h \underset{\sim}{u} - \frac{1}{2} P_0 \operatorname{rot}_h \underset{\sim}{u} [\operatorname{rot}_h \underset{\sim}{u} - \operatorname{div} \underset{\sim}{z}] \right) dx = \|\underset{\approx}{\text{grad}}_h \underset{\sim}{u}\|_0^2.\end{aligned}$$

Hence,

$$\|\underset{\approx}{\varepsilon}_h^*(\underset{\sim}{u})\|_0 \ge \frac{\int_\Omega \underset{\approx}{\varepsilon}_h^*(\underset{\sim}{u}) : \underset{\approx}{\tau}\, dx}{\|\underset{\approx}{\tau}\|_0} \ge K \|\underset{\approx}{\text{grad}}_h \underset{\sim}{u}\|_0. \quad \square$$

The analogue of Theorem 5.1 holds for this modified approximation scheme, and we again get the following optimal-order error estimate.

Theorem 6.3. *Let $\underset{\sim}{u}$ and $\underset{\sim}{u}_h$ be the solutions to Problems* P *and* P_h^1, *respectively, and $\underset{\approx}{\sigma}$ and $\underset{\approx}{\sigma}_h$ defined by* (2.2) *and* (6.2). *If $\underset{\sim}{u} \in \underset{\sim}{H}^2(\Omega)$ and $\underset{\approx}{\sigma} \in \underset{\approx}{H}^1(\Omega)$, then*

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{1,h} + \|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_0 \le Ch(\|\underset{\sim}{u}\|_2 + \|\underset{\approx}{\sigma}\|_1),$$

where C is independent of $\underset{\sim}{u}$ and h, and uniform for $0 \le \nu < 1/2$.

It is interesting to note that $\underset{\approx}{\varepsilon}_h^*$, defined by (6.1), is not a symmetric matrix because of the presence of the projection P_0. In fact, it is possible to give an interpretation of this scheme as a mixed finite element method involving both stresses and displacements, which relaxes the symmetry of the stress tensor through the use of a Lagrange multiplier. Thus, it is similar in spirit to the method proposed in [2]. Using a slight modification of the ideas in [2], we consider the mixed formulation:

Problem M. Find $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega)$, $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$, $\gamma \in L^2(\Omega)$ such that

$$\int_\Omega \mathbf{A}\underset{\approx}{\sigma} : \underset{\approx}{\tau}\, dx - \int_\Omega \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{u} : \underset{\approx}{\tau}\, dx + \int_\Omega \gamma \operatorname{as}(\underset{\approx}{\tau})\, dx = 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

$$\int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{v}\, dx = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}},$$

$$\int_\Omega \operatorname{as}(\underset{\approx}{\sigma})\beta\, dx = 0 \quad \text{for all } \beta \in L^2(\Omega).$$

It is easy to see that if $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2$, $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$, $\gamma \in L^2$ solve Problem M, then $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$ solves Problem P and $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2$ satisfies (2.2). Conversely, if $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$ solves Problem P and $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2$ satisfies (2.2), then (2.1) implies that $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2$, $\underset{\sim}{u} \in \underset{\sim}{\widehat{V}}$, $\gamma = \operatorname{rot} \underset{\sim}{u}/2 \in L^2$ solve Problem M.

To give a reformulation of the approximate problem P_h^1, we first define an approximate space of nonsymmetric stresses by

$$\underset{\approx}{H}_h^0 = \{\underset{\approx}{\tau} : \tau_{ij}|_T \in \mathscr{P}_0(T) \text{ for all } T \in \mathscr{T}_h,\ i, j = 1, 2\}.$$

The approximate mixed formulation is then

Problem M_h^1. Find $\underset{\approx}{\sigma}_h \in \underset{\approx}{H}_h^0$, $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{V}}_h^1$, $\gamma_h \in G_{h'}$ such that

$$\int_\Omega \mathbf{A}\underset{\approx}{\sigma}_h : \underset{\approx}{\tau}\, dx - \int_\Omega \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{u}_h : \underset{\approx}{\tau}\, dx + \int_\Omega \gamma_h \operatorname{as}(\underset{\approx}{\tau})\, dx = 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{H}_h^0, \tag{6.5}$$

$$\int_\Omega \underset{\approx}{\sigma}_h : \underset{\approx}{\operatorname{grad}}_h \underset{\sim}{v}\, dx = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx + \int_{\partial\Omega} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\widehat{V}}_h^1, \tag{6.6}$$

$$\int_\Omega \operatorname{as}(\underset{\approx}{\sigma}_h)\beta = 0 \quad \text{for all } \beta \in G_{h'}. \tag{6.7}$$

We now show the equivalence of Problems P_h^1 and M_h^1.

Lemma 6.4. *Problem* M_h^1 *has a unique solution* $\underset{\approx}{\sigma}_h \in \underset{\approx}{H}_h^0$, $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{V}}_h^1$, $\gamma_h \in G_{h'}$, *where* $\underset{\sim}{u}_h$ *is the unique solution of Problem* P_h^1, $\underset{\approx}{\sigma}_h$ *is given by* (6.2), *and* $\gamma_h = \frac{1}{2}P_0 \operatorname{rot}_h \underset{\sim}{u}_h$.

Proof. To establish existence and uniqueness, we show that zero is the only solution to Problem M_h^1 with zero data. First set $\underset{\approx}{\tau} = \underset{\approx}{\sigma}_h$, $\underset{\sim}{v} = \underset{\sim}{u}_h$, $\beta = \gamma_h$. It follows immediately that

$$\begin{aligned} 0 = \int_\Omega \mathbf{A}\underset{\approx}{\sigma}_h : \underset{\approx}{\sigma}_h\, dx &= \mu(\|\underset{\approx}{\sigma}_h\|_0^2 - \nu \|\operatorname{tr}(\underset{\approx}{\sigma}_h)\|_0^2) \\ &= \mu\left[\left\|\underset{\approx}{\sigma}_h - \frac{1}{2}\operatorname{tr}(\underset{\approx}{\sigma}_h)\underset{\approx}{\delta}\right\|_0^2 + \left(\frac{1}{2} - \nu\right)\|\operatorname{tr}(\underset{\approx}{\sigma}_h)\|_0^2\right], \end{aligned}$$

and hence that

$$\underset{\approx}{\sigma}_h = \frac{1}{2}\operatorname{tr}(\underset{\approx}{\sigma}_h)\underset{\approx}{\delta}.$$

Inserting this result in (6.6), we get

$$\frac{1}{2}\sum_T \int_T \operatorname{tr}(\underset{\approx}{\sigma}_h)\operatorname{div}\underset{\sim}{v}\,dx = 0.$$

We now let $\underset{\sim}{z} = \underset{\sim}{\operatorname{grad}}\, r$, where r satisfies $\Delta r = \operatorname{tr}(\underset{\approx}{\sigma}_h)$ in Ω, $r = 0$ on $\partial\Omega$. Define $\underset{\sim}{z}_h \in \underset{\sim}{M}^1_*$ satisfying for each edge e of $\mathscr{T}_h$ the condition $\int_e(\underset{\sim}{z} - \underset{\sim}{z}_h)\,ds = 0$, and set $\underset{\sim}{v} = \underset{\sim}{z}_h - \overline{\underset{\sim}{z}}_h$, where we again use the notation $\overline{b}$ to denote the mean value of b on Ω. Then it is easy to check that $\underset{\sim}{v} \in \underset{\sim}{\widehat{V}}^1_h$ and satisfies $\operatorname{div}_h \underset{\sim}{v} = \operatorname{tr}(\underset{\approx}{\sigma}_h)$. With this choice of $\underset{\sim}{v}$, it follows immediately that $\operatorname{tr}(\underset{\approx}{\sigma}_h) = 0$, and hence $\underset{\approx}{\sigma}_h = 0$. Inserting this result and choosing $\underset{\approx}{\tau} = \underset{\approx}{\varepsilon}^*_h(\underset{\sim}{u}_h)$, we get that $\|\underset{\approx}{\varepsilon}^*_h(\underset{\sim}{u}_h)\|_0 = 0$, and hence from (6.3) that $\underset{\sim}{u}_h = 0$. Finally, choosing $\underset{\approx}{\tau} = \gamma_h \underset{\approx}{\chi}$ implies that $\gamma_h = 0$. It is now easy to check that $\underset{\sim}{u}_h$, the solution of Problem P^1_h, $\underset{\approx}{\sigma}_h$ given by (6.2), and $\gamma_h = \frac{1}{2}P_0 \operatorname{rot}_h \underset{\sim}{u}_h$, solve Problem M^1_h. □

7. Modified schemes for conforming elements

It should be noted that if a slightly different but analogous modification is made to the usual continuous piecewise linear approximation of the displacement formulation of elasticity, then one also obtains optimal-order error estimates, uniform for $\nu \in [0, 1/2)$. The modified approximation scheme is:

Problem C^1_h. Find $\underset{\sim}{u}_h \in \underset{\sim}{\widehat{W}}^1_h$ such that for all $\underset{\sim}{v} \in \underset{\sim}{\widehat{W}}^1_h$,

$$\mu\left(\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}_h) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\,dx + \frac{\nu}{1-2\nu}\int_\Omega P_0 \operatorname{div}\underset{\sim}{u}_h\, P_0 \operatorname{div}\underset{\sim}{v}\,dx\right) = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx + \int_{\partial\Omega}\underset{\sim}{g}\cdot\underset{\sim}{v}\,ds,$$

where

$$\underset{\sim}{\widehat{W}}^1_h = \left\{\underset{\sim}{v} \in \underset{\sim}{M}^1_0 : \int_\Omega \underset{\sim}{v}\,dx = \int_\Omega \operatorname{rot}\underset{\sim}{v}\,dx = 0\right\}.$$

The loss of accuracy occurring for continuous piecewise quadratic and cubic approximations to the elasticity equations near the incompressible limit can also be eliminated by adding the bubble functions $\underset{\sim}{B}^{k+1}$ to the spaces $\underset{\sim}{M}^k_0$ (as done for the Stokes problem) and then replacing the $\operatorname{div}\underset{\sim}{u}$ term by $P_{k-1}\operatorname{div}\underset{\sim}{u}$, where

P_{k-1} denotes the L^2 projection into M_{-1}^{k-1}. Note that without this projection, the divergence of the bubble functions will not be in this space. We are thus led for $k = 2$ and 3 to the following approximation schemes:

Problem C_h^k. Find $\underset{\sim}{u}_h \in \widehat{\underset{\sim}{W}}{}_h^k$ such that for all $\underset{\sim}{v} \in \widehat{\underset{\sim}{W}}{}_h^k$,

$$\mu\left(\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}_h) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\,dx + \frac{\nu}{1-2\nu}\int_\Omega P_{k-1}\operatorname{div}\underset{\sim}{u}_h\, P_{k-1}\operatorname{div}\underset{\sim}{v}\,dx\right) = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,dx + \int_{\partial\Omega}\underset{\sim}{g}\cdot\underset{\sim}{v}\,ds,$$

where

$$\widehat{\underset{\sim}{W}}{}_h^k = \left\{\underset{\sim}{v} \in \underset{\sim}{M}_0^k + \underset{\sim}{B}^{k+1} : \int_\Omega \underset{\sim}{v}\,dx = \int_\Omega \operatorname{rot}\underset{\sim}{v}\,dx = 0\right\}.$$

The approximate stress $\underset{\approx}{\sigma}_h$ in each case is then defined by

$$\underset{\approx}{\sigma}_h = \mu\left[\underset{\approx}{\varepsilon}(\underset{\sim}{u}_h) + \frac{\nu}{1-2\nu}P_{k-1}\operatorname{div}\underset{\sim}{u}_h\underset{\approx}{\delta}\right]. \tag{7.1}$$

To get optimal-order error estimates for these schemes, we first introduce an approximate pressure $p_h = -\frac{\mu}{1-2\nu}P_{k-1}\operatorname{div}\underset{\sim}{u}_h$ and write the schemes in a form analogous to Problem S_h^k, where $q \in G_{h'}$ for $k = 0$ and $\in M_{-1}^{k-1}$ for $k = 2, 3$, and we use the spaces $\widehat{\underset{\sim}{W}}{}_h^k$ for the velocities. Following the proof of Theorem 5.1 (without the extra term to account for the use of nonconforming elements) and using a slightly modified form of Lemmas 4.2 and 6.2 (since now $\underset{\sim}{v} \in \widehat{\underset{\sim}{W}}{}_h^k$ instead of $\mathring{\underset{\sim}{M}}{}_0^1$ or $\underset{\sim}{W}{}_h^k$) to replace (5.1), we obtain a result analogous to Theorem 5.1. Optimal-order error estimates for $\underset{\sim}{u} - \underset{\sim}{u}_h$ and $\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h$ follow directly from this result, (7.1), and standard approximation theory.

Finally, we compare the number of unknowns used by these methods with the nonconforming methods of the same order. Let v, e, and T denote the number of vertices, edges, and triangles in the triangulation $\mathscr{T}_h$, respectively. In the case of linears, the conforming method has $2v-3$ unknowns as compared to $2e-3$ unknowns for the nonconforming method. Since by Euler's formula, $e - v = T - 1$, the conforming method is simpler. The projection into $G_{h'}$ involves the same amount of work for both methods. In the case of quadratics, it is a choice of adding to $\underset{\sim}{M}_0^2$ the nonconforming space $\underset{\sim}{B}_*^2$ or adding the conforming space $\underset{\sim}{B}^3$ and then using the projection P_1. The number of unknowns is the same. In the case of cubic elements, the conforming method has $2(v+2e+3T)-3$ unknowns, while the nonconforming method has $2(3e + T) - 3$ unknowns. Using Euler's formula, we find that the nonconforming method uses $2(T + 1)$ fewer unknowns. Since no projection is required, the cubic nonconforming method seems simpler.

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