EDGE EFFECTS IN THE REISSNER-MINDLIN PLATE THEORY*

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Abstract. We study the behavior of solutions of five different boundary value problems for the Reissner-Mindlin plate model emphasizing the structure of the dependence of the solutions on the plate thickness. The boundary value problems considered are those modelling hard and soft clamped plates, hard and soft simply supported plates, and free plates. As proven elsewhere, the transverse displacement variable does not exhibit any edge effect, but the rotation vector exhibits a boundary layer for all the boundary value problems. The bending moment tensor and shear force vector have more pronounced boundary layers. The structures of each of these boundary layers are explored in detail. In particular, their strength depends on the type of boundary conditions considered. They are strongest for the soft simply supported and free plates, weakest for the soft clamped plate, and intermediate in the other two cases. For the soft clamped and hard simply supported plate, the boundary layers vanish near a flat boundary, but this is not true for the other boundary value problems. In order to illustrate the theory explicitly, we construct and analyze the exact solution to all the boundary value problems in the special cases of a circular plate and of a semi-infinite plate subject to a particular loading. We also examine the cases of an axisymmetrically loaded circular plate and a uniaxially loaded semi-infinite plate. In these special cases, the edge effects disappear entirely.

Key words. Reissner, Mindlin, plate, boundary layer, edge effect

AMS(MOS) subject classifications (1991 revision). 73K10, 73K25

NOMENCLATURE. Boldface type is used to denote vector quantities, including vector-valued functions and operators whose values are vector-valued functions. Script type is used to denote matrix quantities and sans serif type is used for higher order tensor quantities.

C	tensor of bending moduli
curl	(vector) curl of a scalar function
D	scaled bending modulus, $= E/[12(1-\nu^2)]$
div	(scalar) divergence of a vector function
div	(vector) divergence of a matrix function (applies by rows)
E	Young's modulus
g	scaled loading function, = transverse load density per unit area divided by t^3
grad	gradient of a scalar function
$\mathcal{I}$	2×2 identity matrix
I_1, I_2	modified Bessel functions of order 1 and 2

*Presented at the winter annual meeting of the American Society of Mechanical Engineers, December 10–15, 1989, San Francisco, California. This paper is included in *Analytic and Computational Models of Shells*, published by the A.S.M.E., New York, 1989.

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k shear correction factor

$\mathcal{M}$ tensor of bending moments

$\boldsymbol{n}$ unit vector normal to the boundary, directed outward

$\boldsymbol{s}$ unit vector tangent to the boundary, directed counterclockwise

t plate thickness

tr trace of a matrix

α $= \sqrt{12k}$

β $= \sqrt{12k + t^2}$

Δ Laplace operator

$\mathcal{E}$ (matrix) symmetric part of the gradient of a vector function

$\boldsymbol{\zeta}$ shear force vector in Reissner-Mindlin plate theory

κ curvature of the boundary

λ $= Ek/[2(1+\nu)]$

ν Poisson ratio

ρ, θ boundary fitted coordinates, distance to nearest boundary point and arclength parameter value of that point, respectively

$\boldsymbol{\phi}$ rotation vector in Reissner-Mindlin plate theory

$\boldsymbol{\phi}_0$ $= \mathbf{grad}\, \omega_0$, rotation vector in biharmonic plate theory

$\boldsymbol{\phi}_i$ interior expansion functions for $\boldsymbol{\phi}$

ϕ_ρ radial component of $\boldsymbol{\phi}$ (for circular plate)

ϕ_θ angular component of $\boldsymbol{\phi}$ (for circular plate)

$\boldsymbol{\Phi}_i$ boundary expansion functions for $\boldsymbol{\phi}$

χ cutoff function, identically one near boundary

ω transverse displacement in Reissner-Mindlin plate theory

ω_0 transverse displacement in biharmonic plate theory

ω_i interior expansion functions for ω

Ω region occupied by the midplane of the plate

$\partial\Omega$ the boundary of the region Ω

1. INTRODUCTION. The Reissner-Mindlin model for the bending of an isotropic elastic plate in equilibrium determines ω, the transverse displacement of the midplane, and $\boldsymbol{\phi}$, the rotation of fibers normal to the midplane, as the solution of the partial differential equations

$$-\,\mathbf{div}\, \mathcal{C}\, \mathcal{E}(\boldsymbol{\phi}) - \lambda t^{-2}(\mathbf{grad}\, \omega - \boldsymbol{\phi}) = 0 \text{ in } \Omega, \tag{1}$$

$$-\lambda t^{-2} \operatorname{div}(\mathbf{grad}\, \omega - \boldsymbol{\phi}) = g \text{ in } \Omega. \tag{2}$$

Here Ω is the two-dimensional region occupied by the midsection of the plate, t is the plate thickness, gt^3 is the transverse load force density per unit area, $\lambda = Ek/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, and k the shear correction factor, $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, and the fourth order tensor $\mathcal{C}$ is defined by

$$\mathcal{C}\mathcal{T} = D\left[(1-\nu)\mathcal{T} + \nu \operatorname{tr}(\mathcal{T})\mathcal{I}\right], \quad D = \frac{E}{12(1-\nu^2)},$$

for any 2×2 matrix $\mathcal{T}$ ($\mathcal{I}$ denotes the 2×2 identity matrix). Note that the load g has been scaled so that the solution tends to a nonzero limit as t tends to zero.

Solutions of the equations are minimizers of the energy functional

$$(\boldsymbol{\phi}, \omega) \mapsto \int_\Omega \left[\frac{1}{2}\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\phi}) + \frac{1}{2}\lambda t^{-2}|\,\mathbf{grad}\,\omega - \boldsymbol{\phi}|^2 - g\omega\right] d\boldsymbol{x}.$$

To obtain boundary conditions, we restrict the boundary behavior in the class of functions over which we minimize. For example, if we insist that $\boldsymbol{\phi}\cdot\boldsymbol{n}$ and $\boldsymbol{\phi}\cdot\boldsymbol{s}$ (the normal and tangential components of $\boldsymbol{\phi}$) and ω all vanish on the boundary of Ω, we obtain a model of a clamped or welded plate, which we call *hard clamped.* If we impose $\boldsymbol{\phi}\cdot\boldsymbol{n} = 0$ and $\omega = 0$ on $\partial\Omega$, but do not restrict $\boldsymbol{\phi}\cdot\boldsymbol{s}$, we obtain another model of clamping, which we term *soft clamped.* By standard variational arguments, we derive in this case a third (natural) boundary condition, namely that $\boldsymbol{s}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = 0$ on $\partial\Omega$. In general, we may impose or not each of the three essential boundary conditions $\boldsymbol{\phi}\cdot\boldsymbol{n} = 0$, $\boldsymbol{\phi}\cdot\boldsymbol{s} = 0$, and $\omega = 0$, thereby obtaining eight distinct boundary value problems. We consider in this study the five with the greatest physical significance, which are listed in Table 1.

Table 1 - Boundary conditions for various boundary value problems for the Reissner-Mindlin plate model.

	Essential	*Natural*
hard clamped	$\boldsymbol{\phi}\cdot\boldsymbol{n}=0$ $\boldsymbol{\phi}\cdot\boldsymbol{s}=0$ $\omega=0$	
soft clamped	$\boldsymbol{\phi}\cdot\boldsymbol{n}=0$ $\omega=0$	$\boldsymbol{s}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$
hard simply supported	$\boldsymbol{\phi}\cdot\boldsymbol{s}=0$ $\omega=0$	$\boldsymbol{n}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$
soft simply supported	$\omega=0$	$\boldsymbol{n}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$ $\boldsymbol{s}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$
free		$\boldsymbol{n}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$ $\boldsymbol{s}\cdot\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n}=0$ $\dfrac{\partial w}{\partial n}-\boldsymbol{\phi}\cdot\boldsymbol{n}=0$

For each of the first four boundary value problems, there is a unique solution for any load g. For the free plate, a necessary and sufficient condition for the existence of a solution is that the load satisfy the compatibility conditions

$$\int_\Omega g\,d\boldsymbol{x}=\int_\Omega xg\,d\boldsymbol{x}=\int_\Omega yg\,d\boldsymbol{x}=0. \tag{3}$$

Moreover, in this case, the solution is not unique. If $(\boldsymbol{\phi},\omega)$ is a solution, then so is $(\boldsymbol{\phi}+\mathbf{grad}\,l,\omega+l)$ for any linear function l.

From Eqs. 1 and 2, we deduce that

$$\mathbf{grad}\,\omega-\boldsymbol{\phi}=-t^2/\lambda\,\mathrm{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}),$$
$$\mathrm{div}\,\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})=g.$$

Formally taking the limit as the thickness t tends to zero in the first equation, we obtain Kirchhoff's hypothesis

$$\boldsymbol{\phi}_0=\mathbf{grad}\,\omega_0$$

(where the subscript 0 indicates the limit at $t=0$). Inserting this result in the second equation we then obtain the classical biharmonic equation of plate bending

$$D\,\Delta^2\,\omega_0=g. \tag{4}$$

Solutions of this equation are minimizers of the energy functional

$$\omega_0 \mapsto \int_\Omega \left[\frac{1}{2} C\, \mathcal{E}(\mathbf{grad}\, \omega_0) : \mathcal{E}(\mathbf{grad}\, \omega_0) - g\omega_0\right] d\boldsymbol{x}.$$

To obtain boundary conditions for the biharmonic model we may proceed as we did for the Reissner-Mindlin model, enforcing or not each of the conditions $\omega_0 = 0$ and $\partial\omega_0/\partial n (= \boldsymbol{\phi}_0 \cdot \boldsymbol{n}) = 0$ on $\partial\Omega$. Note that if ω_0 vanishes on $\partial\Omega$ then so does $\boldsymbol{\phi}_0 \cdot \boldsymbol{s} = \partial\omega_0/\partial s$, so this quantity cannot be constrained independently. Consequently, we only obtain four distinct boundary value problems in this way, the three most significant of which are listed in Table 2.

Table 2 - Boundary conditions for various boundary value problems for the biharmonic plate model.

	Essential	*Natural*
clamped	$\omega_0 = 0$ $\dfrac{\partial\omega_0}{\partial n} = 0$	
simply supported	$\omega_0 = 0$	$(1-\nu)\dfrac{\partial^2\omega_0}{\partial n^2} + \nu\,\Delta\,\omega_0 = 0$
free		$\dfrac{\partial}{\partial n}\,\Delta\,\omega_0 + (1-\nu)\dfrac{\partial}{\partial s}(\dfrac{\partial^2\omega_0}{\partial s \partial n} - \kappa\dfrac{\partial\omega_0}{\partial s}) = 0$ $(1-\nu)\dfrac{\partial^2\omega_0}{\partial n^2} + \nu\,\Delta\,\omega_0 = 0$

A fundamental difference between the biharmonic model and the Reissner-Mindlin model is that the solution of the former is independent of the plate thickness t (except through the scaling factor t^3 which we have absorbed into the loading function g), while the solution of the latter depends on the thickness in a complicated way. In particular, the solution exhibits a boundary layer for small t. More precisely, certain derivatives of the rotation vector vary rapidly in a narrow layer around the boundary. The existence of a boundary layer (in some sense) for the Reissner-Mindlin model has been noted by several authors (see Refs. 5, 4, 6, and 7, Chapter 3.5). In Refs. 2 and 3 we analyzed in detail the structure of the dependence of the solution on the plate thickness. These results apply generally, assuming only that the boundary of Ω and the loading function g, are smooth. We briefly recall the principal results here.

Let $\boldsymbol{\phi}$ and ω satisfy Eqs. 1 and 2 and one of the sets of boundary conditions found in Table 1. The transverse displacement admits an asymptotic expansion (as $t \to 0$) of the form

(5) $$\omega \sim \omega_0 + t\omega_1 + t^2\omega_2 + \cdots .$$

The functions ω_0, ω_1, etc., appearing in this expansion are smooth functions on Ω independent of t. There is no degeneration in ω as t tends to zero (that is, ω converges to ω_0 uniformly in the entire domain Ω, and all the derivatives of ω converge uniformly to the corresponding derivatives of ω_0). We term Expansion 5 a *regular expansion.* Its leading term, ω_0, satisfies the biharmonic equation (Eq. 4) and appropriate boundary conditions (i.e., if the boundary conditions for the Reissner-Mindlin problem are hard or soft clamped, then the boundary conditions for the biharmonic problem are clamped; similarly for simply supported and free boundary conditions). The other terms, ω_1, ω_2, etc., of the expansion may be obtained as solutions of other biharmonic problems. Explicit recipes are given in Refs. 2 and 3.

The rotation vector $\boldsymbol{\phi}$ admits an asymptotic expansion of the form

$$\boldsymbol{\phi} \sim (\boldsymbol{\phi}_0 + t\boldsymbol{\phi}_1 + t^2\boldsymbol{\phi}_2 + \cdots) + \chi(\boldsymbol{\Phi}_0 + t\boldsymbol{\Phi}_1 + t^2\boldsymbol{\Phi}_2 + \cdots). \tag{6}$$

Here the first sum on the right hand side is a regular expansion as for ω. The first two terms satisfy $\boldsymbol{\phi}_0 = \mathbf{grad}\,\omega_0$, and $\boldsymbol{\phi}_1 = \mathbf{grad}\,\omega_1$. However, it is generally not true that $\boldsymbol{\phi}_2 = \mathbf{grad}\,\omega_2$. The second sum represents an *edge effect* or *boundary layer.* To describe it we introduce a coordinate system fitted to the boundary. To any point $\boldsymbol{x}$ of Ω which is sufficiently close to the boundary, there corresponds a unique nearest boundary point $\boldsymbol{x}_0$. We associate to $\boldsymbol{x}$ the coordinates ρ and θ giving, respectively, the distance $|\boldsymbol{x} - \boldsymbol{x}_0|$ and the arclength along the boundary from $\boldsymbol{x}_0$ to some fixed reference point on the boundary (see Figure 1). We also associate to $\boldsymbol{x}$ a normal vector $\boldsymbol{n} = \boldsymbol{n}_x$, namely the unit vector in the direction of $\boldsymbol{x}_0$, and a tangential vector $\boldsymbol{s} = \boldsymbol{s}_x$ which is equal to the vector $\boldsymbol{n}_x$ rotated counterclockwise 90°. Note that this construction assumes that $\partial\Omega$ is smooth, that is, has no corners.

Returning to Expansion 6, the functions $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$, etc., have the explicit form

$$\boldsymbol{\Phi}_i = e^{-\sqrt{12k}\rho/t}\boldsymbol{F}_i(\rho/t, \theta)$$

where $\boldsymbol{F}_i$ is a smooth function, independent of t. Each $\boldsymbol{\Phi}_i$ may be determined as the solution of a certain system of ordinary differential equations in the stretched variable ρ/t. Because of the exponential factor, these functions are negligibly small outside a layer about the boundary of width proportional to t. Near the boundary, they vary rapidly in the direction of the normal: $\partial\boldsymbol{\Phi}_i/\partial n = O(1/t)$, $\partial^2\boldsymbol{\Phi}_i/\partial n^2 = O(1/t^2)$, etc.

Since the boundary-fitted coordinates are only valid in a region near the boundary, the functions $\boldsymbol{\Phi}_i$ are only defined in that region. Therefore, we introduce the cutoff function χ which equals zero outside of the region, and which equals one in a region near the boundary. Consequently, the products $\chi\boldsymbol{\Phi}_i$ appearing in the expansion 6 are defined everywhere in Ω. (Actually, for the two particular regions we consider in Sections 3 and 4, the circle and the halfplane, χ may be taken to be identically one.)

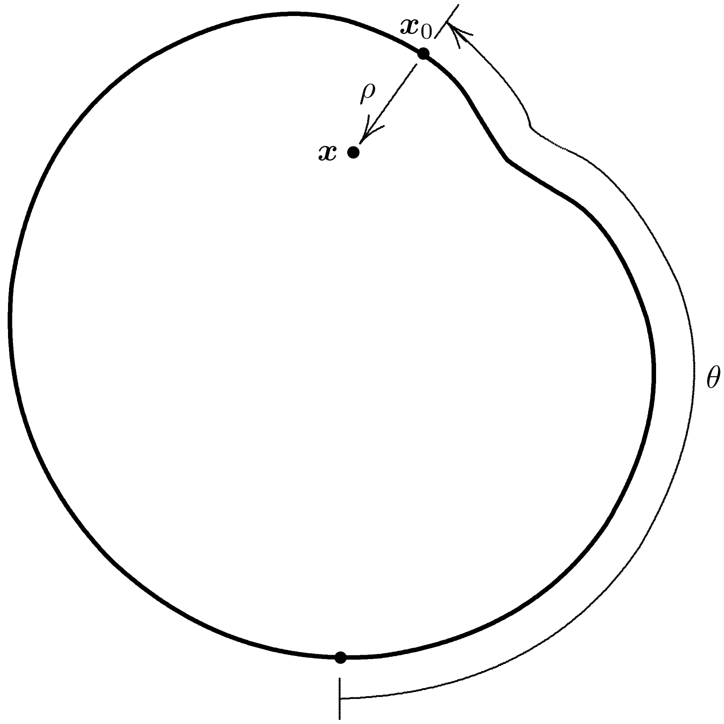

Fig. 1 - Boundary fitted coordinates ρ and θ for the point $\boldsymbol{x}$.

Table 3 - Terms which vanish in the boundary layer expansions 5 and 6 for various boundary value problems.

hard clamped	ω_1	$\boldsymbol{\phi}_1$	$\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2 \cdot \boldsymbol{n}$
soft clamped	ω_1	$\boldsymbol{\phi}_1$	$\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3 \cdot \boldsymbol{n}$
hard simply supported	ω_1	$\boldsymbol{\phi}_1$	$\boldsymbol{\Phi}_0, \boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2 \cdot \boldsymbol{n}$
soft simply supported	–	–	$\boldsymbol{\Phi}_0$	$\boldsymbol{\Phi}_1 \cdot \boldsymbol{n}$
free	–	–	$\boldsymbol{\Phi}_0$	$\boldsymbol{\Phi}_1 \cdot \boldsymbol{n}$

For each of the boundary conditions, certain of the terms in expansions 5 and 6 vanish (no matter for what load or boundary). Moreover, in all cases, the first nonvanishing $\boldsymbol{\Phi}_i$ is purely tangential, i.e., has zero normal component. The vanishing terms are listed in Table 3.

Note that $\boldsymbol{\phi}_1$ and ω_1 vanish for the first three boundary conditions, but not the other two. Thus, in these three cases the difference between the Reissner-Mindlin and biharmonic solutions is of order t^2 in the interior, while in the remaining two cases it is of order t.

The strength of the boundary layer for $\boldsymbol{\phi}$ is determined by the relative orders of the initial terms of the interior and boundary layer expansions. Since we have normalized the load so that the solution is always of order one with respect to t in the interior (i.e., $\boldsymbol{\phi}_0$ is always nonzero), the determining factor is the power of t before the first nonvanishing term of the boundary layer expansions. Thus, from Table 3, we see that the soft simply supported and free plates exhibit the strongest boundary layer, the hard clamped and hard simply supported plates have a weaker layer, and the soft clamped plate has the weakest layer. In each case, the normal component of $\boldsymbol{\phi}$ (recall that the normal direction is defined in a neighborhood of the boundary) has a weaker boundary layer than the tangential component.

Even for the soft simply supported and free plates, the leading term of the boundary layer expansion is of order t. Thus for t small, the boundary layer has a small effect on $\boldsymbol{\phi}$. However it has a more pronounced effect on various quantities derived from $\boldsymbol{\phi}$. For example, the bending moment tensor, $\mathcal{M} = \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})$, depends on the first derivatives of $\boldsymbol{\phi}$ and has a boundary layer of one order higher than that for $\boldsymbol{\phi}$ itself. Thus for the soft simply supported and the free plates, the edge effect on the moments is of the same order as the moments themselves. The shear force is given by the vector

$$\boldsymbol{\zeta} = \lambda t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi})$$

(which we have again scaled to be $O(1)$ in the interior at t tends to zero). Since the first two terms of the interior expansions cancel, we obtain the asymptotic expansion

$$\boldsymbol{\zeta} \sim \lambda\,[(\mathbf{grad}\,\omega_2 - \boldsymbol{\phi}_2) + t(\mathbf{grad}\,\omega_3 - \boldsymbol{\phi}_3) + \cdots] - \chi(t^{-1}\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + \cdots).$$

Thus, for the soft simply supported and free plates, for which $\boldsymbol{\Phi}_1$ is generally nonzero, the boundary layer dominates the shear force vector in the boundary region, and we have a very marked edge effect. For the hard clamped and hard simply supported plates, the edge effect is an order one effect on the shear, and for the soft clamped plate it is an order t effect.

Figure 2 shows the exact solution of the boundary value problem for the soft simply supported circular Reissner-Mindlin plate of radius 1. The load g is taken to be $\cos\theta$, the elastic coefficients are $E = 1$, $\nu = 0.3$. (The exact solution to this problem can be determined as explained in section 3.) Results for the transverse displacement and a component of the rotation vector, the bending moment tensor, and the shear force vector are shown plotted along a radius of the disc for a thick plate, a moderately thick plate, and a thin plate. The radius was taken along the positive x-axis for the displacement and along the positive y-axis in the other three cases. Note that the boundary layer is clearly visible in the large derivative of the bending moments for the thin plate. It is even clearer in the shear, which becomes very large in a tiny boundary layer for the thin plate.

Some of the terms of the boundary layer expansion for $\boldsymbol{\phi}$ involve the curvature of the boundary, and some of these vanish when the curvature is zero. For example, one can compute in the case of the hard simply supported plate (see Ref. 2) that

$$\boldsymbol{\Phi}_2 = -\frac{1}{6k(1-\nu)}\boldsymbol{s}\frac{\partial\,\Delta\,\omega_0}{\partial s}e^{-\sqrt{12k}\rho/t}.$$

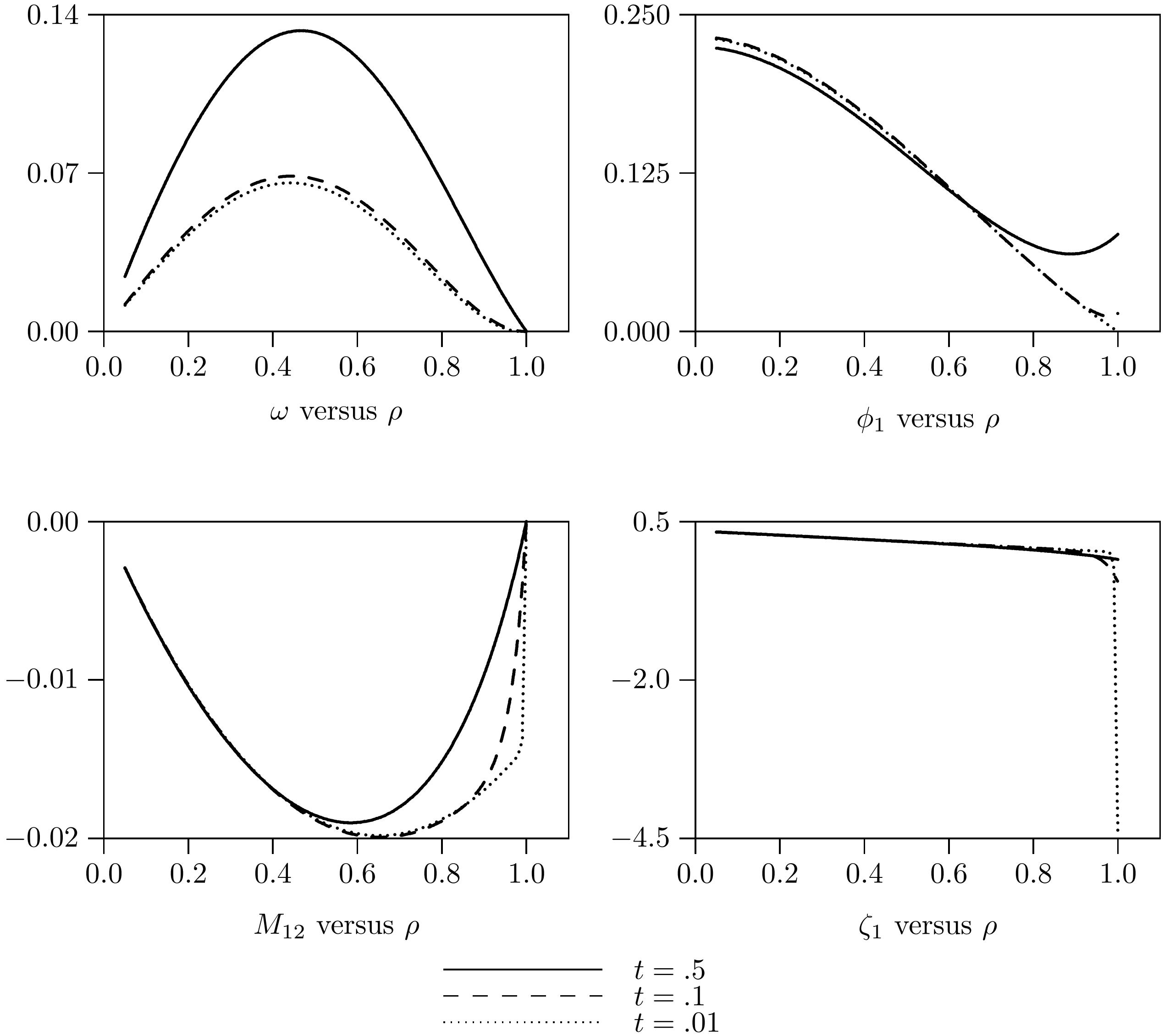

Fig. 2 - Transverse displacement, rotation, bending moment, and shear force along a radius of the unit disc for the soft simply supported Reissner-Mindlin plate.

Now for the hard simply supported plate, ω_0 satisfies the boundary conditions for the simply supported biharmonic plate given in Table 2, whence one easily shows

$$\Delta\,\omega_0 = \kappa(1-\nu)\frac{\partial \omega_0}{\partial n}.$$

Hence, if the curvature κ is identically zero, then $\boldsymbol{\Phi}_2$ vanishes. In fact, it can be shown that in a neighborhood of a flat section of the boundary, all the $\boldsymbol{\Phi}_i$ vanish. For the soft clamped plate problem we can also show that the $\boldsymbol{\Phi}_i$ vanish if κ vanishes identically. However, the solutions of the other boundary value problems experience an edge effect on straight as well as curved boundaries.

The theory just described can be found in more detail, along with a rigorous mathematical justification and a number of applications, in Refs. 2 and 3. In the following section, we illustrate this theory in the case of a circular plate with a particulary simple load, a case in which we are able to derive the exact solution and the expansions explicitly. In Section 4, we consider a semi-infinite plate (the plate problem posed on a halfplane) to illustrate the effect of a flat boundary. In Section 5, we consider briefly the simplifications which arise if the load is axisymmetric on a circular plate or uniaxial on a semi-infinite plate. In these cases, the expansions become extremely simple, and the edge effects disappear altogether.

The determination of exact solutions and their interior and boundary layer expansions, which is given in Sections 3 and 4, involves a great deal of computation which, while in principle elementary, is very cumbersome. In the following section we give a reformulation of the Reissner-Mindlin system which simplifies the construction of exact solutions to some extent. The computations leading to Tables 4, 5, and 6, and the expansions at the end of Sections 3 and 4 were performed using the Mathematica computer algebra system and were independently verified using the Maple computer algebra system.

2. CONSTRUCTION OF EXACT SOLUTIONS. Our basic construction defines a solution to the Reissner-Mindlin equations in terms of three functions, v, m, and q, which satisfy the uncoupled partial differential equations

$$D\,\Delta^2 v = g, \tag{7}$$

$$\Delta m = 0, \tag{8}$$

$$-\frac{t^2}{12k}\,\Delta q + q = 0. \tag{9}$$

We define $\boldsymbol{\phi}$ and ω in terms of these functions by the equations

$$\boldsymbol{\phi} = \mathbf{grad}\, v - \lambda^{-1}t^2(\mathbf{grad}\, m + \mathbf{curl}\, q), \qquad \omega = -\lambda^{-1}Dt^2\,\Delta\, v + v - \lambda^{-1}t^2 m. \tag{10}$$

It follows that,

$$\mathbf{grad}\, w - \boldsymbol{\phi} = -\lambda^{-1}t^2(D\,\mathbf{grad}\,\Delta\, v - \mathbf{curl}\, q),$$

and so

$$-\lambda t^{-2}\,\mathrm{div}(\mathbf{grad}\, w - \boldsymbol{\phi}) = D\,\Delta^2\, v = g.$$

It also follows easily from the definitions that

$$\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) = D\,\mathbf{grad}\,\Delta\, v - \mathbf{curl}\, q,$$

so

$$-\,\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) - \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\phi}) = 0.$$

Thus, for any functions v, m, and q satisfying Eqs. 7–9, we obtain functions $\boldsymbol{\phi}$ and ω which satisfy the Reissner-Mindlin equations, Eqs. 1 and 2.

It remains to satisfy the various boundary conditions described in Section 1. For each of the boundary value problems given in Table 1 we prescribe corresponding boundary conditions on v, m, and q. Since Eq. 7 is fourth order, while Eqs. 8 and 9 are second order, we need to give four boundary conditions. Three of the boundary conditions are clear: we merely substitute the formulas given in Eqs. 10 into the boundary conditions for $\boldsymbol{\phi}$ and ω. Thus, for example, in the hard clamped case we get

$$\begin{aligned}
\frac{\partial v}{\partial n} - \lambda^{-1}t^2\left(\frac{\partial m}{\partial n} - \frac{\partial q}{\partial s}\right) &= 0, \\
\frac{\partial v}{\partial s} - \lambda^{-1}t^2\left(\frac{\partial m}{\partial s} + \frac{\partial q}{\partial n}\right) &= 0, \\
-\lambda^{-1}Dt^2\,\Delta\, v + v - \lambda^{-1}t^2 m &= 0.
\end{aligned}$$

As a fourth boundary condition, we choose $v = 0$ in the hard and soft clamped and hard and soft simply supported cases, and $m = 0$ in the case of the free plate. Note that in the case of the free plate, this boundary condition, together with the differential equation 8, implies that m vanishes identically.

3. THE CIRCULAR PLATE. We now specialize to the case where the domain Ω is the unit circle and employ polar coordinates r and θ. For the soft and hard clamped and soft and hard simply supported plates we take as the load function $g(r,\theta) = \cos\theta$. This load does not satisfy the compatibility conditions given by Eq. 3 necessary for existence of a solution in the case of a free plate, so we use $g(r,\theta) = \cos 2\theta$ in that case. We do not take the simplest load, $g \equiv 1$, since in that special situation there is no boundary layer.

Consistent with the use of polar coordinates, we decompose the rotation vector into its radial and angular components

$$\phi_r = \boldsymbol{\phi}_1\cos\theta + \boldsymbol{\phi}_2\sin\theta, \quad \phi_\theta = -\boldsymbol{\phi}_1\sin\theta + \boldsymbol{\phi}_2\cos\theta.$$

As remarked in Section 1, ϕ_r will have a weaker boundary layer than ϕ_θ.

Written in polar coordinates, the construction of the previous section gives

$$\begin{aligned}
\phi_r &= \frac{\partial v}{\partial r} - \lambda^{-1}t^2\left(\frac{\partial m}{\partial r} - \frac{1}{r}\frac{\partial q}{\partial \theta}\right), \\
\phi_\theta &= \frac{1}{r}\frac{\partial v}{\partial \theta} - \lambda^{-1}t^2\left(\frac{1}{r}\frac{\partial m}{\partial \theta} + \frac{\partial q}{\partial r}\right), \\
\omega &= -D\lambda^{-1}t^2\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right) + v - \lambda^{-1}t^2 m,
\end{aligned}$$

where v, m, and q satisfy Eqs. 7–9.

In the case of the soft and hard clamped and soft and hard simply supported plates, we easily see from the form of the load that

$$v(r,\theta) = V(r)\cos\theta, \quad m(r,\theta) = M(r)\cos\theta, \quad q(r,\theta) = Q(r)\sin\theta,$$

for some functions V, M, Q of one variable. Substituting in Eqs. 7–9 we get

$$\begin{aligned} D\left[V''''(r) + \frac{2}{r}V'''(r) - \frac{3}{r^2}V''(r) + \frac{3}{r^3}V'(r) - \frac{3}{r^4}V(r)\right] &= 1, \\ M''(r) + \frac{1}{r}M'(r) - \frac{1}{r^2}M(r) &= 0, \\ -\frac{t^2}{12k}\left[Q''(r) + \frac{1}{r}Q'(r) - \frac{1}{r^2}Q(r)\right] + Q(r) &= 0. \end{aligned}$$

It is easy to compute the general solution of these ordinary differential equations. Excluding solutions for which the functions or first derivative are unbounded at the origin, we obtain

$$V(r) = r^4/(45D) + ar^3 + br, \quad M(r) = cr, \quad Q(r) = dI_1(\alpha r/t).$$

Here a, b, c, and d are arbitrary constants, $\alpha = \sqrt{12k}$, and I_1 denotes the modified Bessel function of order 1. To determine the constants a, b, c, and d, we substitute these expressions for ϕ and ω into the boundary conditions given in Table 1. This gives three linear equations, to which we append the equation $v = 0$ as explained in the previous section. The resulting system of four linear equations in four unknowns may be solved to give the results in Table 4. The solution for the hard clamped plate is also given in Ref. 4.

Table 4 - Exact solutions to the Reissner-Mindlin equations on the unit disc with $g = \cos\theta$.

	$\phi_r = \left[4r^3/(45D) + 3ar^2 + b - c\lambda^{-1}t^2 + r^{-1}\lambda^{-1}dt^2 I_1(\alpha r/t)\right]\cos\theta$ $\phi_\theta = \left[-r^3/(45D) - ar^2 - b + c\lambda^{-1}t^2 - d\alpha\lambda^{-1}tI_1'(\alpha r/t)\right]\sin\theta$ $\omega = \left[r^4/(45D) - \lambda^{-1}t^2r^2/3 + a(r^3 - 8D\lambda^{-1}rt^2) + br - c\lambda^{-1}t^2 r\right]\cos\theta$
hard clamped	$a = \left[-5Dt^3 I_1(\alpha/t) + \alpha\left(\lambda + 5Dt^2\right) I_1'(\alpha/t)\right]/(2Df)$ $b = \left[7Dt^3 I_1(\alpha/t) - \alpha\left(\lambda + 7Dt^2\right) I_1'(\alpha/t)\right]/(6Df)$ $c = \alpha\lambda I_1'(\alpha/t)/f$ $d = \lambda t/f$ $f = 15\left[4Dt^3 I_1(\alpha/t) - \alpha\left(\lambda + 4Dt^2\right) I_1'(\alpha/t)\right]$
soft clamped	$a = \left[-\left(\alpha^2\lambda + 5\alpha^2 Dt^2 + 10Dt^4\right) I_1(\alpha/t) + 2\alpha t\left(\lambda + 5Dt^2\right) I_1'(\alpha/t)\right]/(2Df)$ $b = \left[\left(\alpha^2\lambda + 7\alpha^2 Dt^2 + 14Dt^4\right) I_1(\alpha/t) - 2\alpha t\left(\lambda + 7Dt^2\right) I_1'(\alpha/t)\right]/(6Df)$ $c = \left[-\alpha^2\lambda I_1(\alpha/t) + 2\alpha\lambda t I_1'(\alpha/t)\right]/f$ $d = 2\lambda t^2/f$ $f = 15\left[\left(\alpha^2\lambda + 4\alpha^2 Dt^2 + 8Dt^4\right) I_1(\alpha/t) - 2\alpha t\left(\lambda + 4Dt^2\right) I_1'(\alpha/t)\right]$
hard simply supported	$a = \left[5Dt^3(1-\nu)I_1(\alpha/t) + \alpha\left(4\lambda + \lambda\nu - 5Dt^2 + 5D\nu t^2\right) I_1'(\alpha/t)\right]/(2Df)$ $b = \left[-7Dt^3(1-\nu)I_1(\alpha/t) - \alpha\left(6\lambda + \lambda\nu - 7Dt^2 + 7D\nu t^2\right) I_1'(\alpha/t)\right]/(6Df)$ $c = -\alpha\lambda(1-\nu)I_1'(\alpha/t)/f$ $d = -\lambda(1-\nu)t/f$ $f = 15\left[-4Dt^3(1-\nu)I_1(\alpha/t) - \alpha\left(3\lambda + \lambda\nu - 4Dt^2 + 4D\nu t^2\right) I_1'(\alpha/t)\right]$
soft simply supported	$a = \left[-\left(4\alpha^2 + \alpha^2\nu + 10t^2\right) I_1(\alpha/t) + 10\alpha t I_1'(\alpha/t)\right]/(2Df)$ $b = \left[\left(6\alpha^2 + \alpha^2\nu + 14t^2\right) I_1(\alpha/t) - 14\alpha t I_1'(\alpha/t)\right]/(6Df)$ $c = \alpha^2(1-\nu)I_1(\alpha/t)/f$ $d = 2\lambda/(Df)$ $f = 15\left[\left(3\alpha^2 + \alpha^2\nu + 8t^2\right) I_1(\alpha/t) - 8\alpha t I_1'(\alpha/t)\right]$

In the case of a free plate, we proceed similarly to obtain the exact solution. In this case v, m, and q have the form

$$v(r,\theta) = V(r)\cos 2\theta, \quad m(r,\theta) = M(r)\cos 2\theta, \quad q(r,\theta) = Q(r)\sin 2\theta,$$

where V, M, and Q solve

$$\begin{aligned} D\left[V''''(r) + \frac{2}{r}V'''(r) - \frac{9}{r^2}V''(r) + \frac{9}{r^3}V'(r)\right] &= 1, \\ M''(r) + \frac{1}{r}M'(r) - \frac{4}{r^2}M(r) &= 0, \\ -\frac{t^2}{12k}\left[Q''(r) + \frac{1}{r}Q'(r) - \frac{4}{r^2}Q(r)\right] + Q(r) &= 0. \end{aligned}$$

Solving, we obtain

$$V(r) = (r^4\log r)/(48D) + ar^4 + br^2, \quad M(r) = cr^2, \quad Q(r) = dI_2(\alpha r/t).$$

Finally, we solve for a, b, c, and d, using the boundary conditions for the free plate and the supplementary boundary condition $m = 0$, to get the results in Table 5.

Table 5 - Exact solution to the Reissner-Mindlin equations on the unit disc with $g = \cos 2\theta$ and free boundary conditions.

$\phi_r = \big[r^3/(48D) + r^3\log r/(12D) + 4ar^3 + 2br - 2cr\lambda^{-1}t^2 + 2dr^{-1}\lambda^{-1}t^2 I_2(\alpha r/t)\big] \times\cos 2\theta$
$\phi_\theta = \big[-r^3\log r/(24D) - 2ar^3 - 2br + 2c\lambda^{-1}t^2 r - d\alpha\lambda^{-1}tI_2'(\alpha r/t)\big]\sin 2\theta$
$\omega = \big\{r^2[-8\lambda^{-1}t^2 + (r^2/D - 12\lambda^{-1}t^2)\log r]/48 + a(r^4 - 12D\lambda^{-1}r^2t^2) + br^2 - c\lambda^{-1}t^2r^2\big\}\cos 2\theta$
$a = \Big(\{-8\lambda + (1-\nu)[3\lambda - 7D(\alpha^2 + 12t^2)]\}I_2(\alpha/t) + 42\alpha D(1-\nu)tI_2'(\alpha/t)\Big)/(24Df)$
$b = \Big(\{4\lambda + 6D(\alpha^2 + 8t^2) - (1-\nu)[\lambda - D(\alpha^2 + 28t^2)]\}I_2(\alpha/t) - 2\alpha D(17 - 11\nu)tI_2'(\alpha/t)\Big)/(8Df)$
$c = 0$
$d = -\lambda(1+2\nu)/f$
$f = 12\Big(\{2\lambda - (1-\nu)[\lambda - D(\alpha^2 + 12t^2)]\}I_2(\alpha/t) - 6\alpha D(1-\nu)tI_2'(\alpha/t)\Big)$

Having computed the exact solution to the various boundary value problems, we can derive asymptotic expansions like those given in Formulas 5 and 6 directly. To do this, first we substitute asymptotic expansions for I_1 and I_1' into the equations given at the top of Table 4 and for I_2 and I_2' into the equations at the top of Table 5. The required expansions are given in Ref. 1, Chapter 9.7 as

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{(4n^2-1^2)}{1!(8z)^1} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8z)^2} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!(8z)^3} + \cdots\right] \text{ as } z \to +\infty,$$

and

$$I_n'(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{(4n^2+1\cdot 3)}{1!(8z)^1} + \frac{(4n^2-1^2)(4n^2+3\cdot 5)}{2!(8z)^2} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2+5\cdot 7)}{3!(8z)^3} + \cdots\right] \text{ as } z \to +\infty.$$

When we substitute these expansions in the formulas for a, b, and c (but not d), the terms $e^{\alpha/t}$ and $\sqrt{2\pi\alpha/t}$ all cancel, leaving the quotient of two power series in t, which can be reduced to a single power series by standard manipulations. For example, for the hard clamped plate we get

$$a \sim \frac{\alpha\lambda - 7\lambda t/8 + [5\alpha D + 57\lambda/(128\alpha)]t^2 + \cdots}{-30\alpha D\lambda + 105D\lambda t/4 - [120\alpha D^2 + 855D\lambda/(64\alpha)]t^2 + \cdots}$$
$$\sim -\frac{1}{30D} - \frac{t^2}{30\lambda} + \frac{t^3}{30\alpha\lambda} + \cdots .$$

From the form of ω, in particular the fact that it is independent of d, we can see that ω will admit a regular (power series) expansion in t for all the boundary value problems.

On the other hand, when we substitute these expansions in the formula for d, the exponential and square root terms remain. Now d enters the formulas for ϕ_r and ϕ_θ multiplied by $I_n(\alpha r/t)$ and $I_n'(\alpha r/t)$, respectively. Consequently, the terms involving $\sqrt{t}$ will cancel, while the exponential terms combine to give a factor of $e^{-\alpha(1-r)/t} = e^{-\sqrt{12k}\rho/t}$. Thus the terms involving d, and these alone, determine the boundary layer expansion. Since this term includes a factor of t^2 for ϕ_r for all the boundary value problems, it will always be the case that the radial components of $\boldsymbol{\Phi}_0$ and and $\boldsymbol{\Phi}_1$ vanish. For ϕ_θ, the corresponding factor is the first power of t, so only $\boldsymbol{\Phi}_0$ need have vanishing tangential component in general. For the hard clamped and hard simply supported boundary conditions, d contains an additional power of t, so in addition the radial component of $\boldsymbol{\Phi}_2$ and the tangential component of $\boldsymbol{\Phi}_1$ also vanish. For the soft clamped case, there is a factor of t^2 in d, so the radial components of $\boldsymbol{\Phi}_2$ and $\boldsymbol{\Phi}_3$ and the tangential components of $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ vanish in this case.

In fact, using elementary manipulations, we can compute as many terms of the asymptotic expansions of $\boldsymbol{\phi}$ and ω as desired. The first several terms are given below. (For the hard clamped plate these results are also basically contained in Ref. 4.)

Hard clamped

$$\phi_r \sim \left\{ \left[\frac{1-9r^2+8r^3}{90D} + \frac{(1-r^2)t^2}{5D\alpha^2(1-\nu)} + \cdots \right] + e^{-\alpha(1-r)/t} \left[\frac{-2t^3}{15D\alpha^3(1-\nu)} + \cdots \right] \right\} \cos\theta$$

$$\phi_\theta \sim \left\{ \left[\frac{-1+3r^2-2r^3}{90D} - \frac{(3-r^2)t^2}{15D\alpha^2(1-\nu)} + \cdots \right] + e^{-\alpha(1-r)/t} \left[\frac{2t^2}{15D\alpha^2(1-\nu)} + \cdots \right] \right\} \sin\theta$$

$$\omega \sim \left\{ \frac{r-3r^3+2r^4}{90D} + \frac{(11r-10r^2-r^3)t^2}{15D\alpha^2(1-\nu)} + \cdots \right\} \cos\theta$$

Soft clamped

$$\phi_r \sim \left\{ \left[\frac{1-9r^2+8r^3}{90D} + \frac{(1-r^2)t^2}{5D\alpha^2(1-\nu)} + \cdots \right] + e^{-\alpha(1-r)/t} \left[\frac{4t^4}{15D\alpha^4(1-\nu)} + \cdots \right] \right\} \cos\theta$$

$$\phi_\theta \sim \left\{ \left[\frac{-1+3r^2-2r^3}{90D} - \frac{(3-r^2)t^2}{15D\alpha^2(1-\nu)} + \cdots \right] + e^{-\alpha(1-r)/t} \left[\frac{-4t^3}{15D\alpha^3(1-\nu)} + \cdots \right] \right\} \sin\theta$$

$$\omega \sim \left\{ \frac{r-3r^3+2r^4}{90D} + \frac{(11r-10r^2-r^3)t^2}{15D\alpha^2(1-\nu)} + \cdots \right\} \cos\theta$$

Hard simply supported

$$\phi_r \sim \left\{ \frac{6+\nu-36r^2-9\nu r^2+24r^3+8\nu r^3}{90D(3+\nu)} - \frac{(5+3\nu+3r^2-3\nu r^2)t^2}{15D\alpha^2(3+\nu)^2} + \cdots \right.$$
$$\left. + e^{-\alpha(1-r)/t} \left[\frac{2t^3}{15D\alpha^3(3+\nu)} + \cdots \right] \right\} \cos\theta$$

$$\phi_\theta \sim \left\{ \frac{-6-\nu+12r^2+3\nu r^2-6r^3-2\nu r^3}{90D(3+\nu)} - \frac{(-5-3\nu-r^2+\nu r^2)t^2}{15D\alpha^2(3+\nu)^2} + \cdots \right.$$
$$\left. + e^{-\alpha(1-r)/t} \left[\frac{-2t^2}{15D\alpha^2(3+\nu)} + \cdots \right] \right\} \sin\theta$$

$$\omega \sim \left\{ \frac{6r+\nu r-12r^3-3\nu r^3+6r^4+2\nu r^4}{90D(3+\nu)} \right.$$
$$\left. + \frac{(r-r^2)(91+58\nu+11\nu^2+r-2\nu r+\nu^2 r)t^2}{15D\alpha^2(1-\nu)(3+\nu)^2} + \cdots \right\} \cos\theta$$

Soft simply supported

$$\phi_r \sim \left\{ \frac{6+\nu-36r^2-9\nu r^2+24r^3+8\nu r^3}{90D(3+\nu)} + \frac{(1-\nu)(1-3r^2)t}{15D\alpha(3+\nu)^2} + \cdots \right.$$
$$\left. + e^{-\alpha(1-r)/t}\left[\frac{2t^2}{15D\alpha^2(3+\nu)} + \cdots\right]\right\}\cos\theta$$

$$\phi_\theta \sim \left\{ \frac{-6-\nu+12r^2+3\nu r^2-6r^3-2\nu r^3}{90D(3+\nu)} + \frac{(1-\nu)(r^2-1)t}{15D\alpha(3+\nu)^2} + \cdots \right.$$
$$\left. + e^{-\alpha(1-r)/t}\left[\frac{-2t}{15D\alpha(3+\nu)} + \cdots\right]\right\}\sin\theta$$

$$\omega \sim \left\{ \frac{6r+\nu r-12r^3-3\nu r^3+6r^4+2\nu r^4}{90D(3+\nu)} + \frac{(1-\nu)(r-r^3)t}{15D\alpha(3+\nu)^2} + \cdots \right\}\cos\theta$$

Free

$$\phi_r \sim \left\{ \frac{3(17-4\nu-\nu^2)r-(1-\nu)(29+3\nu)r^3}{144D(1-\nu)(3+\nu)} + \frac{r^3\log r}{12D} + \frac{(1+2\nu)r(\nu+r^2-\nu r^2)t}{3\alpha D(1-\nu)(3+\nu)^2} + \cdots \right.$$
$$\left. + e^{-\alpha(1-r)/t}\left[\frac{-(1+2\nu)t^2}{3\alpha^2 D(1-\nu)(3+\nu)} + \cdots\right]\right\}\cos 2\theta$$

$$\phi_\theta \sim \left\{ \frac{3(-17+4\nu+\nu^2)r+(1-\nu)(19+3\nu)r^3}{144D(1-\nu)(3+\nu)} - \frac{r^3\log r}{24D} - \frac{(1+2\nu)r(2\nu+r^2-\nu r^2)t}{6\alpha D(1-\nu)(3+\nu)^2} \right.$$
$$\left. + \cdots \quad + e^{-\alpha(1-r)/t}\left[\frac{(1+2\nu)t}{6\alpha D(1-\nu)(3+\nu)} + \cdots\right]\right\}\sin 2\theta$$

$$\omega \sim \left\{ \frac{3(17-4\nu-\nu^2)r^2-(1-\nu)(19+3\nu)r^4}{288D(1-\nu)(3+\nu)} \right.$$
$$\left. + \frac{r^4\log r}{48D} + \frac{(1+2\nu)r^2(2\nu+r^2-\nu r^2)t}{12\alpha D(1-\nu)(3+\nu)^2} + \cdots \right\}\cos 2\theta$$

4. THE SEMI-INFINITE PLATE. In this section we consider the semi-infinite plate, i.e., the case when the domain Ω is equal to the half plane $y > 0$. For the loading function we take $g(x, y) = \cos x$. This load is compatible with the existence of a solution for the free plate, so we may use it for all five boundary conditions. We again use the construction of Section 2.

Based on the form of the load, we seek v, m, and q as

$$v(x, y) = V(y)\cos x, \quad m(x, y) = M(y)\cos x, \quad q(x, y) = Q(y)\sin x,$$

where $V(y)$, $M(y)$, and $Q(y)$ satisfy

$$\begin{aligned} D\left[V''''(y) - 2V''(y) + V(y)\right] &= 1, \\ M''(y) - M(y) &= 0, \\ -\frac{t^2}{12k}\left[Q''(y) - Q(y)\right] + Q(y) &= 0. \end{aligned}$$

Solving, we obtain

$$V(y) = 1/D + ae^{-y} + bye^{-y}, \quad M(y) = ce^{-y}, \quad Q(y) = de^{-\beta y/t},$$

where $\beta = \sqrt{12k + t^2}$. Solving for the unknown constants from the boundary conditions in Table 1 and the supplementary boundary condition ($v = 0$ or $m = 0$), we obtain the results given in Table 6.

Finally, straightforward algebraic manipulations give the following asymptotic expansions.

Hard clamped

$$\phi_1 \sim \left\{\left[\frac{e^{-y} + ye^{-y} - 1}{D} - \frac{2(1+y)e^{-y}t^2}{D\alpha^2(1-\nu)} + \cdots\right] + e^{-\alpha y/t}\left[\frac{2t^2}{D\alpha^2(1-\nu)} + \cdots\right]\right\}\sin x$$

$$\phi_2 \sim \left\{\left[\frac{ye^{-y}}{D} - \frac{2ye^{-y}t^2}{D\alpha^2(1-\nu)} + \cdots\right] + e^{-\alpha y/t}\left[\frac{2t^3}{D\alpha^3(1-\nu)} + \cdots\right]\right\}\cos x$$

$$\omega \sim \left[\frac{1 - e^{-y} - ye^{-y}}{D} + \frac{2(1 - e^{-y} + ye^{-y})t^2}{D\alpha^2(1-\nu)} + \cdots\right]\cos x$$

Soft clamped

$$\phi_1 \sim \left[\frac{e^{-y} + ye^{-y} - 1}{D} - \frac{2(1+y)e^{-y}t^2}{D\alpha^2(1-\nu)} + \cdots\right]\sin x$$

$$\phi_2 \sim \left[\frac{ye^{-y}}{D} - \frac{2ye^{-y}t^2}{D\alpha^2(1-\nu)} + \cdots\right]\cos x$$

$$\omega \sim \left[\frac{1 - e^{-y} - ye^{-y}}{D} + \frac{2(1 - e^{-y} + ye^{-y})t^2}{D\alpha^2(1-\nu)} + \cdots\right]\cos x$$

Table 6 - Exact solutions to the Reissner-Mindlin equations on the upper halfplane with $g = \cos x$.

$\phi_1 =$	$[-1/D - ae^{-y} - bye^{-y} + c\lambda^{-1}t^2e^{-y} - d\beta\lambda^{-1}te^{-\beta y/t}]\sin x$
$\phi_2 =$	$[-ae^{-y} + b(1-y)e^{-y} + c\lambda^{-1}t^2e^{-y} - d\lambda^{-1}t^2e^{-\beta y/t}]\cos x$
$\omega =$	$[1/D + \lambda^{-1}t^2 + ae^{-y} + b(2D\lambda^{-1}t^2 + y)e^{-y} - c\lambda^{-1}t^2e^{-y}]\cos x$
hard clamped	$a = -1/D$ $b = (-\beta\lambda/D - \beta t^2 + t^3)/f$ $c = -\beta\lambda/f$ $d = -\lambda t/f$ $f = \beta\lambda + 2\beta Dt^2 - 2Dt^3$
soft clamped	$a = -1/D$ $b = -(\lambda + Dt^2)/(Df)$ $c = -\lambda/f$ $d = 0$ $f = \lambda + 2Dt^2$
hard simply supported	$a = -1/D$ $b = -1/(2D)$ $c = 0$ $d = 0$
soft simply supported	$a = -1/D$ $b = \{2\beta\lambda\nu t + (t-\beta)^2[\lambda + D(1-\nu)t^2]\}/(2Df)$ $c = -\beta\lambda(1-\nu)t/f$ $d = -\lambda[\lambda + D(1-\nu)t^2]/(Df)$ $f = -\lambda(\beta^2 + t^2) + (1-\nu)t[\beta\lambda - Dt(\beta - t)^2]$
free	$a = \nu[\lambda - D(\beta^2 + t^2)]/(Df)$ $b = \lambda\nu/(Df)$ $c = 0$ $d = 2\lambda\nu/f$ $f = -2\lambda + (1-\nu)[\lambda - D(\beta - t)^2]$

Hard simply supported

$$\phi_1 = \frac{2e^{-y} + ye^{-y} - 2}{2D} \sin x$$

$$\phi_2 = \frac{e^{-y} + ye^{-y}}{2D} \cos x$$

$$\omega \sim \left[\frac{2 - 2e^{-y} - ye^{-y}}{2D} + \frac{2(1 - e^{-y})t^2}{D\alpha^2(1-\nu)} \right] \cos x$$

Soft simply supported

$$\phi_1 \sim \left\{ \left[\frac{2e^{-y} + ye^{-y} - 2}{2D} - \frac{(1-\nu)ye^{-y}t}{2D\alpha} + \cdots \right] + e^{-\alpha y/t} \left[-\frac{t}{D\alpha} + \cdots \right] \right\} \sin x$$

$$\phi_2 \sim \left\{ \left[\frac{e^{-y} + ye^{-y}}{2D} + \frac{(1-\nu)(1-y)e^{-y}t}{2D\alpha} + \cdots \right] + e^{-\alpha y/t} \left[\frac{-t^2}{D\alpha^2} + \cdots \right] \right\} \cos x$$

$$\omega \sim \left[\frac{2 - 2e^{-y} - ye^{-y}}{2D} + \frac{(1-\nu)ye^{-y}t}{2D\alpha} + \cdots \right] \cos x$$

Free

$$\phi_1 \sim \left\{ \left[-\frac{(1-\nu)(3+\nu) + \nu(1+\nu-y+\nu y)e^{-y}}{D(1-\nu)(3+\nu)} - \frac{4\nu(1+\nu-y+\nu y)e^{-y}t}{\alpha D(1-\nu)(3+\nu)^2} + \cdots \right] \right. \\ \left. + e^{-\alpha y/t} \left[\frac{4\nu t}{\alpha D(1-\nu)(3+\nu)} + \cdots \right] \right\} \sin x$$

$$\phi_2 \sim \left\{ \left[-\frac{\nu(2-y+\nu y)e^{-y}}{D(1-\nu)(3+\nu)} - \frac{4\nu(2-y+\nu y)e^{-y}t}{\alpha D(1-\nu)(3+\nu)^2} + \cdots \right] \right. \\ \left. + e^{-\alpha y/t} \left[\frac{4\nu t^2}{\alpha^2 D(1-\nu)(3+\nu)} + \cdots \right] \right\} \cos x$$

$$\omega \sim \left[\frac{(1-\nu)(3+\nu) + \nu(1+\nu-y+\nu y)e^{-y}}{D(1-\nu)(3+\nu)} + \frac{4\nu(1+\nu-y+\nu y)e^{-y}t}{\alpha D(1-\nu)(3+\nu)^2} + \cdots \right] \cos x$$

5. Axisymmetric and Uniaxial Loads. The general construction given in Section 2 simplifies considerably for a circular plate with load independent of θ or for a semi-infinite plate with load independent of x. In this case v turns out to be identical with ω_0, the solution to the corresponding biharmonic problem, m is constant, and q is zero. Since q vanishes, there is no edge effect.

Suppose that the plate domain Ω is either the disc or the halfplane, and that the load g depends only on r or y, respectively, in the two cases. Let ω_0 be the solution of one of the boundary value problems for the biharmonic equation listed in Table 2. Then ω_0 also depends only on r or y. Set

$$\boldsymbol{\phi} = \mathbf{grad}\, \omega_0, \qquad \omega = \omega_0 - \frac{t^2}{\lambda}(D\,\Delta\,\omega_0 - m),$$

where m equals zero for the free plate and equals the constant value of $\Delta\,\omega_0$ on $\partial\Omega$ in the other cases. It is easy to see that $\boldsymbol{\phi}$ and ω satisfy the Reissner-Mindlin equations. If ω_0 satisfies the boundary conditions for a clamped biharmonic plate, then $\boldsymbol{\phi}$ and ω satisfy the boundary conditions for *both* the hard and soft clamped Reissner-Mindlin plates. Indeed, that $\boldsymbol{\phi}\cdot\boldsymbol{s} = \boldsymbol{\phi}\cdot\boldsymbol{n} = 0$ follows immediately from the boundary conditions for ω_0. Combining these with our choice of m also implies that ω vanishes on $\partial\Omega$. Finally,

$$\boldsymbol{s}\cdot \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = D(1-\nu)\left(\frac{\partial^2\omega_0}{\partial s\partial n} - \kappa\frac{\partial\omega_0}{\partial s}\right),$$

which vanishes because ω_0 and $\partial\omega_0/\partial n$ are constant on $\partial\Omega$.

In a similar way, it is easy to see that if ω_0 satisfies the boundary value problem for a simply supported biharmonic plate, then $\boldsymbol{\phi}$ and ω satisfy the boundary conditions for both the hard and soft simply supported Reissner-Mindlin plates, while if ω_0 satisfies the free plate biharmonic problem, then $\boldsymbol{\phi}$ and ω satisfy the free plate Reissner-Mindlin problem.

Note that for these loadings, the rotation vector $\boldsymbol{\phi}$ is independent of t and agrees with that of the biharmonic theory, while the transverse displacement ω differs from ω_0 only by the addition of a term of order t^2. The interior expansion for ω reduces to two terms. Let us also note that the problem of a circular plate with a uniformly distributed load is often used as a benchmark problem for Reissner-Mindlin solvers. We see here that it is a rather atypical problem.

6. CONCLUSIONS. We have described the detailed structure of the dependence of the solution to the Reissner-Mindlin plate model on the plate thickness. This theory, which was derived and justified for a general smoothly bounded plate by the authors in Refs. 2 and 3, is exemplified here through explicit computations in the case of a circular plate and a semi-infinite plate. These cases conform in detail to the predicted behavior, confirming the sharpness of the general theory. In particular, the transverse displacement exhibits no edge effect for any of the boundary value problems considered, while the rotation vector has a weak boundary layer whose strength depends on the particular boundary value problem. The weakest layer occurs for the soft clamped plate, the strongest for the free and soft simply supported plates, and the hard clamped and hard simply supported plates are intermediate. The moment tensor and, especially, the shear force vector, exhibit stronger edge effects than the rotation vector in all cases. In the case of a flat boundary, the edge effect disappears for the soft clamped and hard simply supported plates, but not in the other cases. The case of an axisymmetric or uniaxial loading is quite special: in this case the edge effect disappears entirely for all the boundary value problems considered.

ACKNOWLEDGEMENT. This research was supported by the National Science Foundation through grants DMS-86-01489 (Arnold) and DMS-87-03354 (Falk).

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