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A FORTIN OPERATOR FOR TWO-DIMENSIONAL TAYLOR-HOOD ELEMENTS *, **

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Abstract. A standard method for proving the inf-sup condition implying stability of finite element approximations for the stationary Stokes equations is to construct a Fortin operator. In this paper, we show how this can be done for two-dimensional triangular and rectangular Taylor-Hood methods, which use continuous piecewise polynomial approximations for both velocity and pressure.

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1. Introduction

In this paper, we consider the approximation of the stationary Stokes equations

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = 0 \quad \text{on } \partial\Omega,$$

by elements of Taylor-Hood type where Ω is a polygon in $\mathbb{R}^2$ (when triangular elements are considered) or a union of rectangles in $\mathbb{R}^2$ (when rectangular elements are considered). The construction of the Fortin operator will be given in detail for the case of triangular elements. The extension to rectangular elements is discussed briefly in the final section of the paper. More specifically, for triangular elements and $k = 2, 3$, the velocity vector $\boldsymbol{u}$ is approximated in the space $\boldsymbol{V}_{0,h}^k = \boldsymbol{V}_h^k \cap \boldsymbol{H}_0^1(\Omega)$, where $\boldsymbol{V}_h^k$ is the space of continuous piecewise polynomial vectors of total degree $\leq k$ and the pressure p is approximated in the space Q_h^{k-1} consisting of continuous piecewise polynomials of total degree $\leq k-1$. The stability of these pairs depends on verification of the classical inf-sup condition

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_{0,h}} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q \,\mathrm{d}x}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geq \gamma \|q\|_{L^2(\Omega)} \quad \text{for all } q \in Q_h, \tag{1.1}$$

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where γ is a constant independent of the mesh size h. If (1.1) is satisfied, the general theory of saddle-point problems developed by Babuška and Brezzi then implies the quasi-optimal error estimate

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \le C \inf(\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|p-q\|_{L^2(\Omega)}),$$

where the inf is taken over all $\boldsymbol{v} \in \boldsymbol{V}_{0,h}$ and all $q_h \in Q_h$.

For many stable pairs $(\boldsymbol{V}_{0,h}, Q_h)$ for the Stokes problem, the inf-sup condition is established by constructing a Fortin operator Π mapping $\boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{V}_{0,h}$ and satisfying

$$\int_\Omega \operatorname{div}(\boldsymbol{v}-\Pi\boldsymbol{v})\, q \,\mathrm{d}x, \quad q \in Q_h, \qquad \|\Pi \boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \le C \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}. \tag{1.2}$$

Using the inf-sup condition for the continuous problem, it is then easy to establish the discrete inf-sup condition (1.1), *i.e.*, for $q \in Q_h$,

$$\bar{\gamma}\|q\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q \,\mathrm{d}x}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \le C \sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)} \frac{\int_\Omega \operatorname{div} \Pi\boldsymbol{v}\, q \,\mathrm{d}x}{\|\Pi\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \le C \sup_{\boldsymbol{v}\in \boldsymbol{V}_{0,h}} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q \,\mathrm{d}x}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}},$$

which is the discrete inf-sup condition with $\gamma = \bar{\gamma}/C$.

In the case of Taylor-Hood type elements, this approach has not been used, possibly because it is not so obvious how to construct the Fortin operator, and stability has been established by using a number of other approaches. Of course, once one has a stability analysis, the existence of a Fortin operator follows directly. Our aim in this paper, however, is not to prove the existence of a Fortin operator, but to construct it by using suitable degrees of freedom.

The first error analysis of the $(k = 2)$ Taylor-Hood method was given by Bercovier and Pironneau [1]. Their approach was to show that the Taylor-Hood spaces satisfy a modified form of the inf-sup condition (1.1), namely,

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_{0,h}} \frac{\int_\Omega \operatorname{div} \boldsymbol{v}\, q \,\mathrm{d}x}{\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}} \ge \gamma \|\nabla q\|_{L^2(\Omega)} \quad \text{for all } q \in Q_h.$$

Using this stability result, they obtained optimal order error estimates of the form

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\boldsymbol{L}^2(\Omega)} + h\|\nabla(p-p_h)\|_{L^2(\Omega)} \le Ch^2(\|\boldsymbol{u}\|_{\boldsymbol{H}^3(\Omega)} + \|p\|_{H^2(\Omega)}).$$

Later, Verfürth [9] showed that if the modified stability condition holds, then so does (1.1). Stability for the Taylor-Hood method has also been established using the macro-element technique (see [6] for the case $k = 2$, [8] for the case $k = 3$, and [2,3] for general $k \ge 2$ in both two and three dimensions). For $k \ge 4$, Scott and Vogelius [7] have shown, that except for some exceptional meshes, the combination $\boldsymbol{V}_{0,h}^k$- $\tilde{Q}_h^{k-1}$ (*i.e.*, discontinuous pressures) satisfy the stability condition (1.1). It was then shown in [4] that when $\tilde{Q}_h^{k-1}$ is replaced by Q_h^{k-1}, the stability condition (1.1) is satisfied under a milder restriction on the meshes. The method to be used in this paper is most closely related to the presentation in Brezzi-Fortin [5] for the case $k = 2$ and its generalization to the case $k = 3$ in Brezzi-Falk [4]. Given the result of [7] for $k \ge 4$, these are the most interesting cases.

In the derivation given below, we show how to construct Fortin operators, Π, for the two pairs of Taylor-Hood elements (corresponding to $k = 2$ and 3). For some applications, it will also be convenient to construct Π so that it satisfies the optimal order approximation properties

$$\|\boldsymbol{v}-\Pi\boldsymbol{v}\|_s \le Ch^{r-s}\|\boldsymbol{v}\|_r, \qquad s = 0, 1, \qquad 1 \le r \le k+1.$$

We begin our construction by following the approach described in Brezzi-Fortin [5], which constructs the function $\Pi\boldsymbol{v}$ in two pieces, *i.e.*, $\Pi\boldsymbol{v} = \Pi_1\boldsymbol{v} + \Pi_2\boldsymbol{v}$. For $\Pi_1\boldsymbol{v}$, we choose a Fortin operator associated with the $\boldsymbol{P}_k - P_{k-2}$ Stokes element, *i.e.*, $\Pi_1\boldsymbol{v} \in \boldsymbol{V}^k_{0,h}$ satisfies

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})\tilde{q}\,\mathrm{d}x, \quad \tilde{q} \in \tilde{Q}_h^{k-2}, \qquad \|\Pi_1\boldsymbol{v}\|_1 \le C\|v\|_1,$$

where $\tilde{Q}_h^{k-2}$ denotes the space of discontinuous piecewise polynomials of degree $\le k-2$. We note that $\Pi_1\boldsymbol{v}$ can be constructed to also satisfy the error estimate

$$\|\boldsymbol{v} - \Pi_1\boldsymbol{v}\|_s \le Ch^{r-s}\|\boldsymbol{v}\|_r, \quad s = 0,1, \quad 1 \le r \le k+1.$$

For example, we could define $\Pi_1\boldsymbol{v}$ to satisfy for each triangle T, with vertices a, and edges e,

$$\Pi_1\boldsymbol{v}(a) = (R_h\boldsymbol{v})(a), \qquad \int_e (\boldsymbol{v} - \Pi_1\boldsymbol{v}) \cdot \boldsymbol{p}_{k-2}\,\mathrm{d}s = 0, \qquad \int_T (\boldsymbol{v} - \Pi_1\boldsymbol{v}) \cdot \boldsymbol{p}_{k-3}\,\mathrm{d}x = 0,$$

where $\boldsymbol{p}_i$ denotes vector polynomials of degree $\le i$ and $R_h\boldsymbol{v}$ denotes the Clement interpolant of $\boldsymbol{v}$.

Let $\Pi_0 q$ be a suitable approximation to q in $\tilde{Q}_h^{k-2}$ to be chosen later. To satisfy (1.2), we then need to construct $\Pi_2\boldsymbol{v}$ to satisfy:

$$\int_\Omega \operatorname{div}\Pi_2\boldsymbol{v} q\,\mathrm{d}x = \int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})\,q\,\mathrm{d}x = \int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})(q - \Pi_0 q)\,\mathrm{d}x, \quad q \in Q_h^{k-1}. \tag{1.3}$$

For both $k = 2$ and $k = 3$, the construction of $\Pi_2\boldsymbol{v}$ will rely on the use of appropriate quadrature formulas. When $k = 2$, we will use the midpoint rule formula, exact for polynomials of degree ≤ 2, *i.e.*, for $\phi \in P_2(T)$,

$$\int_T \phi\,\mathrm{d}x = \frac{|T|}{3}\sum_{i<j}\phi(a_{ij}), \tag{1.4}$$

where a_{ij} denotes the midpoint of the edge e_{ij} and $|T|$ the area of T. When $k = 3$, we will use the following quadrature formula (*cf.* [4]), exact for polynomials of degree ≤ 4, *i.e.*, for $\phi \in P_4(T)$,

$$\int_T \phi\,\mathrm{d}x = |T|\Big(\omega_1\phi(a_{123}) + \omega_2\sum_{i=1}^{3}\phi(a_i) + \omega_3\sum_{\substack{i,j=1\\ i\ne j}}^{3}\phi(a_{iij})\Big), \tag{1.5}$$

where $\omega_1 = 9/20$, $\omega_2 = -1/60$, $\omega_3 = 1/10$, a_i denote the vertices of T, a_{123} the centroid, and on each edge $e_{ij} = [a_i, a_j]$, $a_{iij} = (1/2+\theta)a_i + (1/2-\theta)a_j$, where $\theta = 1/\sqrt{12}$.

To make clear the basic idea of the construction of $\Pi_2\boldsymbol{v}$, we will first consider for both $k = 2$ and $k = 3$ a simpler case, when the space $\boldsymbol{H}_0^1(\Omega)$ is replaced by the space $\boldsymbol{H}_n^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$. We then define $\boldsymbol{V}^k_{n,h} = \boldsymbol{V}^k_h \cap \boldsymbol{H}^1_n(\Omega)$. Thus, the remainder of the paper consists of four sections on triangular elements, detailing the construction of the Fortin operator in the spaces, $\boldsymbol{V}^2_{n,h}$, $\boldsymbol{V}^2_{0,h}$, $\boldsymbol{V}^3_{n,h}$, and $\boldsymbol{V}^3_{0,h}$, respectively, and a final section indicating how these ideas can be applied to rectangular elements.

2. Construction of a Fortin operator in $\boldsymbol{V}^2_{n,h}$

To satisfy (1.3), we define $\Pi_2\boldsymbol{v}$ to be zero at all the vertices of $\mathcal{T}_h$ and $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}$ to be zero at the midpoints of all edges of triangles in $\mathcal{T}_h$, where $\mathcal{T}_h$ is a triangulation of the domain Ω by triangles of maximum diameter h. Here $\boldsymbol{n}$ is the unit normal to an edge and $\boldsymbol{t}$ the counterclockwise unit tangent vector along the edge. Thus, it remains to determine $\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}$ at the midpoint of all edges in $\mathcal{T}_h$.

To do so, we consider an arbitrary triangle $T\in\mathcal{T}_h$, and let a_i, $i=1,3$ denote the vertices of T, e_{ij} denote the edge joining the vertices a_i and a_j (with length $|e_{ij}|$ and midpoint a_{ij}), and $\boldsymbol{t}_{ij}$ denote the unit tangent along e_{ij} in the direction from a_i to a_j. Using the definition of $\Pi_2\boldsymbol{v}$, the midpoint quadrature rule (1.4), and the fact that ∇q is constant, we obtain

$$\begin{aligned}\int_T \operatorname{div}\Pi_2\boldsymbol{v}\, q\,\mathrm{d}x &= \int_{\partial T}\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}\, q\,\mathrm{d}s - \int_T \Pi_2\boldsymbol{v}\cdot\nabla q\,\mathrm{d}x\\ &= -\sum_{e_{ij}\in T}\frac{|T|}{3}(\Pi_2\boldsymbol{v}\cdot\nabla q)(a_{ij}) = -\sum_{e_{ij}\in T}\frac{|T|}{3}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{ij}).(\nabla q\cdot\boldsymbol{t}_{ij}),\end{aligned}\tag{2.1}$$

where the vanishing of the boundary integral is a consequence of the fact that on each edge $\boldsymbol{v}\cdot\boldsymbol{n}$ is a quadratic polynomial vanishing at three points and thus is identically zero on each edge. Letting M_I and M_B denote the set of interior and boundary edges in $\mathcal{T}_h$, respectively, and summing over all $T\in\mathcal{T}_h$, we get

$$\begin{aligned}\int_\Omega \operatorname{div}\Pi_2\boldsymbol{v}\, q\,\mathrm{d}x = \sum_T\int_T \operatorname{div}\Pi_2\boldsymbol{v}\, q\,\mathrm{d}x &= -\frac{1}{3}\sum_{e_{ij}\in M_B}|T_{ij}|(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{ij}).(\nabla q\cdot\boldsymbol{t}_{ij})\\ &\quad-\frac{1}{3}\sum_{e_{ij}\in M_I}(|T_{1ij}|+|T_{2ij}|)(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{ij}).(\nabla q\cdot\boldsymbol{t}_{ij}),\end{aligned}\tag{2.2}$$

where for an edge $e_{ij}\in M_I$, T_{1ij} and T_{2ij} denote the two triangles sharing this common edge and for $e_{ij}\in M_B$, T_{ij} is the triangle with e_{ij} as an edge.

We next consider the term $\int_\Omega \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x$ and, abandoning the approach described in [5], show that this can also be written as a summation involving the terms $(\nabla q\cdot\boldsymbol{t}_{ij})(a_{ij})$. We choose $\Pi_0 q$ to be the L^2 projection of q into $\tilde{Q}^0_h$ and observe that using barycentric coordinates on the triangle T,

$$\begin{aligned}3(q-\Pi_0 q) &= \sum_{i=1}^3 q(a_i)[3\lambda_i(x)-1] = q(a_1)[(\lambda_1-\lambda_2)+(\lambda_1-\lambda_3)]\\ &\quad+q(a_2)[(\lambda_2-\lambda_1)+(\lambda_2-\lambda_3)]+q(a_3)[(\lambda_3-\lambda_1)+(\lambda_3-\lambda_2)]\\ &=[q(a_2)-q(a_1)](\lambda_2-\lambda_1)+[q(a_3)-q(a_2)](\lambda_3-\lambda_2)+[q(a_1)-q(a_3)](\lambda_1-\lambda_3)\\ &=(\nabla q\cdot\boldsymbol{t}_{12})|e_{12}|(\lambda_2-\lambda_1)+(\nabla q\cdot\boldsymbol{t}_{23})|e_{23}|(\lambda_3-\lambda_2)+(\nabla q\cdot\boldsymbol{t}_{31})|e_{31}|(\lambda_1-\lambda_3).\end{aligned}$$

Hence,

$$\begin{aligned}\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x &= \frac{|e_{12}|}{3}(\nabla q\cdot\boldsymbol{t}_{12})\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(\lambda_2-\lambda_1)\,\mathrm{d}x\\ &\quad+\frac{|e_{23}|}{3}(\nabla q\cdot\boldsymbol{t}_{23})\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(\lambda_3-\lambda_2)\,\mathrm{d}x+\frac{|e_{31}|}{3}(\nabla q\cdot\boldsymbol{t}_{31})\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(\lambda_1-\lambda_3)\,\mathrm{d}x.\end{aligned}\tag{2.3}$$

Summing over all $T \in \mathcal{T}_h$, we get

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\,\mathrm{d}x = \sum_T \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\,\mathrm{d}x$$
$$= \sum_{e_{ij}\in M_I} \frac{|e_{ij}|}{3}(\nabla q \cdot \boldsymbol{t}_{ij}) \int_{T_{1ij}\cup T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_j - \lambda_i)\,\mathrm{d}x$$
$$+ \sum_{e_{ij}\in M_B} \frac{|e_{ij}|}{3}(\nabla q \cdot \boldsymbol{t}_{ij}) \int_{T_{ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_j - \lambda_i)\,\mathrm{d}x, \tag{2.4}$$

where T_{1ij}, T_{2ij}, and T_{ij} are defined as above.

Hence, from (2.2) and (2.4), it is clear that (1.3) will be satisfied if for $e_{ij} \in M_I$, we choose

$$(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij}) = -\frac{|e_{ij}|}{|T_{1ij}| + |T_{2ij}|} \int_{T_{1ij}\cup T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_j - \lambda_i)\,\mathrm{d}x,$$

and for $e_{ij} \in M_B$, we choose

$$(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij}) = -\frac{|e_{ij}|}{|T_{ij}|} \int_{T_{ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_j - \lambda_i)\,\mathrm{d}x.$$

To estimate the norm of $\Pi_2 \boldsymbol{v}$, we first note that

$$\begin{aligned}
|(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij})| &\le \frac{|e_{ij}|}{|T_{1ij}| + |T_{2ij}|} \|\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|_{T_{1ij}\cup T_{2ij}} \|\lambda_j - \lambda_i\|_{T_{1ij}\cup T_{2ij}} \\
&\le \frac{|e_{ij}|}{|T_{1ij}| + |T_{2ij}|} \|\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|_{T_{1ij}\cup T_{2ij}} \frac{1}{\sqrt{6}} (|T_{1ij}| + |T_{2ij}|)^{1/2} \\
&\le C \|\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|_{T_{1ij}\cup T_{2ij}}.
\end{aligned}$$

An easy scaling argument shows that

$$\|\Pi_2 \boldsymbol{v}\| \le Ch \|\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|, \qquad \|\Pi_2 \boldsymbol{v}\|_1 \le C \|\operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\| \le C \|\boldsymbol{v}\|_1.$$

Combining these results, we see that the operator $\Pi = \Pi_1 + \Pi_2$ satisfies (1.2). Finally, we observe that an estimate for $\|\boldsymbol{v} - \Pi \boldsymbol{v}\|_s$, $s = 0, 1$, follows easily from the previous results, *i.e.*, for $1 \le r \le 3$,

$$\|\boldsymbol{v} - \Pi \boldsymbol{v}\|_s \le \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_s + \|\Pi_2 \boldsymbol{v}\|_s \le \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_s + Ch^{1-s} \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_1 \le Ch^{r-s} \|\boldsymbol{v}\|_r.$$

3. Construction of a Fortin operator in $\boldsymbol{V}^2_{0,h}$

To construct a Fortin operator for functions that vanish on $\partial\Omega$, we will need to distinguish among several types of triangles: those that have no edges lying on $\partial\Omega$ which we designate $\mathcal{T}_h^0$, those that have one edge lying on $\partial\Omega$ which we designate $\mathcal{T}_h^1$, and those that have two edges lying on $\partial\Omega$ which we designate $\mathcal{T}_h^2$. Thus $\mathcal{T}_h = \mathcal{T}_h^0 \cup \mathcal{T}_h^1 \cup \mathcal{T}_h^2$. The issue in this case is that since $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t} = 0$ on $\partial\Omega$, we no longer have the degrees of freedom $(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij})$ at our disposal in equation (2.2) to deal with the terms $(\nabla q \cdot \boldsymbol{t}_{ij})(a_{ij})$ in (2.4), when a_{ij} is the midpoint of a boundary edge. The remedy, following ideas from other proofs of stability of the Taylor-Hood element, is to eliminate the terms $(\nabla q \cdot \boldsymbol{t}_{ij})(a_{ij})$ (when $e_{ij} \in M_B$) from equation (2.4) and to introduce the additional degrees of freedom $(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{ij})(a_{ij})$ at the midpoints of the edges not lying on $\partial\Omega$ of triangles in $\mathcal{T}_h^2$. It will be convenient for the construction to also define $\mathcal{T}_h^3$ to be the set of triangles sharing a common edge with triangles in $\mathcal{T}_h^2$, and denote by M_N the set of edges common to triangles in $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$. We shall assume that $\mathcal{T}_h^2 \cap \mathcal{T}_h^3$ is empty (so the mesh must consist of more than two triangles).

In this more general case, we will again choose $\Pi_2 \boldsymbol{v}$ to be zero at all the vertices of $\mathcal{T}_h$. Now $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}$ will be zero at the midpoints of boundary edges and $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}$ will be zero at the midpoints of all edges of triangles in $\mathcal{T}_h$ with the exception of the edges in M_N. Thus, we need to determine $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}$ at the midpoints of all edges not lying on $\partial\Omega$ and $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}$ at the midpoints of edges in M_N. For triangles in $\mathcal{T}_h^0 \cup \mathcal{T}_h^1 - \mathcal{T}_h^3$, we can again use formula (2.1), noting that for triangles in $\mathcal{T}_h^1$, the term $(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t})(a_{ij})$ will be zero if a_{ij} is a midpoint of a boundary edge. For triangles in $\mathcal{T}_h^2 \cup \mathcal{T}_h^3$, we need to use a modified version of (2.1). Let $T_3 \in \mathcal{T}_h^3$ have edges e_{ij}, with midpoints a_{ij}, unit tangents $\boldsymbol{t}_{ij}$, and outward unit normals $\boldsymbol{n}_{ij}$. Suppose first that T_3 has edges in common with only one triangle $T_2 \in \mathcal{T}_h^2$, and denote that common edge by e_{23}. Now, since $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}$ vanishes along all the edges of $T_2 \cup T_3$ except e_{12}, we get by the midpoint quadrature rule:

$$\begin{aligned}\int_{T_2} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x + \int_{T_3} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x &= -\int_{T_2} \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x - \int_{T_3} \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x \\ &= -\frac{|T_2| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{23})(a_{23})(\nabla q \cdot \boldsymbol{t}_{23}) - \frac{|T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{13})(a_{13})(\nabla q \cdot \boldsymbol{t}_{13}) \\ &\quad - \frac{|T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{12})(a_{12})(\nabla q \cdot \boldsymbol{t}_{12}) - \frac{|T_2| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{23})(a_{23})(\nabla q \cdot \boldsymbol{n}_{23}).\end{aligned} \tag{3.1}$$

Summing over all $T \in \mathcal{T}_h$, we get

$$\begin{aligned}\int_{\Omega} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x = &-\frac{1}{3} \sum_{e_{ij} \in M_I} (|T_{1ij}| + |T_{2ij}|)(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij}).(\nabla q \cdot \boldsymbol{t}_{ij}) \\ &- \frac{1}{3} \sum_{e_{ij} \in M_N} (|T_{1ij}| + |T_{2ij}|)(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{ij})(a_{ij}).(\nabla q \cdot \boldsymbol{n}_{ij}).\end{aligned} \tag{3.2}$$

We next consider the case when a triangle in $\mathcal{T}_h^3$ could have edges in common with two triangles in $\mathcal{T}_h^2$. This would include the case of a mesh with three triangles. If $T_1 \in \mathcal{T}_h^2$ and $T_3 \in \mathcal{T}_h^3$ share the common edge e_{13} and $T_2 \in \mathcal{T}_h^2$ and $T_3 \in \mathcal{T}_h^3$ share the common edge e_{23}, then a simple modification of (3.1) gives the following:

$$\begin{aligned}&\int_{T_1} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x + \int_{T_2} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x + \int_{T_3} \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x = -\int_{T_1} \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x - \int_{T_2} \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x - \int_{T_3} \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x \\ &\qquad = -\frac{|T_2| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{23})(a_{23})(\nabla q \cdot \boldsymbol{t}_{23}) - \frac{|T_1| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{13})(a_{13})(\nabla q \cdot \boldsymbol{t}_{13}) \\ &- \frac{|T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{12})(a_{12})(\nabla q \cdot \boldsymbol{t}_{12}) - \frac{|T_2| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{23})(a_{23})(\nabla q \cdot \boldsymbol{n}_{23}) - \frac{|T_1| + |T_3|}{3} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{13})(a_{13})(\nabla q \cdot \boldsymbol{n}_{13}).\end{aligned}$$

Formula (3.2) then remains unchanged.

We now turn to the modification of formula (2.3), beginning with triangles in $\mathcal{T}_h^1$, where we denote by e_{23} the edge lying on $\partial\Omega$ and a_{23} the midpoint of that edge. Using the facts that ∇q is constant on each triangle and $|e_{12}|\boldsymbol{t}_{12} + |e_{23}|\boldsymbol{t}_{23} + |e_{31}|\boldsymbol{t}_{31} = 0$, we may rewrite (2.3) as:

$$\begin{aligned}\int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q) \,\mathrm{d}x &= \frac{|e_{12}|}{3} (\nabla q \cdot \boldsymbol{t}_{12}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(2\lambda_2 - \lambda_1 - \lambda_3) \,\mathrm{d}x \\ &\quad + \frac{|e_{31}|}{3} (\nabla q \cdot \boldsymbol{t}_{31}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_1 + \lambda_2 - 2\lambda_3) \,\mathrm{d}x \\ &= \frac{|e_{12}|}{3} (\nabla q \cdot \boldsymbol{t}_{12}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(3\lambda_2 - 1) \,\mathrm{d}x + \frac{|e_{31}|}{3} (\nabla q \cdot \boldsymbol{t}_{13}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(3\lambda_3 - 1) \,\mathrm{d}x.\end{aligned}$$

For triangles in $\mathcal{T}_h^2$, denote by e_{23} the edge that does not lie on $\partial\Omega$ and a_{23} the midpoint of that edge. Again using the fact that ∇q is constant on each triangle, we get from (2.3)

$$\int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\, \mathrm{d}x = \frac{|e_{23}|}{3} \nabla q \cdot \boldsymbol{t}_{23} \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_3 - \lambda_2)\, \mathrm{d}x$$
$$+ \frac{|e_{12}|}{3} \nabla q \cdot [(\boldsymbol{t}_{12} \cdot \boldsymbol{t}_{23})\boldsymbol{t}_{23} + (\boldsymbol{t}_{12} \cdot \boldsymbol{n}_{23})\boldsymbol{n}_{23}] \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_2 - \lambda_1)\, \mathrm{d}x$$
$$+ \frac{|e_{31}|}{3} \nabla q \cdot [(\boldsymbol{t}_{31} \cdot \boldsymbol{t}_{23})\boldsymbol{t}_{23} + (\boldsymbol{t}_{31} \cdot \boldsymbol{n}_{23})\boldsymbol{n}_{23}] \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_1 - \lambda_3)\, \mathrm{d}x.$$

Combining terms, we may write this in the form

$$\int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\, \mathrm{d}x = \frac{1}{3}(\nabla q \cdot \boldsymbol{t}_{23}) \Big[|e_{23}| \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_3 - \lambda_2)\, \mathrm{d}x$$
$$+ |e_{12}|(\boldsymbol{t}_{12} \cdot \boldsymbol{t}_{23}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_2 - \lambda_1)\, \mathrm{d}x + |e_{31}|(\boldsymbol{t}_{31} \cdot \boldsymbol{t}_{23}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_1 - \lambda_3)\, \mathrm{d}x \Big]$$
$$+ \frac{1}{3}(\nabla q \cdot \boldsymbol{n}_{23}) \Big[|e_{12}|(\boldsymbol{t}_{12} \cdot \boldsymbol{n}_{23}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_2 - \lambda_1)\, \mathrm{d}x$$
$$+ |e_{31}|(\boldsymbol{t}_{31} \cdot \boldsymbol{n}_{23}) \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(\lambda_1 - \lambda_3)\, \mathrm{d}x \Big].$$

Summing over all $T \in \mathcal{T}_h$, we now obtain

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\, \mathrm{d}x = \sum_T \int_T \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})(q - \Pi_0 q)\, \mathrm{d}x$$
$$= \sum_{e_{ij} \in M_I} \frac{1}{3}(\nabla q \cdot \boldsymbol{t}_{ij}) \left(\int_{T_{1ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \phi_{1ij}\, \mathrm{d}x + \int_{T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \phi_{2ij}\, \mathrm{d}x \right),$$
$$+ \sum_{e_{ij} \in M_N} \frac{1}{3}(\nabla q \cdot \boldsymbol{n}_{ij}) \left(\int_{T_{1ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \psi_{1ij}\, \mathrm{d}x + \int_{T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \psi_{2ij}\, \mathrm{d}x \right),$$

where T_{1ij} and T_{2ij} are again the two triangles sharing the common edge e_{ij}, λ_i now denotes the continuous piecewise linear function that is equal to one at vertex a_i and zero at the other vertices of $T_{1ij} \cup T_{2ij}$, and for $m = 1, 2$,

$$\phi_{mij} = |e_{ij}|(\lambda_j - \lambda_i), \qquad \psi_{mij} = 0, \qquad T_{mij} \in \mathcal{T}_h^0,$$
$$\phi_{mij} = |e_{ij}|(3\lambda_j - 1), \qquad \psi_{mij} = 0, \qquad T_{mij} \in \mathcal{T}_h^1,$$

and for $T_{mij} \in \mathcal{T}_h^2$,

$$\phi_{mij} = |e_{ij}|(\lambda_j - \lambda_i) + |e_{ki}|(\boldsymbol{t}_{ki} \cdot \boldsymbol{t}_{ij})(\lambda_i - \lambda_k) + |e_{jk}|(\boldsymbol{t}_{jk} \cdot \boldsymbol{t}_{ij})(\lambda_k - \lambda_j),$$
$$\psi_{mij} = |e_{ki}|(\boldsymbol{t}_{ki} \cdot \boldsymbol{n}_{ij})(\lambda_i - \lambda_k) + |e_{jk}|(\boldsymbol{t}_{jk} \cdot \boldsymbol{n}_{ij})(\lambda_k - \lambda_j),$$

where e_{ki} and e_{kj} are the other two edges of T_{mij}.

It is then immediate that we will satisfy (1.2) by choosing

$$(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij}) = -\frac{1}{|T_{1ij}| + |T_{2ij}|} \left(\int_{T_{1ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \phi_{1ij} \,\mathrm{d}x + \int_{T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \phi_{2ij} \,\mathrm{d}x \right),$$
$$(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{ij})(a_{ij}) = -\frac{1}{|T_{1ij}| + |T_{2ij}|} \left(\int_{T_{1ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \psi_{1ij} \,\mathrm{d}x + \int_{T_{2ij}} \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v}) \psi_{2ij} \,\mathrm{d}x \right),$$

for a_{ij} the midpoint of an edge in M_I and M_N, respectively.

An estimate for the norm of $\Pi_2 \boldsymbol{v}$ may be obtained by a slight modification of the procedure used in the previous section, *i.e.*, we have for $a_{ij} \in M_I$,

$$|(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{ij})| \le C \| \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|_{T_{1ij} \cup T_{2ij}},$$

and for $a_{ij} \in M_N$,

$$|(\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}_{ij})(a_{ij})| \le C \| \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|_{T_{1ij} \cup T_{2ij}}.$$

Again, an easy scaling argument shows that

$$\|\Pi_2 \boldsymbol{v}\| \le Ch \| \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\|, \qquad \|\Pi_2 \boldsymbol{v}\|_1 \le C \| \operatorname{div}(\boldsymbol{v} - \Pi_1 \boldsymbol{v})\| \le C \|\boldsymbol{v}\|_1.$$

Combining these results, we see that the operator $\Pi = \Pi_1 + \Pi_2$ satisfies (1.2). Finally, we observe that an estimate for $\|\boldsymbol{v} - \Pi \boldsymbol{v}\|_s$, $s = 0, 1$, follows easily from the previous results, *i.e.*, for $1 \le r \le 3$,

$$\|\boldsymbol{v} - \Pi \boldsymbol{v}\|_s \le \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_s + \|\Pi_2 \boldsymbol{v}\|_s \le \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_s + Ch^{1-s} \|\boldsymbol{v} - \Pi_1 \boldsymbol{v}\|_1 \le Ch^{r-s} \|\boldsymbol{v}\|_r.$$

4. Construction of a Fortin operator in $\boldsymbol{V}_{n,h}^3$

To satisfy (1.3), we define $\Pi_2 \boldsymbol{v}$ to be zero at all the vertices of $\mathcal{T}_h$, $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}$ to be zero at the points a_{iij} (defined in the quadrature formula (1.5)) on the edges of $\mathcal{T}_h$ and $\Pi_2 \boldsymbol{v}$ to be zero at the centroid of each triangle in $\mathcal{T}_h$. Thus, it remains to determine $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}$ at the points a_{iij} on each edge e_{ij} of $\mathcal{T}_h$.

Since $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n} \in P_3$ on each edge, and vanishes at four points on each edge, $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on each edge. Hence, using the above definitions, and the quadrature formula (1.5), we get

$$\begin{aligned} \int_T \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x = -\int_T \Pi_2 \boldsymbol{v} \cdot \nabla q \,\mathrm{d}x = -|T| \Big(\omega_1 (\Pi_2 \boldsymbol{v} \cdot \nabla q)(a_{123}) + \omega_2 \sum_{i=1}^3 (\Pi_2 \boldsymbol{v} \cdot \nabla q)(a_i) \\ + \omega_3 \sum_{\substack{i,j=1 \\ i \ne j}}^3 (\Pi_2 \boldsymbol{v} \cdot \nabla q)(a_{iij}) \Big) = -|T| \, \omega_3 \sum_{\substack{i,j=1 \\ i \ne j}}^3 (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a_{iij}) (\nabla q \cdot \boldsymbol{t}_{ij})(a_{iij}). \end{aligned} \tag{4.1}$$

Summing over all $T \in \mathcal{T}_h$, and letting $A_{ij} = a_{iij} \cup a_{jji}$, $i \ne j$, we get

$$\begin{aligned} \int_\Omega \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x = \sum_T \int_T \operatorname{div} \Pi_2 \boldsymbol{v} q \,\mathrm{d}x = -\omega_3 \sum_{e_{ij} \in M_B} |T_{ij}| \sum_{a \in A_{ij}} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a) (\nabla q \cdot \boldsymbol{t}_{ij})(a) \\ - \omega_3 \sum_{e_{ij} \in M_I} (|T_{1ij}| + |T_{2ij}|) \sum_{a \in A_{ij}} (\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}_{ij})(a) (\nabla q \cdot \boldsymbol{t}_{ij})(a). \end{aligned} \tag{4.2}$$

We next consider the term $\int_\Omega \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x$ and show that this can also be written as a summation involving the terms $(\nabla q\cdot \boldsymbol{t}_{ij})(a_{iij})$. In this case, we let $\Pi_0 q\in \tilde{Q}_h^1$ denote the piecewise linear function that interpolates q at the vertices of T. Since $q-\Pi_0 q=0$ at the vertices of T, and in the triangle T, $(\Pi_0 q)(a_{ik}) = [(\Pi_0 q)(a_i)+(\Pi_0 q)(a_k)]/2$, we get for points in T, that

$$\begin{aligned} q-\Pi_0 q &= 2\lambda_1\lambda_2[2q(a_{12})-q(a_1)-q(a_2)]+2\lambda_2\lambda_3[2q(a_{23})-q(a_2)-q(a_3)]+2\lambda_1\lambda_3[2q(a_{13})-q(a_1)-q(a_3)]\\ &= -2[\lambda_1\lambda_2\Delta^2 q(a_{12})+\lambda_2\lambda_3\Delta^2 q(a_{23})+\lambda_1\lambda_3\Delta^2 q(a_{13})] = -2\sum_{1\le i<j\le 3}\Delta^2 q(a_{ij})\lambda_i\lambda_j \end{aligned}$$

where $\Delta^2 q(a_{ij}) = q(a_i)+q(a_j)-2q(a_{ij})$. Hence,

$$\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x = -2\sum_{1\le i<j\le 3}\Delta^2 q(a_{ij})\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\lambda_i\lambda_j\,\mathrm{d}x. \tag{4.3}$$

Now since q_{tt} is constant on each edge, we easily obtain from simple Taylor expansions that

$$\Delta^2 q(a_{ij}) = |e_{ij}|^2 q_{tt}/4 = \frac{|e_{ij}|}{8\theta}[\nabla q\cdot \boldsymbol{t}_{ij}(a_{jji})-\nabla q\cdot \boldsymbol{t}_{ij}(a_{iij})]. \tag{4.4}$$

Inserting this result and summing over all $T\in\mathcal{T}_h$, we get

$$\begin{aligned} \int_\Omega \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x &= \sum_T\int_T \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x\\ &= -\sum_{e_{ij}\in M_I}\frac{|e_{ij}|}{4\theta}[\nabla q\cdot \boldsymbol{t}_{ij}(a_{iij})-\nabla q\cdot \boldsymbol{t}_{ij}(a_{jji})]\int_{T_{1ij}\cup T_{2ij}}\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\lambda_i\lambda_j\,\mathrm{d}x\\ &\quad -\sum_{e_{ij}\in M_B}\frac{|e_{ij}|}{4\theta}[\nabla q\cdot \boldsymbol{t}_{ij}(a_{iij})-\nabla q\cdot \boldsymbol{t}_{ij}(a_{jji})]\int_{T_{ij}}\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\lambda_i\lambda_j\,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Hence, from (4.2) and (4.5), it is clear that (1.3) will be satisfied if for each $e_{ij}\in M_I$, we choose

$$(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{jji}) = -(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{iij}) = -\frac{\int_{T_{1ij}\cup T_{2ij}}\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\lambda_i\lambda_j\,\mathrm{d}x}{4\theta\omega_3(|T_{1ij}|+|T_{2ij}|)},$$

and for $e_{ij}\in M_B$, we choose

$$(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{jji}) = -(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a_{iij}) = -\frac{\int_{T_{ij}}\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\lambda_i\lambda_j\,\mathrm{d}x}{4\theta\omega_3(|T_{ij}|}\cdot$$

Applying estimates similar to those used for the space $\boldsymbol{V}^2_{n,h}$, we obtain

$$\begin{gathered} \|\Pi_2\boldsymbol{v}\|\le Ch\|\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\|,\qquad \|\Pi_2\boldsymbol{v}\|_1\le C\|\operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})\|\le C\|\boldsymbol{v}\|_1,\\ \|\boldsymbol{v}-\Pi\boldsymbol{v}\|_s\le Ch^{r-s}\|\boldsymbol{v}\|_r,\quad s=0,1,\quad 1\le r\le 4. \end{gathered} \tag{4.6}$$

5. Construction of a Fortin operator in $\boldsymbol{V}^3_{0,h}$

As in the case of $\boldsymbol{V}^2_{0,h}$, we need to consider several types of triangles and modify the definition of $\Pi_2\boldsymbol{v}$, since we no longer have the degrees of freedom $\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij}(a_{iij})$ for edges e_{ij} lying on $\partial\Omega$. Following the procedure for the case $\boldsymbol{V}^2_{0,h}$, we eliminate the terms $(\nabla q\cdot\boldsymbol{t}_{ij})(a_{iij})$ for e_{ij} lying on $\partial\Omega$ from (4.5) and introduce the additional degrees of freedom $(\Pi_2 \boldsymbol{v}\cdot\boldsymbol{n}_{ij})(a_{iij})$ on the edges not lying on $\partial\Omega$ of triangles in $\mathcal{T}_h^2$ and $(\Pi_2\boldsymbol{v})(a_{123})$ for triangles in $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$.

In this more general case, we will again choose $\Pi_2\boldsymbol{v}$ to be zero at all the vertices of $\mathcal{T}_h$. Now $\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}$ will be zero at the points a_{iij} of boundary edges and we choose $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}$ to be zero at the points a_{iij} of all edges of triangles in $\mathcal{T}_h$ with the exception of the edges that are common to triangles in $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$. Finally, we choose $\Pi_2\boldsymbol{v}$ to be zero at the centroids of all triangles in $\mathcal{T}_h^0$. Thus, we need to determine $\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}$ at the points a_{iij} of all edges not lying on $\partial\Omega$, $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}$ at the points a_{iij} of edges common to triangles in $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$, and $\Pi_2\boldsymbol{v}$ at the centroids a_{123} of all triangles in $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$. For triangles in $\mathcal{T}_h^0 - \mathcal{T}_h^3$, we can again use formula (4.1), while for triangles in $\mathcal{T}_h^1 - \mathcal{T}_h^3$, we have from (4.1),

$$\int_T \operatorname{div}\Pi_2\boldsymbol{v}q\,\mathrm{d}x = -|T|\omega_1(\Pi_2\boldsymbol{v}\cdot\nabla q)(a_{123}) - |T|\omega_3 \sum_{e_{ij}\notin\partial\Omega}\sum_{a\in A_{ij}} (\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a)(\nabla q\cdot\boldsymbol{t}_{ij})(a).$$

For triangles in $\mathcal{T}_h^2 \cup \mathcal{T}_h^3$, we need to use a modified version of (4.1). Let $T_3 \in \mathcal{T}_h^3$ have edges e_{ij}. Suppose first that T_3 has edges in common with only one triangle $T_2 \in \mathcal{T}_h^2$, and denote that common edge by e_{23}. Let a_{123}^i denote the centroid of T_i. Now, since $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}$ vanishes along all the edges of $T_2 \cup T_3$ except e_{23}, we get by the quadrature formula (1.5) that

$$\begin{aligned}\int_{T_2}\operatorname{div}\Pi_2\boldsymbol{v}q\,\mathrm{d}x + \int_{T_3}\operatorname{div}\Pi_2\boldsymbol{v}q\,\mathrm{d}x &= -\int_{T_2}\Pi_2\boldsymbol{v}\cdot\nabla q\,\mathrm{d}x - \int_{T_3}\Pi_2\boldsymbol{v}\cdot\nabla q\,\mathrm{d}x\\ &= -|T_2|\omega_1(\Pi_2\boldsymbol{v}\cdot\nabla q)(a_{123}^2) - \omega_3(|T_2|+|T_3|)\sum_{a\in A_{23}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{23})(a)(\nabla q\cdot\boldsymbol{t}_{23})(a)\\ &\quad - \omega_3|T_3|\sum_{a\in A_{13}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{13})(a)(\nabla q\cdot\boldsymbol{t}_{13})(a) - \omega_3|T_3|\sum_{a\in A_{12}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{12})(a)(\nabla q\cdot\boldsymbol{t}_{12})(a)\\ &\quad - \omega_3(|T_2|+|T_3|)\sum_{a\in A_{23}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}_{23})(a)(\nabla q\cdot\boldsymbol{n}_{23})(a) - |T_3|\omega_1(\Pi_2\boldsymbol{v}\cdot\nabla q)(a_{123}^3),\end{aligned}$$

where the last term is not needed if $T_3 \in \mathcal{T}_h^0$. Summing over all $T \in \mathcal{T}_h$, we get

$$\begin{aligned}\int_\Omega \operatorname{div}\Pi_2\boldsymbol{v}q\,\mathrm{d}x = &-\omega_3\sum_{e_{ij}\in M_I}(|T_{1ij}|+|T_{2ij}|)\sum_{a\in A_{ij}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}_{ij})(a)(\nabla q\cdot\boldsymbol{t}_{ij})(a)\\ &-\omega_3\sum_{e_{ij}\in M_N}(|T_{1ij}|+|T_{2ij}|)\sum_{a\in A_{ij}}(\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}_{ij})(a)(\nabla q\cdot\boldsymbol{n}_{ij})(a) - \omega_1\sum_{T\in\mathcal{T}_h^1\cup\mathcal{T}_h^2}|T|(\Pi_2\boldsymbol{v}\cdot\nabla q)(a_{123}).\end{aligned}$$

We note that if a triangle in $\mathcal{T}_h^3$ has edges in common with two triangles in $\mathcal{T}_h^2$, then using an argument analogous to the one used for $\boldsymbol{V}_{0,h}^2$, the above formula will still be valid.

We next consider the modification of formulas (4.3) and (4.5) required for triangles in $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$. We begin by observing that since ∇q is a linear function, ∇q is completely determined by its values at the two points a_{iij}, a_{jji} on any edge e_{ij}, together with its value at a_{123}. It is easy to check that

$$\begin{aligned}\nabla q = &\frac{1}{4\theta}[\lambda_i(2\theta+1)+\lambda_j(2\theta-1)-4\theta\lambda_k]\nabla q(a_{iij})\\ &+\frac{1}{4\theta}[\lambda_i(2\theta-1)+\lambda_j(2\theta+1)-4\theta\lambda_k]\nabla q(a_{jji}) + 3\lambda_k\nabla q(a_{123}),\end{aligned} \tag{5.1}$$

where λ_k is the barycentric coordinate that is equal to zero on the edge e_{ij}.

Now let $T \in \mathcal{T}_h^1$, and denote by e_{23} the edge lying on $\partial\Omega$. Using the formula

$$|e_{12}|\boldsymbol{t}_{12} + |e_{23}|\boldsymbol{t}_{23} + |e_{31}|\boldsymbol{t}_{31} = 0,$$

and the above result, we obtain

$$\begin{aligned}
|e_{23}|\nabla q \cdot \boldsymbol{t}_{23} &= -|e_{12}|\nabla q \cdot \boldsymbol{t}_{12} - |e_{31}|\nabla q \cdot \boldsymbol{t}_{31} \\
&= -|e_{12}|\bigg\{ \frac{1}{4\theta}[\lambda_1(2\theta+1) + \lambda_2(2\theta-1) - 4\theta\lambda_3]\nabla q(a_{112}) \cdot \boldsymbol{t}_{12} \\
&\quad + \frac{1}{4\theta}[\lambda_1(2\theta-1) + \lambda_2(2\theta+1) - 4\theta\lambda_3]\nabla q(a_{221}) \cdot \boldsymbol{t}_{12} + 3\lambda_3 \nabla q(a_{123}) \cdot \boldsymbol{t}_{12} \bigg\} \\
&\quad - |e_{31}|\bigg\{ \frac{1}{4\theta}[\lambda_3(2\theta+1) + \lambda_1(2\theta-1) - 4\theta\lambda_2]\nabla q(a_{331}) \cdot \boldsymbol{t}_{31} \\
&\quad + \frac{1}{4\theta}[\lambda_3(2\theta-1) + \lambda_1(2\theta+1) - 4\theta\lambda_2]\nabla q(a_{113}) \cdot \boldsymbol{t}_{31} + 3\lambda_2 \nabla q(a_{123}) \cdot \boldsymbol{t}_{31} \bigg\}.
\end{aligned}$$

Since $\lambda_1(a_{223}) - \lambda_1(a_{332}) = 0$, $\lambda_2(a_{223}) - \lambda_2(a_{332}) = 2\theta$, $\lambda_3(a_{223}) - \lambda_3(a_{332}) = -2\theta$, and $1/(4\theta) = 3\theta$, we have

$$\begin{aligned}
|e_{23}|[\nabla q \cdot \boldsymbol{t}_{23}(a_{223}) - \nabla q \cdot \boldsymbol{t}_{23}(a_{332})] &= |e_{23}|(\nabla q \cdot \boldsymbol{t}_{23})(a_{223} - a_{332}) \\
&= |e_{12}|(\nabla q(a_{112}) \cdot \boldsymbol{t}_{12})3\theta(2\theta-1) - |e_{12}|(\nabla q(a_{221}) \cdot \boldsymbol{t}_{12})3\theta(2\theta+1) \\
&\quad + |e_{31}|(\nabla q(a_{331}) \cdot \boldsymbol{t}_{31})3\theta(2\theta+1) - |e_{31}|(\nabla q(a_{113}) \cdot \boldsymbol{t}_{31})3\theta(2\theta-1) \\
&\quad + |e_{12}|(\nabla q(a_{123}) \cdot \boldsymbol{t}_{12})6\theta - |e_{13}|(\nabla q(a_{123}) \cdot \boldsymbol{t}_{31})6\theta.
\end{aligned}$$

Using this formula, $|e_{23}|[\nabla q \cdot \boldsymbol{t}_{23}(a_{223}) - \nabla q \cdot \boldsymbol{t}_{23}(a_{332})]$ may be eliminated from formula (4.5) on edges $e_{ij} \in M_B$ for triangles in $\mathcal{T}_h^1$ by introducing additional terms containing one of the following expressions:

$$\nabla q \cdot \boldsymbol{t}_{12}(a_{112}), \quad \nabla q \cdot \boldsymbol{t}_{12}(a_{221}), \quad \nabla q \cdot \boldsymbol{t}_{31}(a_{113}), \quad \nabla q \cdot \boldsymbol{t}_{31}(a_{331}), \quad \nabla q(a_{123}).$$

We next consider a triangle $T \in \mathcal{T}_h^2$. In this case, let e_{23} be the edge not lying on $\partial\Omega$. We then want to write the quantities

$$\nabla q \cdot \boldsymbol{t}_{12}(a_{112}), \quad \nabla q \cdot \boldsymbol{t}_{12}(a_{221}), \quad \nabla q \cdot \boldsymbol{t}_{31}(a_{331}), \quad \nabla q \cdot \boldsymbol{t}_{31}(a_{113})$$

in terms of the quantities

$$\nabla q(a_{223}), \quad \nabla q(a_{332}), \quad \nabla q(a_{123}).$$

This follows directly from (5.1) by choosing $i = 2$, $j = 3$, and $k = 1$. Hence, these quantities can also be eliminated from (4.5) on edges $e_{ij} \in M_B$ for triangles in $\mathcal{T}_h^2$.

Inserting these results, we can then satisfy (1.3) by obvious choices, analogous to those in the previous section, of the quantities $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{t}$ at the points a_{iij} of all edges not lying on $\partial\Omega$, $\Pi_2 \boldsymbol{v} \cdot \boldsymbol{n}$ at the points a_{iij} of edges common to triangles in $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$, and $\Pi_2 \boldsymbol{v}$ at the centroids a_{123} of all triangles in $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$. As in the discussion for $\boldsymbol{V}_{0,h}^2$, these changes do not affect the estimates given in (4.6).

6. Construction of a Fortin operator on rectangles

In this final section, we show how the ideas previously developed can be extended to rectangles. To keep the treatment brief, we consider here only approximations to $\boldsymbol{u}$ in the space $\boldsymbol{V}_{n,h}^k = \boldsymbol{V}_h^k \cap \boldsymbol{H}_n^1(\Omega)$, (rather than $\boldsymbol{V}_{0,h}^k = \boldsymbol{V}_h^k \cap \boldsymbol{H}_0^1(\Omega)$), since as we have seen in the previous sections, the extension to zero boundary conditions is quite technical. Rather than change notation, we now use $\boldsymbol{V}_h^k$ to denote the space of continuous piecewise polynomial vectors of degree $\leq k$ in each variable and Q_h^{k-1}, the approximating space for p, to denote the space of continuous piecewise polynomials of degree $\le k-1$ in each variable. We again set $\Pi\boldsymbol{v} = \Pi_1\boldsymbol{v} + \Pi_2\boldsymbol{v}$, where now $\Pi_1\boldsymbol{v}$ is a Fortin operator associated with the $\boldsymbol{Q}_k - \tilde{Q}_{k-2}$ Stokes element, *i.e.*, $\Pi_1\boldsymbol{v} \in \boldsymbol{V}_{0,h}^k$ satisfies

$$\int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})\tilde{q}\,\mathrm{d}x, \quad \tilde{q} \in \tilde{Q}_h^{k-2}, \qquad \|\Pi_1\boldsymbol{v}\|_1 \le C\|v\|_1,$$

where $\tilde{Q}_h^{k-2}$ now denotes the space of discontinuous piecewise polynomials of degree $\le k-2$ in each variable. We note that $\Pi_1\boldsymbol{v}$ can be constructed to also satisfy the error estimate

$$\|\boldsymbol{v} - \Pi_1\boldsymbol{v}\|_s \le Ch^{r-s}\|\boldsymbol{v}\|_r, \quad s = 0, 1, \quad 1 \le r \le k+1.$$

For example, we could define $\Pi_1\boldsymbol{v}$ to satisfy for each rectangle K, with vertices a, and edges e,

$$\Pi_1\boldsymbol{v}(a) = (R_h\boldsymbol{v})(a), \qquad \int_e (\boldsymbol{v} - \Pi_1\boldsymbol{v})\cdot\boldsymbol{p}_{k-2}\,\mathrm{d}s = 0, \qquad \int_K (\boldsymbol{v} - \Pi_1\boldsymbol{v})\cdot\boldsymbol{p}_{k-2}\,\mathrm{d}x = 0,$$

where $\boldsymbol{p}_i$ denotes vector polynomials of degree $\le i$ in each variable and $R_h\boldsymbol{v}$ denotes the Clement interpolant of $\boldsymbol{v}$.

Let $\Pi_0 q$ be a suitable approximation to q in $\tilde{Q}_h^{k-2}$ to be chosen later. To satisfy (1.2), we again need to construct $\Pi_2\boldsymbol{v}$ to satisfy:

$$\int_\Omega \operatorname{div}\Pi_2\boldsymbol{v}q\,\mathrm{d}x = \int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})\,q\,\mathrm{d}x = \int_\Omega \operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v})(q - \Pi_0 q)\,\mathrm{d}x, \quad q \in Q_h^{k-1}. \tag{6.1}$$

As in the case of triangular elements, we shall make use of a suitable quadrature formula. In this case, it is the two-dimensional Gauss-Lobatto formula, exact for polynomials of degree $\le 2k-1$ is each variable, and given on the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_i]$ by:

$$\begin{aligned}\frac{1}{|R_{ij}|}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(x,y)\,\mathrm{d}y\,\mathrm{d}x &= H_0^2[f(x_{i-1},y_{j-1}) + f(x_i,y_{j-1}) + f(x_{i-1},y_j) + f(x_i,y_j)]\\ &+ H_0\sum_{m=1}^{k-1} H_m[f(x_{i,m},y_{j-1}) + f(x_{i,m},y_j) + f(x_{i-1},y_{j,m}) + f(x_i,y_{j,m}] + \sum_{m=1}^{k-1}\sum_{l=1}^{k-1} H_m H_l f(x_{i,m},y_{j,l}),\end{aligned} \tag{6.2}$$

where the H_i denote the Gauss-Lobatto weights, and $x_{i,1},\ldots,x_{i,k-1}$ and $y_{j,1},\ldots,y_{j,k-1}$ denote the interior Gauss-Lobatto points in the intervals $[x_{i-1},x_i]$ and $[y_{j-1},y_i]$, respectively.

To satisfy (6.1), we define $\Pi_2\boldsymbol{v}$ to be zero at all the vertices of $\mathcal{T}_h$ and $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n}$ to be zero at the interior Gauss-Lobatto points on the edges of $\mathcal{T}_h$. Thus, it remains to determine $\Pi_2\boldsymbol{v}\cdot\boldsymbol{t}$ at the interior Gauss-Lobatto points on each edge e_{ij} of $\mathcal{T}_h$ and $\Pi_2\boldsymbol{v}$ at the interior Gauss-Lobatto points of each rectangle in $\mathcal{T}_h$.

Since $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n} \in P_k$ on each edge and vanishes at $k+1$ points on each edge, $\Pi_2\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on each edge. Hence, using the quadrature formula (6.2), we get

$$\begin{aligned}\int_{R_{ij}} \operatorname{div}\Pi_2\boldsymbol{v}\,q\,\mathrm{d}x = -\int_{R_{ij}} \Pi_2\boldsymbol{v}\cdot\nabla q\,\mathrm{d}x &= -|R_{ij}|H_0\sum_{m=1}^{k-1} H_m\big[(\Pi_2\boldsymbol{v}\cdot\boldsymbol{t})(x_{i,m},y_{j-1})(\nabla q\cdot\boldsymbol{t})(x_{i,m},y_{j-1})\\ &+ (\Pi_2\boldsymbol{v}\cdot\boldsymbol{t})(x_{i,m},y_j)(\nabla q\cdot\boldsymbol{t})(x_{i,m},y_j) + (\Pi_2\boldsymbol{v}\cdot\boldsymbol{t})(x_{i-1},y_{j,m})(\nabla q\cdot\boldsymbol{t}))(x_{i-1},y_{j,m})\\ &+ (\Pi_2\boldsymbol{v}\cdot\boldsymbol{t})(x_i,y_{j,m})(\nabla q\cdot\boldsymbol{t})(x_i,y_{j,m})\big] - |R_{ij}|\sum_{m=1}^{k-1}\sum_{l=1}^{k-1} H_m H_l(\Pi_2\boldsymbol{v}\cdot\nabla q)(x_{i,m},y_{j,l}).\end{aligned}$$

Since the analysis is similar to the case of triangles, we now present only some of the main calculations, further simplifying the presentation by restricting our attention to the two lowest order cases, $k = 2$ and $k = 3$.

Letting M_I and M_B denote the set of interior and boundary edges in $\mathcal{T}_h$, respectively, and summing over all $R_{ij} \in \mathcal{T}_h$, we get for $k = 2$,

$$\int_\Omega \operatorname{div} \Pi_2 \boldsymbol{v}\, q \,\mathrm{d}x = \sum_{R_{ij}\in\mathcal{T}_h} \int_{R_{ij}} \operatorname{div} \Pi_2 \boldsymbol{v}\, q \,\mathrm{d}x = -H_0 H_1 \sum_{e_{ij}\in M_B} |R_{ij}|(\Pi_2 \boldsymbol{v}\cdot \boldsymbol{t}_{ij})(a_{ij}).(\nabla q \cdot \boldsymbol{t}_{ij})$$
$$- H_0 H_1 \sum_{e_{ij}\in M_I} (|R_{1ij}| + |R_{2ij}|)(\Pi_2 \boldsymbol{v}\cdot \boldsymbol{t}_{ij})(a_{ij})(\nabla q\cdot \boldsymbol{t}_{ij}) - H_1^2 \sum_{R_{ij}\in\mathcal{T}_h} |R_{ij}|(\Pi_2\boldsymbol{v}\cdot\nabla q)(x_{i,1}, y_{j,1}), \quad (6.3)$$

where a_{ij} denotes the midpoint of the edge e_{ij}, R_{1ij} and R_{2ij} denote the two rectangles sharing this common edge $e_{ij} \in M_I$, and R_{ij} is the rectangle with edge $e_{ij} \in M_B$. For $k = 3$, we obtain a similar expression with two interior Gauss-Lobatto points per edge and four interior Gauss-Lobatto points in each rectangle.

As in the case of triangles, the main new idea in this paper is to show that the term $\int_\Omega \operatorname{div}(\boldsymbol{v}-\Pi_1\boldsymbol{v})(q-\Pi_0 q)\,\mathrm{d}x$ can also be written as a summation involving the terms $(\nabla q \cdot \boldsymbol{t}_{ij})(a_{ij})$ (*i.e.*, the tangential derivative of q at the interior Gauss-Lobatto points along rectangle edges), and ∇q at the Gauss-Lobatto points interior to the rectangles in $\mathcal{T}_h$. This is done by showing that $q - \Pi_0 q$ can be written in this form.

For the case $k = 2$ we choose, as in the case of triangles, $\Pi_0 q$ to be the L^2 projection into piecewise constants, *i.e.*, on the rectangle R,

$$\Pi_0 q = \frac{1}{4|R|}[q(x_{i-1}, y_{j-1}) + q(x_i, y_{j-1}) + q(x_{i-1}, y_j) + q(x_i, y_j)].$$

To simplify computations, we consider the unit square. Then

$$\begin{aligned}
8[q - \Pi_0 q] &= 2q(0,0)[4(1-x)(1-y) - 1] + 2q(1,0)[4x(1-y) - 1] + 2q(0,1)[4(1-x)y - 1] + 2q(1,1)[4xy - 1]\\
&= 2q(0,0)[(1-2x)(1-2y) - 2x - 2y + 2] + 2q(1,0)[(2x-1)(1-2y) + 2x - 2y]\\
&\quad + 2q(0,1)[(1-2x)(2y-1) + 2y - 2x] + 2q(1,1)[(2x-1)(2y-1) + 2y + 2x - 2]\\
&= [q(1,0) - q(0,0)][(2x-1)(1-2y) + 2(2x-1)] + [q(1,1) - q(1,0)][(2x-1)(2y-1) + 2(2y-1)]\\
&\quad + [q(0,1) - q(1,1)][(1-2x)(2y-1) + 2(1-2x)] + [q(0,0) - q(0,1)][(1-2x)(1-2y) + 2(1-2y)]\\
&= (\nabla q\cdot \boldsymbol{t})(1/2, 0)(2x-1)(3-2y) + (\nabla q\cdot \boldsymbol{t})(1, 1/2)(2x+1)(2y-1)\\
&\quad + (\nabla q\cdot \boldsymbol{t})(1/2, 1)(1-2x)(2y+1) + (\nabla q\cdot \boldsymbol{t})(0, 1/2)(1-2y)(3-2x),
\end{aligned}$$

where $\boldsymbol{t}$ denotes the counterclockwise unit tangent vector to R. Hence, with this choice of $\Pi_0 q$, $q - \Pi_0 q$ will have the desired form. Since the term ∇q at the center of the rectangle does not occur in this expression, we may choose $\Pi_2 \boldsymbol{v} = 0$ at this point. The remainder of the analysis is similar to the case of triangles.

For the case $k = 3$, we choose $\Pi_0 q$ to be the piecewise bilinear interpolant of q. Then, again performing calculations on the unit square, we have

$$\begin{aligned}
q - \Pi_0 q &= 2x(1-x)(1-y)(1-2y)[2q(1/2,0) - q(0,0) - q(1,0)] + 2x(1-x)y(2y-1)[2q(1/2,1) - q(0,1) - q(1,1)]\\
&\quad + 2(1-x)(1-2x)y(1-y)[2q(0,1/2) - q(0,0) - q(0,1)] + 2x(2x-1)y(1-y)[2q(1,1/2) - q(1,0) - q(1,1)]\\
&\quad + 4x(1-x)y(1-y)[4q(1/2,1/2) - q(0,0) - q(1,0) - q(0,1) - q(1,1)].
\end{aligned}$$

Terms of the form $2q(1/2,0) - q(0,0) - q(1,0)$ are handled just as in the case of triangles by using (4.4) to write them as the difference of the tangential derivative of q at the two interior Gauss-Lobatto points along the edge. Thus, it remains to show that $4q(1/2,1/2) - q(0,0) - q(1,0) - q(0,1) - q(1,1)$ can be written in terms of the tangential derivative of q at the two interior Gauss-Lobatto points along the rectangle edges and ∇q at the Gauss-Lobatto points interior to the rectangle.

First write

$$4q(1/2,1/2) - q(0,0) - q(1,0) - q(0,1) - q(1,1) = [2q(1/2,1/2) - q(1/2,0) - q(1/2,1)]$$
$$+ [2q(1/2,1/2) - q(0,1/2) - q(1,1/2)] + \frac{1}{2}\big\{[2q(1/2,0) - q(0,0) - q(1,0)]$$
$$+ [2q(1/2,1) - q(0,1) - q(1,1)] + [2q(0,1/2) - q(0,0) - q(0,1)] + [2q(1,1/2) - q(1,0) - q(1,1)]\big\}.$$

The last four terms as handled as above, so we only have to deal with the first two terms. Since $q \in Q^2(R)$, the function $2q(x,1/2) - q(x,0) - q(x,1)$ is a quadratic in x, so can be written as a linear combination of its values at the four Gauss-Lobatto points on $[0,1]$, *i.e.*, at $x = a_0 = 0$ and $x = a_3 = 1$, and the two interior Gauss-Lobatto points, which we denote a_1 and a_2. As above, we can then write $2q(a_i,1/2) - q(a_i,0) - q(a_i,1)$ as a linear combination of $q_y(a_i,a_1)$ and $q_y(a_i,a_2)$. When $i = 0$ or $i = 3$, these will be tangential derivatives on the boundary of the rectangle. The other term is handled in a similar manner. Combining these results, we see that the term $4q(1/2,1/2) - q(0,0) - q(1,0) - q(0,1) - q(1,1)$ can be written in terms of the tangential derivatives of q at the interior Gauss-Lobatto points on the boundary of the rectangle and the gradient of q at the Gauss-Lobatto points in the interior of R. The remainder of the derivation follows as in the case of triangles.

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