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MIXED FINITE ELEMENT APPROXIMATION OF THE VECTOR LAPLACIAN WITH DIRICHLET BOUNDARY CONDITIONS

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We consider the finite element solution of the vector Laplace equation on a domain in two dimensions. For various choices of boundary conditions, it is known that a mixed finite element method, in which the rotation of the solution is introduced as a second unknown, is advantageous, and appropriate choices of mixed finite element spaces lead to a stable, optimally convergent discretization. However, the theory that leads to these conclusions does not apply to the case of Dirichlet boundary conditions, in which both components of the solution vanish on the boundary. We show, by computational example, that indeed such mixed finite elements do not perform optimally in this case, and we analyze the suboptimal convergence that does occur. As we indicate, these results have implications for the solution of the biharmonic equation and of the Stokes equations using a mixed formulation involving the vorticity.

Keywords: Vector Laplacian; Hodge Laplacian; mixed finite elements.

AMS Subject Classification: 65N30

1. Introduction

We consider the vector Laplace equation (Hodge Laplace equation for 1-forms) on a two-dimensional domain Ω. That is, given a vector field $\mathbf{f}$ on Ω, we seek a vector field $\mathbf{u}$ such that

$$\operatorname{curl}\operatorname{rot}\mathbf{u} - \operatorname{grad}\operatorname{div}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega. \tag{1.1}$$

(Notations are detailed at the end of this Introduction.) A weak formulation of a boundary value problem for this equation seeks the solution $\mathbf{u}$ in a subspace $H \subset H(\operatorname{rot}) \cap H(\operatorname{div})$ satisfying

$$(\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in H. \tag{1.2}$$

If H is taken to be $\mathring{H}(\operatorname{rot}) \cap H(\operatorname{div})$, the variational formulation implies Eq. (1.1) together with the electric boundary conditions

$$\mathbf{u} \cdot \mathbf{s} = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Magnetic boundary conditions, $\mathbf{u} \cdot \mathbf{n} = 0$, $\operatorname{rot}\mathbf{u} = 0$, result if instead the subspace H in the weak formulation is taken to be $H(\operatorname{rot}) \cap \mathring{H}(\operatorname{div})$. (The terms electric and magnetic are derived from the close relation of the Hodge Laplacian and Maxwell's equations.) If the domain Ω is simply-connected, both these boundary value problems are well-posed. (Otherwise, H contains a finite-dimensional subspace consisting of vector fields which satisfy the boundary conditions and have vanishing rotation and divergence with dimension equal to the number of holes in the domain, and each problem can be rendered well-posed by replacing H with the orthogonal complement of this space.)

Even when the domain is simply connected, finite element methods based on (1.2) are problematic. For example, on a non-convex polygon, approximations using continuous piecewise linear functions converge to a function different from the solution of the boundary value. See §2.3.2 of Ref. 2 for more details. A convergent finite element method can be obtained by discretizing a *mixed* formulation with a *stable* choice of elements. The mixed formulation for the electric boundary value problem seeks $\sigma \in H^1$, $\mathbf{u} \in H(\operatorname{div})$ such that

$$(\sigma, \tau) - (\mathbf{u}, \operatorname{curl}\tau) = 0, \quad \tau \in H^1,$$

$$(\operatorname{curl}\sigma, \mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in H(\operatorname{div}).$$

On a simply connected domain, this problem has a unique solution for any L^2 vector field $\mathbf{f}$; $\mathbf{u}$ solves (1.1) and (1.3) and $\sigma = \operatorname{rot} u$. To discretize, we choose finite element spaces $\Sigma_h \subset H^1$, $V_h \subset H(\operatorname{div})$, indexed by a sequence of positive numbers h tending to 0, and determine $\sigma_h \in \Sigma_h$, $\mathbf{u}_h \in V_h$ by

$$(\sigma_h, \tau) - (\mathbf{u}_h, \operatorname{curl}\tau) = 0, \quad \tau \in \Sigma_h, \tag{1.4}$$

$$(\operatorname{curl}\sigma_h, \mathbf{v}) + (\operatorname{div}\mathbf{u}_h, \operatorname{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in V_h. \tag{1.5}$$

In order to obtain a stable numerical method, the finite element spaces Σ_h and V_h must be chosen appropriately. A stable method is obtained by choosing Σ_h to be the Lagrange elements of any degree $r \geq 1$ and V_h to be the Raviart–Thomas elements of the same degree r (where the case $r = 1$ refers to the lowest-order Raviart–Thomas elements). In the notation of finite element exterior calculus,[2] $\Sigma_h \times V_h = \mathcal{P}_r\Lambda^0 \times \mathcal{P}_r^-\Lambda^1$, and the hypotheses required by that theory (the spaces belong to a subcomplex of the Hilbert complex $H^1 \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$ with bounded cochain projections) are satisfied. From this it follows that the mixed finite element method is stable and convergent. Similar considerations apply to the magnetic boundary value problem, where the finite element spaces are $\mathring{\Sigma}_h = \Sigma_h \cap \mathring{H}^1$ and $\mathring{V}_h = V_h \cap \mathring{H}(\text{div})$ and the relevant Hilbert complex is $\mathring{H}^1 \xrightarrow{\text{curl}} \mathring{H}(\text{div}) \xrightarrow{\text{div}} L^2$. Another possible choice is to take Σ_h to be Lagrange elements of degree $r > 1$ and V_h to be Brezzi–Douglas–Marini elements of degree $r - 1$ (i.e. $\Sigma_h \times V_h = \mathcal{P}_r\Lambda^0 \times \mathcal{P}_{r-1}\Lambda^1$). This case is similar, and will not be discussed further here.

We turn now to the main consideration of the current paper, which is Eq. (1.1) with Dirichlet boundary conditions $\mathbf{u} = 0$ on $\partial\Omega$. This problem may of course be treated in the weak formulation (1.2) with $H = \mathring{H}^1(\Omega; \mathbb{R}^2)$. In this case we may integrate by parts and rewrite the bilinear form in terms of the gradient (which, when applied to a vector, is matrix-valued):

$$(\text{rot}\,\mathbf{u}, \text{rot}\,\mathbf{v}) + (\text{div}\,\mathbf{u}, \text{div}\,\mathbf{v}) = (\text{grad}\,\mathbf{u}, \text{grad}\,\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathring{H}^1(\Omega; \mathbb{R}^2).$$

Thus the weak formulation (1.2) is just

$$(\text{grad}\,\mathbf{u}, \text{grad}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathring{H}^1(\Omega; \mathbb{R}^2), \tag{1.6}$$

for which the discretization using Lagrange or similar finite elements is completely standard.

However, one might consider using a mixed method analogous to (1.4) and (1.5) for the Dirichlet boundary value problem in the hope of getting a better approximation of $\sigma = \text{rot}\,\mathbf{u}$, or when Dirichlet boundary conditions are imposed on part of the boundary and electric and/or magnetic boundary conditions are imposed on another part of the boundary. In fact, as we discuss in Secs. 4 and 5, a mixed approach to the vector Laplacian with Dirichlet boundary conditions is implicitly used in certain approaches to the solution of the Stokes equations which introduce the vorticity, and in certain mixed methods for the biharmonic equation. In the mixed formulation of the Dirichlet problem for the vector Laplacian, the vanishing of the normal component is an essential boundary condition, while the vanishing of the tangential component arises as a natural boundary condition. No boundary conditions are imposed on the variable σ. Thus, we define $\mathring{V}_h = V_h \cap \mathring{H}(\text{div})$, and seek $\sigma_h \in \Sigma_h$, $\mathbf{u}_h \in \mathring{V}_h$ satisfying

$$(\sigma_h, \tau) - (\mathbf{u}_h, \text{curl}\,\tau) = 0, \quad \tau \in \Sigma_h, \tag{1.7}$$

$$(\text{curl}\,\sigma_h, \mathbf{v}) + (\text{div}\,\mathbf{u}_h, \text{div}\,\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathring{V}_h. \tag{1.8}$$

Note that $\operatorname{curl}\Sigma_h \not\subseteq \mathring{V}_h$, so there is no Hilbert complex available in this case, and the theory of Ref. 2 does not apply. This suggests that there may be difficulties with stability and convergence of the mixed method (1.7) and (1.8). In the next section, we exhibit computational examples demonstrating that this pessimism is well founded. The convergence of the mixed method for the Dirichlet boundary value problem is severely suboptimal (while it is optimal for electric and magnetic boundary conditions). Thus, the difficulties arising from the loss of the Hilbert complex structure are real, not an artifact of the theory.

However, the computations indicate that even for Dirichlet boundary conditions, the mixed method does converge, albeit in a suboptimal manner. While we do not recommend the mixed formulation for the Dirichlet problem, in Sec. 3 we prove convergence at the suboptimal rates that are observed and, in so doing, clarify the sources of the suboptimality. Theorem 3.1 summarizes the main results of our analysis, and the remainder of the section develops the tools needed to establish them.

This analysis of the mixed finite element approximation of the vector Laplacian has implications for the analysis of mixed methods for other important problems: for the biharmonic equation using the Ciarlet–Raviart mixed formulation, and for the Stokes equations using a mixed formulation involving the vorticity, velocity, and pressure, or, equivalently, using a stream function-vorticity formulation. As a simple consequence of our analysis of the vector Laplacian, we are able to analyze mixed methods for these problems, elucidating the suboptimal rates of convergence observed for them, and establishing convergence at the rates that do occur. Some of the estimates we obtain are already known, while others improve on existing estimates. The biharmonic problem is addressed in Sec. 4 and the Stokes equations in Sec. 5.

We end this Introduction with a summary of the main notations used in the paper. For sufficiently smooth scalar-valued and vector-valued functions σ and $\mathbf{u}$, respectively, we use the standard calculus operators

$$\operatorname{grad}\sigma = \left(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}\right), \quad \operatorname{curl}\sigma = \left(\frac{\partial\sigma}{\partial y}, -\frac{\partial\sigma}{\partial x}\right), \quad \operatorname{div}\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y},$$
$$\operatorname{rot}\mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}.$$

We use the standard Lebesgue and Sobolev spaces $L^p(\Omega)$, $H^l(\Omega)$, $W^l_p(\Omega)$, and also the spaces $H(\operatorname{div},\Omega)$ and $H(\operatorname{rot},\Omega)$ consisting of L^2 vector fields $\mathbf{u}$ with $\operatorname{div}\mathbf{u}$ in L^2 or $\operatorname{rot}\mathbf{u} \in L^2$, respectively. Since the domain Ω will usually be clear from context, we will abbreviate these spaces as L^p, H^l, $H(\operatorname{div})$, etc. For vector-valued functions in a Lebesgue or Sobolev space, we may use notations like $H^l(\Omega;\mathbb{R}^2)$, although when there is little chance of confusion we will abbreviate this to simply H^l. The closure of $C_0^\infty(\Omega)$ in H^1, $H(\operatorname{div})$, and $H(\operatorname{rot})$, are denoted $\mathring{H}^1$, $\mathring{H}(\operatorname{div})$, $\mathring{H}(\operatorname{rot})$. Note that if $\mathbf{u} \in H(\operatorname{div})$, then the normal trace $\mathbf{u}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$ and $\mathring{H}(\operatorname{div}) = \{\mathbf{u} \in H(\operatorname{div}) \,|\, \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$. Similarly, $\mathring{H}(\operatorname{rot}) = \{\mathbf{u} \in H(\operatorname{rot}) \,|\, \mathbf{u}\cdot\mathbf{s} = 0 \text{ on } \partial\Omega\}$. We write $(\cdot,\cdot)$ for the $L^2(\Omega)$ inner product (of either scalar- or vector-valued functions) and $\|\cdot\|$ for the corresponding norm.

We shall also need the dual space of $\mathring{H}(\mathrm{div})$, the space $\mathring{H}(\mathrm{div})'$, normed by

$$\|\mathbf{v}\|_{\mathring{H}(\mathrm{div})'} := \sup_{\mathbf{w}\in\mathring{H}(\mathrm{div})} \frac{(\mathbf{v},\mathbf{w})}{\|\mathbf{w}\|_{H(\mathrm{div})}}. \tag{1.9}$$

Clearly,

$$L^2(\Omega;\mathbb{R}^2) \subset \mathring{H}(\mathrm{div})' \subset H^{-1}(\Omega;\mathbb{R}^2) \tag{1.10}$$

with continuous inclusions.

2. Some Numerical Results

We begin by considering the solution of the Hodge Laplacian (1.1) with electric boundary conditions (1.3) using the mixed method (1.4), (1.5). For the space Σ_h, we use Lagrange finite elements of degree $r \geq 1$ and for the space V_h, Raviart–Thomas elements of degree r (consisting locally of certain polynomials of degree $\leq r$, including all those of degree $\leq r-1$). These are stable elements and a complete analysis has been given in Ref. 2. Assuming that the solution is smooth, it follows from Theorem 3.11 of that reference that the following rates of convergence, each optimal, hold:

$$\|u-u_h\| = O(h^r), \qquad \|\mathrm{div}(u-u_h)\| = O(h^r),$$
$$\|\sigma-\sigma_h\| = O(h^{r+1}), \quad \|\mathrm{grad}(\sigma-\sigma_h)\| = O(h^r).$$

Table 1 shows the results of a computation with $r=2$. Note that the computed rates of convergence are precisely as expected. In the test problem displayed, the domain is $\Omega = (0,1)\times(0,1)$ and the exact solution is $\mathbf{u} = (\cos\pi x\sin\pi y, 2\sin\pi x\cos\pi y)$. The meshes used for computation were obtained by dividing the square into $n\times n$ subsquares, $n = 1,2,4,\ldots,128$, and dividing each subsquare into two triangles with the positively sloped diagonal. Only the result for the four finest meshes are shown. Very similar results were obtained for the case of magnetic boundary conditions, and for a sequence of nonuniform meshes, and also for other values of $r \geq 1$.

The situation in the case of Dirichlet boundary conditions is very different. In Table 2 we consider the problem with exact solution $\mathbf{u} = (\sin\pi x\sin\pi y, \sin\pi x\sin\pi y)$. The finite element spaces are as for the computation of

Table 1. L^2 errors and convergence rates for degree 2 mixed finite element approximation of the vector Laplacian with electric boundary conditions.

$\|\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|$	Rate	$\|\sigma-\sigma_h\|$	Rate	$\|\mathrm{curl}(\sigma-\sigma_h)\|$	Rate
2.14e-03	1.99	1.17e-02	1.99	2.16e-04	3.03	2.63e-02	1.98
5.37e-04	1.99	2.93e-03	2.00	2.70e-05	3.00	6.60e-03	1.99
1.34e-04	2.00	7.33e-04	2.00	3.37e-06	3.00	1.65e-03	2.00
3.36e-05	2.00	1.83e-04	2.00	4.16e-07	3.02	4.14e-04	2.00

Table 2. L^2 errors and convergence rates for degree 2 mixed finite element approximation of the vector Laplacian with Dirichlet boundary conditions.

$\|\mathbf{u} - \mathbf{u}_h\|$	Rate	$\|\mathrm{div}(\mathbf{u} - \mathbf{u}_h)\|$	Rate	$\|\sigma - \sigma_h\|$	Rate	$\|\mathrm{curl}(\sigma - \sigma_h)\|$	Rate
1.22e-03	2.01	1.55e-02	1.58	1.90e-02	1.62	2.53e+00	0.63
3.05e-04	2.00	5.33e-03	1.54	6.36e-03	1.58	1.68e+00	0.60
7.63e-05	2.00	1.85e-03	1.52	2.18e-03	1.54	1.14e+00	0.56
1.91e-05	2.00	6.49e-04	1.51	7.58e-04	1.52	7.89e-01	0.53

Table 1, except that the boundary condition of vanishing normal trace is imposed in the Raviart–Thomas space V_h. Note that the L^2 rate of convergence for σ is not the optimal value of 3, but rather roughly 3/2. The L^2 rate of convergence of $\mathrm{curl}\,\sigma$ (i.e. the H^1 rate of convergence of σ) is also suboptimal by roughly 3/2: it converges only as $h^{1/2}$. For $\mathbf{u}$, the L^2 convergence rate is the optimal 2, but the convergence rate for $\mathrm{div}\,\mathbf{u}$ is suboptimal by 1/2.

We have carried out similar computations for $r = 3$ and 4 and for nonuniform meshes and the results are all very similar: degradation of the rate of convergence by 3/2 for σ and $\mathrm{curl}\,\sigma$, and by 1/2 for $\mathrm{div}\,u$. However, the case $r = 1$ is different. There we saw no degradation of convergence rates for uniform meshes, but for nonuniform meshes σ converged in L^2 with rate suboptimal by 1 and $\mathrm{curl}\,\sigma$ did not converge at all.

The moral of this is that the mixed finite element method using the standard elements is indeed strongly tied to the underlying Hilbert complex structure which is not present for the vector Laplacian with Dirichlet boundary conditions, and the method is not appropriate for this problem. Nonetheless the experiments suggest that the method does converge, albeit at a degraded rate. In the next section, we develop the theory needed to prove that this is indeed so, and also to indicate where the lack of Hilbert complex structure leads to the suboptimality of the method.

3. Error Analysis

Theorem 3.1, which is the primary result of this section, establishes convergence of the mixed method for the Dirichlet problem at the suboptimal rates observed in the previous section. In it we assume that Ω is a convex polygon endowed with a shape-regular and quasi-uniform family of triangulations of mesh size h. We continue to denote by $\Sigma_h \subset H^1$ and $V_h \subset H(\mathrm{div})$ the Lagrange and Raviart–Thomas finite element spaces of some fixed degree $r \geq 1$, respectively, with $\mathring{V}_h = V_h \cap \mathring{H}(\mathrm{div})$.

Theorem 3.1. *Let* $\mathbf{u}$ *denote the solution of the vector Laplace equation* (1.1) *with Dirichlet boundary condition* $\mathbf{u} = 0$, *and let* $\sigma = \mathrm{rot}\,\mathbf{u}$. *There exist unique* $\sigma_h \in \Sigma_h$, $\mathbf{u}_h \in \mathring{V}_h$ *satisfying the mixed method* (1.7) *and* (1.8). *If the polynomial degree* $r \geq 2$, *then the following estimates hold for* $2 \leq l \leq r$ (*whenever the norms on the* *right-hand side are finite*):

$$\|\mathbf{u}-\mathbf{u}_h\| \le Ch^l\|\mathbf{u}\|_l,$$

$$\|\operatorname{div}(\mathbf{u}-\mathbf{u}_h)\| + \|\sigma-\sigma_h\| + h\|\operatorname{curl}(\sigma-\sigma_h)\|$$
$$\le Ch^{l-1/2}(|\ln h|\|\mathbf{u}\|_{W^l_\infty} + \|\mathbf{u}\|_{l+1/2}).$$

If $r = 1$, *the estimates are*:

$$\|\mathbf{u}-\mathbf{u}_h\| \le Ch|\ln h|^2(|\ln h|\|\mathbf{u}\|_{W^1_\infty} + \|\mathbf{u}\|_2),$$

$$\|\operatorname{div}(\mathbf{u}-\mathbf{u}_h)\| + \|\sigma-\sigma_h\| + h\|\operatorname{curl}(\sigma-\sigma_h)\| \le Ch^{1/2}(|\ln h|\|\mathbf{u}\|_{W^1_\infty} + h^{1/2}\|\mathbf{u}\|_2).$$

Note that the error estimate for $\mathbf{u}$ is optimal order (modulo the logarithm when $r = 1$), while (again modulo the logarithm), the estimate for $\operatorname{div} u$ is suboptimal by 1/2 order, and the estimates for σ and $\operatorname{curl}\sigma$ are suboptimal by 3/2 order. This is as observed in the experiments reported above. Above and throughout, we use C to denote a generic constant independent of h, whose values may differ at different occurrences.

The proof of this theorem is rather involved. Without the Hilbert complex structure, the numerical method is not only less accurate, but also harder to analyze. The analysis will proceed in several steps. First, in Sec. 3.2, we will establish the well-posedness of the continuous problem, not in the space $H^1 \times \mathring{H}(\operatorname{div})$, but rather using a larger space than H^1 with weaker norm for σ. Next, in Sec. 3.3, we mimic the well-posedness proof on the discrete level to obtain stability of the discrete problem, but with a mesh-dependent norm on Σ_h. This norm is even weaker than the norm used for the continuous problem, which may be seen as the cause of the loss of accuracy. To continue the analysis, we then introduce projection operators into $\mathring{V}_h$ and Σ_h and develop bounds and error estimates for them in Sec. 3.4. In Sec. 3.5 we combine these with the stability result to obtain basic error estimates for the scheme, and we improve the error estimate for $\mathbf{u}_h$ in Sec. 3.6 using duality.

3.1. *Preliminaries*

First we recall two forms of the Poincaré–Friedrichs inequality:

$$\|\tau\| \le C_P\|\operatorname{curl}\tau\|, \quad \tau \in \mathring{H}^1, \quad \|\psi\| \le C_P\|\operatorname{grad}\psi\|, \quad \psi \in \hat{H}^1. \tag{3.1}$$

Here $\hat{H}^1$ denotes the subspace of functions in H^1 with zero mean. Similarly, we will use $\hat{L}^2$ to denote the zero mean subspace of L^2.

Next we recall the Hodge decomposition. The space $L^2(\Omega;\mathbb{R}^2)$ admits a decomposition into the orthogonal closed subspaces $\operatorname{curl} H^1$ and $\operatorname{grad}\mathring{H}^1$, or, alternatively, into the subspaces $\operatorname{curl}\mathring{H}^1$ and $\operatorname{grad} H^1$. The decomposition of a given $\mathbf{v} \in L^2$ according to either of these may be computed by solving appropriate boundary value problems. For example, we may compute the unique $\rho \in \mathring{H}^1$ and $\phi \in \hat{H}^1$ such that

$$\mathbf{v} = \operatorname{curl}\rho + \operatorname{grad}\phi, \tag{3.2}$$

by a Dirichlet problem and a Neumann problem for the scalar Poisson equation, respectively:

$$(\operatorname{curl}\rho, \operatorname{curl}\tau) = (\mathbf{v}, \operatorname{curl}\tau), \quad \tau \in \mathring{H}^1,$$
$$(\operatorname{grad}\phi, \operatorname{grad}\psi) = (\mathbf{v}, \operatorname{grad}\psi), \quad \psi \in \hat{H}^1. \tag{3.3}$$

Clearly, $\|\operatorname{grad}\phi\| \le \|\mathbf{v}\|$. If $\mathbf{v} \in \mathring{H}(\operatorname{div})$, then ϕ satisfies the Neumann problem

$$\Delta\phi = \operatorname{div}\mathbf{v} \quad \text{in } \Omega, \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \phi\, dx = 0,$$

so, by elliptic regularity, $\|\phi\|_2 \le C\|\operatorname{div}\mathbf{v}\|$ if the domain is convex, and $\|\phi\|_1 \le C\|\operatorname{div}\mathbf{v}\|$ for any domain.

We shall need analogous results on the discrete level. To this end, let S_h denote the space of piecewise polynomials of degree at most $r-1$, with no imposed inter-element continuity. Then the divergence operator maps V_h onto S_h and also maps $\mathring{V}_h$ onto $\hat{S}_h$, the codimension one subspace consisting of functions with mean value zero. The former pair of spaces is used to solve the Dirichlet problem for the Poisson equation, and the latter is used to solve the Neumann problem. Each pair forms part of a short exact sequence:

$$0 \to \hat{\Sigma}_h \xrightarrow{\operatorname{curl}} V_h \xrightarrow{\operatorname{div}} S_h \to 0 \quad \text{and} \quad 0 \to \mathring{\Sigma}_h \xrightarrow{\operatorname{curl}} \mathring{V}_h \xrightarrow{\operatorname{div}} \hat{S}_h \to 0, \tag{3.4}$$

respectively.

The usual Raviart–Thomas approximate solution to the Poisson equation $\Delta\phi = g$ with Dirichlet boundary condition $\phi = 0$ is then: find $\mathbf{v}_h \in V_h$, $\phi_h \in S_h$ such that

$$(\mathbf{v}_h, \mathbf{w}) + (\operatorname{div}\mathbf{w}, \phi_h) = 0, \quad \mathbf{w} \in V_h, \quad (\operatorname{div}\mathbf{v}_h, \psi) = (g, \psi), \quad \psi \in S_h.$$

Define the operator $\operatorname{grad}_h : S_h \to V_h$ by

$$(\operatorname{grad}_h\phi, \mathbf{w}) = -(\phi, \operatorname{div}\mathbf{w}), \quad \phi \in S_h, \quad \mathbf{w} \in V_h.$$

From the stability of the mixed method, we obtain the discrete Poincaré inequality $\|\phi\| \le \bar{C}_P\|\operatorname{grad}_h\phi\|$, $\phi \in S_h$, with $\bar{C}_P$ independent of h. The solution $(\mathbf{v}_h, \phi_h) \in V_h \times S_h$ of the mixed method may be characterized by

$$(\operatorname{grad}_h\phi_h, \operatorname{grad}_h\psi) = -(g, \psi), \quad \psi \in S_h$$

and $\mathbf{v}_h = \operatorname{grad}_h\phi_h$.

Corresponding to the first sequence in (3.4), we have the discrete Hodge decomposition

$$V_h = \operatorname{curl}\Sigma_h + \operatorname{grad}_h S_h, \tag{3.5}$$

and corresponding to the second, the alternate discrete Hodge decomposition

$$\mathring{V}_h = \operatorname{curl}\mathring{\Sigma}_h + \mathring{\operatorname{grad}}_h S_h, \tag{3.6}$$

where $\mathring{\operatorname{grad}}_h : S_h \to \mathring{V}_h$ is defined by

$$(\mathring{\operatorname{grad}}_h\phi, \mathbf{w}) = -(\phi, \operatorname{div}\mathbf{w}), \quad \phi \in S_h, \quad \mathbf{w} \in \mathring{V}_h.$$

Both of the discrete Hodge decompositions can be characterized by finite element computations. For example, in analogy to (3.3), for given $\mathbf{v} \in \mathring{V}_h$ we may compute the unique $\rho_h \in \mathring{\Sigma}_h$ and $\phi_h \in \hat{S}_h$ such that $\mathbf{v} = \operatorname{curl}\rho_h + \operatorname{grad}_h^\circ \phi_h$ from the following finite element systems (one primal, one mixed):

$$(\operatorname{curl}\rho_h, \operatorname{curl}\tau) = (\mathbf{v}, \operatorname{curl}\tau), \qquad \tau \in \mathring{\Sigma}_h,$$
$$\big(\operatorname{grad}_h^\circ \phi_h, \operatorname{grad}_h^\circ \psi\big) = (\mathbf{v}, \operatorname{grad}_h^\circ \psi), \quad \psi \in \hat{S}_h.$$

3.2. *Well-posedness of the continuous formulation*

As a first step towards analyzing the mixed method, we obtain well-posedness of a mixed formulation of the continuous boundary value problem for the vector Laplacian. To do so, we need to introduce a larger space than H^1 for the scalar variable, namely

$$\Sigma = \{\tau \in L^2 : \operatorname{curl}\tau \in \mathring{H}(\operatorname{div})'\},$$

with norm $\|\tau\|_\Sigma^2 = \|\tau\|^2 + \|\operatorname{curl}\tau\|^2_{\mathring{H}(\operatorname{div})'}$ (see (1.9)). The space Σ has appeared before in studies of the vorticity-velocity-pressure and stream function-vorticity formulations of the Stokes problem,[10] and an equivalent space (at least for domains with $C^{1,1}$ boundary) has been used.[4] The bilinear form for the mixed formulation is

$$B(\rho, \mathbf{w}; \tau, \mathbf{v}) = (\rho, \tau) - \langle \operatorname{curl}\tau, \mathbf{w}\rangle + \langle \operatorname{curl}\rho, \mathbf{v}\rangle + (\operatorname{div}\mathbf{w}, \operatorname{div}\mathbf{v}),$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathring{H}(\operatorname{div})'$ and $\mathring{H}(\operatorname{div})$ (or more generally between a Hilbert space and its dual). Often, we will tacitly use the fact that if τ is in H^1, then $\langle \operatorname{curl}\tau, \mathbf{w}\rangle = (\operatorname{curl}\tau, \mathbf{w})$. Clearly,

$$|B(\rho, \mathbf{w}; \tau, \mathbf{v})| \le 2\big(\|\rho\|_\Sigma^2 + \|\mathbf{w}\|^2_{H(\operatorname{div})}\big)^{1/2}\big(\|\tau\|_\Sigma^2 + \|\mathbf{v}\|^2_{H(\operatorname{div})}\big)^{1/2},$$
$$\rho, \tau \in \Sigma, \quad \mathbf{w}, \mathbf{v} \in \mathring{H}(\operatorname{div}),$$

so B is bounded on $(\Sigma \times \mathring{H}(\operatorname{div})) \times (\Sigma \times \mathring{H}(\operatorname{div}))$. For $\tau \in \Sigma$, we define $\tau_0 \in \mathring{H}^1$ by

$$(\operatorname{curl}\tau_0, \operatorname{curl}\psi) = \langle \operatorname{curl}\tau, \operatorname{curl}\psi\rangle, \quad \psi \in \mathring{H}^1.$$

Taking $\psi = \tau_0$ shows that

$$\|\operatorname{curl}\tau_0\| \le \|\operatorname{curl}\tau\|_{\mathring{H}(\operatorname{div})'} \le \|\tau\|_\Sigma, \quad \tau \in \Sigma. \tag{3.7}$$

It is also true that

$$\|\tau\|_\Sigma \le C(\|\tau\| + \|\operatorname{curl}\tau_0\|), \quad \tau \in \Sigma. \tag{3.8}$$

To see this, define $\phi \in \hat{L}^2$ by

$$(\phi, \operatorname{div}\mathbf{v}) = \langle \operatorname{curl}\tau, \mathbf{v}\rangle - (\operatorname{curl}\tau_0, \mathbf{v}), \quad \mathbf{v} \in \mathring{H}(\operatorname{div}). \tag{3.9}$$

This is well-defined, since $\operatorname{div} \mathring{H}(\operatorname{div}) = \hat{L}^2$, and, if $\operatorname{div} \mathbf{v}$ vanishes, then $\mathbf{v} = \operatorname{curl} \psi$ for some $\psi \in \mathring{H}^1$, so the right-hand side vanishes as well. Clearly,

$$\langle \operatorname{curl} \tau, \mathbf{v} \rangle = (\operatorname{curl} \tau_0, \mathbf{v}) + (\phi, \operatorname{div} \mathbf{v}) \le (\|\operatorname{curl} \tau_0\| + \|\phi\|)\|\mathbf{v}\|_{H(\operatorname{div})}, \quad \mathbf{v} \in \mathring{H}(\operatorname{div}).$$

Choosing $\mathbf{v} \in \mathring{H}^1$ in (3.9) with $\operatorname{div} \mathbf{v} = \phi$ and $\|\mathbf{v}\|_1 \le C\|\phi\|$, we get $\|\phi\| \le C\|\operatorname{curl}(\tau - \tau_0)\|_{-1}$. This implies $\|\operatorname{curl} \tau\|_{\mathring{H}(\operatorname{div})'} \le C(\|\tau\| + \|\operatorname{curl} \tau_0\|)$, thus establishing (3.8). We conclude from (3.7) and (3.8) that the norm $\tau \mapsto \|\tau\| + \|\operatorname{curl} \tau_0\|$ is an equivalent norm on Σ.

Assuming that $\mathbf{f} \in L^2$ (or even $\mathring{H}(\operatorname{div})'$), we now give a mixed variational formulation of the continuous problem. We seek $\sigma \in \Sigma$, $\mathbf{u} \in \mathring{H}(\operatorname{div})$, such that

$$(\sigma, \tau) - \langle \operatorname{curl} \tau, \mathbf{u} \rangle = 0, \quad \tau \in \Sigma,$$
$$\langle \operatorname{curl} \sigma, \mathbf{v} \rangle + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathring{H}(\operatorname{div}).$$

We note that, if $\mathbf{u} \in \mathring{H}^1(\Omega; \mathbb{R}^2)$ is the solution of the standard variational formulation (1.6) and $\sigma = \operatorname{rot} \mathbf{u}$, then σ, $\mathbf{u}$ solve this mixed variational formulation. Indeed, $\mathbf{u} \in \mathring{H}^1(\Omega; \mathbb{R}^2) \subset \mathring{H}(\operatorname{div})$, $\sigma \in L^2$, and, for $\mathbf{v} \in \mathring{H}^1(\Omega; \mathbb{R}^2)$

$$\langle \operatorname{curl} \sigma, \mathbf{v} \rangle = (\sigma, \operatorname{rot} \mathbf{v}) = (\operatorname{rot} \mathbf{u}, \operatorname{rot} \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}).$$

This implies that $\operatorname{curl} \sigma \in \mathring{H}(\operatorname{div})'$, so $\sigma \in \Sigma$, and, extending to $\mathbf{v} \in \mathring{H}(\operatorname{div})$ by density, that the second equation above holds. Finally

$$(\sigma, \tau) = (\operatorname{rot} \mathbf{u}, \tau) = \langle \mathbf{u}, \operatorname{curl} \tau \rangle$$

for all $\tau \in L^2$, so the first equation holds.

In the next theorem, we establish well-posedness of the mixed variational problem by proving the inf–sup condition for B, following the approach of Ref. 1. Note that the theorem establishes well-posedness of the more general problem where the zero on the right-hand side of the first equation is replaced by the linear functional $\langle g, \tau \rangle$, where $g \in \Sigma'$, and we allow $\mathbf{f} \in \mathring{H}(\operatorname{div})'$.

Theorem 3.2. *There exist constants $c > 0$, $C < \infty$ such that, for any $(\rho, \mathbf{w}) \in \Sigma \times \mathring{H}(\operatorname{div})$, there exists $(\tau, \mathbf{v}) \in \Sigma \times \mathring{H}(\operatorname{div})$ with*

$$B(\rho, \mathbf{w}; \tau, \mathbf{v}) \ge c\big(\|\rho\|_\Sigma^2 + \|\mathbf{w}\|_{H(\operatorname{div})}^2\big), \tag{3.10}$$

$$\|\tau\|_\Sigma + \|\mathbf{v}\|_{H(\operatorname{div})} \le C(\|\rho\|_\Sigma + \|\mathbf{w}\|_{H(\operatorname{div})}). \tag{3.11}$$

Moreover, if $\mathbf{w} \in \operatorname{curl} \mathring{H}^1$, then we may choose $\mathbf{v} \in \operatorname{curl} \mathring{H}^1$.

Proof. Define $\rho_0 \in \mathring{H}^1$ by $(\operatorname{curl} \rho_0, \operatorname{curl} \psi) = \langle \operatorname{curl} \rho, \operatorname{curl} \psi \rangle$, $\psi \in \mathring{H}^1$. Next, use the Hodge decomposition to write $\mathbf{w}$ in the form $\mathbf{w} = \operatorname{curl} \mu + \operatorname{grad} \phi$, with $\mu \in \mathring{H}^1$ and $\phi \in \hat{H}^1$, and recall that

$$\|\operatorname{grad} \phi\| \le C\|\operatorname{div} \mathbf{w}\|. \tag{3.12}$$

We then choose

$$\tau = \rho - \delta\mu, \quad \mathbf{v} = \mathbf{w} + \operatorname{curl}\rho_0,$$

where δ is a constant to be chosen. Hence,

$$\begin{aligned} B(\rho, \mathbf{w}; \tau, \mathbf{v}) &= \|\rho\|^2 - \delta(\rho,\mu) - \langle \operatorname{curl}\rho, \mathbf{w}\rangle + \delta(\operatorname{curl}\mu, \mathbf{w}) \\ &\quad + \langle \operatorname{curl}\rho, \mathbf{w}\rangle + \langle \operatorname{curl}\rho, \operatorname{curl}\rho_0\rangle \|\operatorname{div}\mathbf{w}\|^2 \\ &= \|\rho\|^2 + \delta\|\operatorname{curl}\mu\|^2 - \delta(\rho,\mu) + \|\operatorname{curl}\rho_0\|^2 + \|\operatorname{div}\mathbf{w}\|^2. \end{aligned}$$

Recalling the constant C_P in the Poincaré inequality (3.1) and choosing δ sufficiently small, we obtain

$$\begin{aligned} B(\rho, \mathbf{w}; \tau, \mathbf{v}) &\geq \frac{1}{2}\|\rho\|^2 + (\delta - \delta^2 C_P^2/2)\|\operatorname{curl}\mu\|^2 + \|\operatorname{curl}\rho_0\|^2 + \|\operatorname{div}\mathbf{w}\|^2 \\ &\geq c\big(\|\rho\|_\Sigma^2 + \|\mathbf{w}\|_{H(\operatorname{div})}^2\big), \end{aligned}$$

where we have used the facts that $\|\mathbf{w}\|^2 = \|\operatorname{curl}\mu\|^2 + \|\operatorname{grad}\phi\|^2$, (3.12), and (3.8) in the last step. This establishes (3.10).

To establish (3.11), we observe that

$$\|\mathbf{v}\|_{H(\operatorname{div})} \leq \|\mathbf{w}\|_{H(\operatorname{div})} + \|\operatorname{curl}\rho_0\| \leq \|\mathbf{w}\|_{H(\operatorname{div})} + \|\rho\|_\Sigma$$

by (3.7), while

$$\|\tau\|_\Sigma \leq \|\rho\|_\Sigma + \delta\|\mu\|_\Sigma \leq \|\rho\|_\Sigma + \delta\|\mu\|_1 \leq C(\|\rho\|_\Sigma + \|\mathbf{w}\|),$$

since $\|\mu\|_1 \leq C\|\operatorname{curl}\mu\| \leq C\|\mathbf{w}\|$.

To establish the final claim, we observe that if $\mathbf{w} \in \operatorname{curl}\mathring{H}^1$, then obviously $\mathbf{v} = \mathbf{w} + \operatorname{curl}\rho_0 \in \operatorname{curl}\mathring{H}^1$. □

Remark 3.1. Had we posed the weak formulation using the space $H^1 \times \mathring{H}(\operatorname{div})$ instead of $\Sigma \times \mathring{H}(\operatorname{div})$, we would not have obtained a well-posed problem.

3.3. *Stability of the discrete formulation*

In this section, we establish the stability of the mixed method (1.7) and (1.8), guided by the arguments used for the continuous problem in the preceding subsection. Analogous to the norm on Σ, we begin by defining a norm on Σ_h by $\|\tau\|_{\Sigma_h}^2 = \|\tau\|^2 + \|\operatorname{curl}\tau\|_{\mathring{V}_h'}^2$, $\tau \in \Sigma_h$, where

$$\|\mathbf{v}\|_{\mathring{V}_h'} := \sup_{\mathbf{w}\in\mathring{V}_h} \frac{(\mathbf{v},\mathbf{w})}{\|\mathbf{w}\|_{H(\operatorname{div})}}.$$

The bilinear form is bounded on the finite element spaces in this norm:

$$|B(\rho,\mathbf{w};\tau,\mathbf{v})| \le 2\big(\|\rho\|_{\Sigma_h}^2 + \|\mathbf{w}\|_{H(\mathrm{div})}^2\big)^{1/2}\big(\|\tau\|_{\Sigma_h}^2 + \|\mathbf{v}\|_{H(\mathrm{div})}^2\big)^{1/2}, \qquad \rho,\tau\in\Sigma_h, \quad \mathbf{w},\mathbf{v}\in\mathring{V}_h.$$

For $\tau\in\Sigma_h$, we define $\tau_0\in\mathring{\Sigma}_h$ by

$$(\operatorname{curl}\tau_0,\operatorname{curl}\psi) = (\operatorname{curl}\tau,\operatorname{curl}\psi), \quad \psi\in\mathring{\Sigma}_h.$$

The discrete analogue of (3.7) again follows by choosing $\psi=\operatorname{curl}\tau_0$:

$$\|\operatorname{curl}\tau_0\| \le \|\operatorname{curl}\tau\|_{\mathring{V}_h'} \le \|\tau\|_{\Sigma_h}, \quad \tau\in\Sigma_h.$$

Next we establish the discrete analogue of (3.8), that is,

$$\|\tau\|_{\Sigma_h} \le C(\|\tau\| + \|\operatorname{curl}\tau_0\|), \quad \tau\in\Sigma_h. \tag{3.13}$$

To see this, define $\phi\in\hat{S}_h$ by

$$(\phi,\operatorname{div}\mathbf{v}) = (\operatorname{curl}\tau,\mathbf{v}) - (\operatorname{curl}\tau_0,\mathbf{v}), \quad \mathbf{v}\in\mathring{V}_h.$$

This is well-defined, since $\operatorname{div}\mathring{V}_h = \hat{S}_h$, and, if $\operatorname{div}\mathbf{v}$ vanishes, then $\mathbf{v}=\operatorname{curl}\psi$ for some $\psi\in\mathring{\Sigma}_h$, so the right-hand side vanishes as well. It follows that $\|\operatorname{curl}\tau\|_{\mathring{V}_h'} \le \|\operatorname{curl}\tau_0\| + \|\phi\|$. To bound $\|\phi\|$, as in the continuous case, we choose $\mathbf{v}\in\mathring{H}^1$ with $\operatorname{div}\mathbf{v}=\phi$ and $\|\mathbf{v}\|_1\le C\|\phi\|$. In the discrete case, we also introduce $\Pi_h^V\mathbf{v}$, the canonical projection of $\mathbf{v}$ into the Raviart–Thomas space $\mathring{V}_h$ (see (3.22)), so $\operatorname{div}\Pi_h^V\mathbf{v} = P_{S_h}\operatorname{div}\mathbf{v} = \phi$ and $\|\mathbf{v}-\Pi_h^V\mathbf{v}\|\le Ch\|\mathbf{v}\|_1$. Then

$$\begin{aligned}\|\phi\|^2 &= \big(\phi,\operatorname{div}\Pi_h^V\mathbf{v}\big) = \big(\operatorname{curl}\tau,\Pi_h^V\mathbf{v}\big) - \big(\operatorname{curl}\tau_0,\Pi_h^V\mathbf{v}\big)\\ &= \big(\operatorname{curl}\tau,\Pi_h^V\mathbf{v}-\mathbf{v}\big) + \big(\operatorname{curl}\tau,\mathbf{v}\big) - \big(\operatorname{curl}\tau_0,\Pi_h^V\mathbf{v}\big)\\ &\le Ch(\|\tau\|_1 + \|\operatorname{curl}\tau\|_{-1} + \|\operatorname{curl}\tau_0\|)\|\mathbf{v}\|_1.\end{aligned}$$

Using the inverse inequality $\|\tau\|_1\le Ch^{-1}\|\tau\|$ and the fact that $\|\mathbf{v}\|_1\le\|\phi\|$, gives the bound $\|\phi\|\le C(\|\tau\| + \|\operatorname{curl}\tau_0\|)$, and implies (3.13).

With this choice of norm, stability of the finite element approximation scheme is established by an argument precisely analogous to that used in the proof of Theorem 3.2, simply using the Σ_h norm, the discrete gradient operator $\operatorname{grad}_h^\circ$, the discrete Hodge decomposition (3.6), the estimate (3.13), and the discrete Poincaré inequality, instead of their continuous counterparts.

Theorem 3.3. *There exist constants $c>0$, $C<\infty$, independent of h, such that, for any $(\rho,\mathbf{w})\in\Sigma_h\times\mathring{V}_h$, there exists $(\tau,\mathbf{v})\in\Sigma_h\times\mathring{V}_h$ with*

$$B(\rho,\mathbf{w};\tau,\mathbf{v}) \ge c\big(\|\rho\|_{\Sigma_h}^2 + \|\mathbf{w}\|_{H(\mathrm{div})}^2\big),$$

$$\|\tau\|_{\Sigma_h} + \|\mathbf{v}\|_{H(\mathrm{div})} \le C(\|\rho\|_{\Sigma_h} + \|\mathbf{w}\|_{H(\mathrm{div})}).$$

Moreover, if $\mathbf{w}\in\operatorname{curl}\mathring{\Sigma}_h$, then we may choose $\mathbf{v}\in\operatorname{curl}\mathring{\Sigma}_h$.

Remark 3.2. Note that $\|\tau\|_{\Sigma_h} \le \|\tau\|_{\Sigma}$ for $\tau \in \Sigma_h$, but, in general equality does not hold. Had we used the Σ norm instead of the Σ_h norm on the discrete level, we would not have been able to establish stability.

3.4. *Projectors*

Our error analysis will be based on the approximation and orthogonality properties of certain projection operators into the finite element spaces:

$$P_{S_h} : L^2 \to S_h, \quad P_{\Sigma_h} : H^1 \to \Sigma_h, \quad P_{\mathring{\Sigma}_h} : \mathring{H}^1 \to \mathring{\Sigma}_h, \quad P_{\mathring{V}_h} : \mathring{H}(\mathrm{div}) \to \mathring{V}_h.$$

For P_{S_h}, we simply take the L^2 projection. By standard approximation theory,

$$\|s - P_{S_h} s\|_{L^p} \le C h^l \|s\|_{W^l_p}, \quad 0 \le l \le r, \quad 1 \le p \le \infty.$$

For P_{Σ_h} and $P_{\mathring{\Sigma}_h}$, we use elliptic projections. Namely, for any $\tau \in H^1$,

$$(\mathrm{curl}\, P_{\Sigma_h}\tau, \mathrm{curl}\, \rho) = (\mathrm{curl}\, \tau, \mathrm{curl}\, \rho), \quad \rho \in \Sigma_h, \quad (P_{\Sigma_h}\tau, 1) = (\tau, 1)$$

and, for any $\tau \in \mathring{H}^1$

$$\big(\mathrm{curl} P_{\mathring{\Sigma}_h}\tau, \mathrm{curl}\, \rho\big) = (\mathrm{curl}\, \tau, \mathrm{curl}\, \rho), \quad \rho \in \mathring{\Sigma}_h.$$

Then, by standard estimates,

$$\|\sigma - P_{\Sigma_h}\sigma\| + h\|\sigma - P_{\Sigma_h}\sigma\|_1 \le C h^l \|\sigma\|_l, \quad 1 \le l \le r+1. \tag{3.14}$$

Moreover,

$$(\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathbf{v}) \le C h \|\mathrm{curl}(\sigma - P_{\Sigma_h}\sigma)\| \|\mathrm{div}\, \mathbf{v}\|, \quad \mathbf{v} \in V_h, \quad \sigma \in H^1. \tag{3.15}$$

To prove this last estimate, we use the discrete Hodge decomposition (3.5) to write $\mathbf{v} = \mathrm{curl}\, \gamma_h + \mathrm{grad}_h\, \psi_h$, with $\gamma_h \in \hat{\Sigma}_h$ and $\psi_h \in S_h$. As explained in Sec. 3.1, the pair $(\mathrm{grad}_h\, \psi_h, \psi_h) \in V_h \times S_h$ is the mixed approximation of $(\mathrm{grad}\, \psi, \psi)$ where $\psi \in \mathring{H}^1$ solves $\Delta\psi = \mathrm{div}\, \mathbf{v}$ in Ω. Since Ω is convex, $\|\psi\|_2 \le C\|\mathrm{div}\, \mathbf{v}\|$. Therefore,

$$\begin{aligned}
(\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathbf{v}) &= (\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathrm{curl}\, \gamma_h + \mathrm{grad}_h\, \psi_h) \\
&= (\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathrm{grad}_h\, \psi_h) \\
&= (\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathrm{grad}_h\, \psi_h - \mathrm{grad}\, \psi) \\
&\le C h \|\mathrm{curl}(\sigma - P_{\Sigma_h}\sigma)\| \|\psi\|_2 \le C h \|\mathrm{curl}(\sigma - P_{\Sigma_h}\sigma)\| \|\mathrm{div}\, \mathbf{v}\|.
\end{aligned}$$

For $P_{\mathring{\Sigma}_h}\tau$, $\tau \in \mathring{H}^1$, we will use the W^1_p estimate (due to Nitsche[15] for $r \ge 2$ and Rannacher and Scott[16] for $r = 1$; cf. also Theorem 8.5.3 of Ref. 5):

$$\|\tau - P_{\mathring{\Sigma}_h}\tau\|_{W^1_p} \le C h^{l-1} \|\tau\|_{W^l_p}, \quad 1 \le l \le r+1, \quad 2 \le p \le \infty, \tag{3.16}$$

which holds with constant C independent of p as well as h.

We define the fourth projection operator, $P_{\mathring{V}_h} : \mathring{H}(\mathrm{div}) \to \mathring{V}_h$, by the equations

$$\left(P_{\mathring{V}_h}\mathbf{v}, \mathrm{curl}\,\tau + \mathrm{grad}_h^{\circ} s\right) = (\mathbf{v}, \mathrm{curl}\,\tau) - (\mathrm{div}\,\mathbf{v}, s), \quad \tau \in \mathring{\Sigma}_h, \ \ s \in S_h.$$

In view of the discrete Hodge decomposition (3.6), $P_{\mathring{V}_h}\mathbf{v} \in \mathring{V}_h$ is well-defined for any $\mathbf{v} \in \mathring{H}(\mathrm{div})$. It may be characterized as well by the equations

$$\left(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}, \mathrm{curl}\,\tau\right) = 0, \quad \tau \in \mathring{\Sigma}_h, \tag{3.17}$$

$$\left(\mathrm{div}\left[\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\right], s\right) = 0, \quad s \in S_h. \tag{3.18}$$

Similar projectors have been used elsewhere.[7] The properties of $P_{\mathring{V}_h}$ are summarized in the following theorem.

Theorem 3.4. *For* $\mathbf{v} \in \mathring{H}(\mathrm{div})$ *and* $U \in \mathring{H}^1$,

$$\mathrm{div}\, P_{\mathring{V}_h}\mathbf{v} = P_{S_h}\,\mathrm{div}\,\mathbf{v}, \quad P_{\mathring{V}_h}\,\mathrm{curl}\,U = \mathrm{curl}\,P_{\mathring{\Sigma}_h}U. \tag{3.19}$$

Moreover, the following estimates hold

$$\left\|\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\right\|_{L^p} \le Cph^l\|\mathbf{v}\|_{W_p^l}, \quad 1 \le l \le r, \ 2 \le p < \infty, \tag{3.20}$$

$$\left\|\mathrm{div}(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v})\right\| \le Ch^l\|\mathrm{div}\,\mathbf{v}\|_l, \quad 0 \le l \le r, \tag{3.21}$$

whenever the norm on the right-hand side is finite.

Proof. The first commutativity property in (3.19) is immediate from (3.18), and the divergence estimate (3.21) follows immediately. For the second commutativity property, we note that $\mathrm{curl}\,P_{\mathring{\Sigma}_h}U \in \mathring{V}_h$ and that, if we set $\mathbf{v} = \mathrm{curl}\,U$ and replace $P_{\mathring{V}_h}\mathbf{v}$ by $\mathrm{curl}\,P_{\mathring{\Sigma}_h}U$, then the defining Eqs. (3.17) and (3.18) are satisfied.

To prove the L^p estimate (3.20), we follow the proof of corresponding results for mixed finite element approximation of second-order elliptic problems by Durán.[11] First, we introduce the canonical interpolant $\Pi_h^V : H^1(\Omega; \mathbb{R}^2) \to V_h$ into the Raviart–Thomas space, defined through the degrees of freedom

$$\mathbf{v} \mapsto \int_e \mathbf{v}\cdot\mathbf{n} w\,ds, \quad w \in \mathcal{P}_{r-1}(e), \quad \mathbf{v} \mapsto \int_T \mathbf{v}\cdot\mathbf{w}\,dx, \quad \mathbf{w} \in \mathcal{P}_{r-2}(T), \tag{3.22}$$

where e ranges over the edges of the mesh and T over the triangles. Then

$$\left\|\mathbf{v} - \Pi_h^V\mathbf{v}\right\|_{L^p} \le Ch^l\|\mathbf{v}\|_{W_p^l}, \quad 1 \le l \le r, \quad 1 \le p \le \infty, \tag{3.23}$$

and, since $\mathrm{div}\,\Pi_h^V\mathbf{v} = P_{S_h}\,\mathrm{div}\,\mathbf{v}$,

$$\left\|\mathrm{div}\left(\mathbf{v} - \Pi_h^V\mathbf{v}\right)\right\|_{L^p} \le Ch^l\|\mathrm{div}\,\mathbf{v}\|_{W_p^l}, \quad 0 \le l \le r, \quad 1 \le p \le \infty. \tag{3.24}$$

Writing $\mathbf{v} - P_{\mathring{V}_h}\mathbf{v} = (\mathbf{v} - \Pi_h^V\mathbf{v}) + (\Pi_h^V\mathbf{v} - P_{\mathring{V}_h}\mathbf{v})$, it thus remains to bound the second term. From (3.19), $\operatorname{div}(P_{\mathring{V}_h}\mathbf{v} - \Pi_h^V\mathbf{v}) = 0$, so $P_{\mathring{V}_h}\mathbf{v} - \Pi_h^V\mathbf{v} = \operatorname{curl}\rho_h$ for some $\rho_h \in \mathring{\Sigma}_h$. Applying the decomposition (3.2), we have $\mathbf{v} - \Pi_h^V\mathbf{v} = \operatorname{curl}\rho + \operatorname{grad}\psi$ for some $\rho \in \mathring{H}^1$ and $\psi \in \hat{H}^1$. From (3.17),

$$(\operatorname{curl}\rho_h, \operatorname{curl}\tau) = (\operatorname{curl}\rho, \operatorname{curl}\tau), \quad \tau \in \mathring{\Sigma}_h.$$

Thus, $\rho_h = P_{\mathring{\Sigma}_h}\rho$ and so satisfies the bound $\|\operatorname{curl}\rho_h\|_{L^p} \le C\|\operatorname{curl}\rho\|_{L^p}$ given above in (3.16).

Since

$$(\operatorname{curl}\rho, \operatorname{curl}\tau) = (\mathbf{v} - \Pi_h^V\mathbf{v}, \operatorname{curl}\tau) = (\operatorname{rot}(\mathbf{v} - \Pi_h^V\mathbf{v}), \tau), \quad \tau \in \mathring{H}^1,$$

$\rho \in \mathring{H}^1$ satisfies $-\Delta\rho = \operatorname{rot}(\mathbf{v} - \Pi_h^V\mathbf{v})$. Using the elliptic regularity result of Corollary 1 of Ref. 13, we have for $1 < p < \infty$ that

$$\|\rho\|_{W_p^1} \le C_p\|\operatorname{rot}(\mathbf{v} - \Pi_h^V\mathbf{v})\|_{W^{-1,p}} \le C_p\|\mathbf{v} - \Pi_h^V\mathbf{v}\|_{L^p}.$$

Following the proof of that result, the dependence of the constant C_p on p arises from the use of the Marcinkiewicz interpolation theorem for interpolating between a weak L^1 and an L^2 estimate. Using the explicit bound on the constant in this theorem found in Theorem VIII.9.2 of Ref. 8, it follows directly that $C_p \le Cp$, where C is a constant independent of p. We remark that this regularity result requires the assumed convexity of Ω, and does not hold for all $1 < p < \infty$ if Ω is only Lipschitz.[14] Estimate (3.20) follows by combining these results and applying (3.23). □

Theorem 3.5 below gives one more property of $P_{\mathring{V}_h}$, inspired by an idea of Scholz.[17] To prove it we need a simple lemma.

Lemma 3.1. *Let ρ be a piecewise polynomial function with respect to some triangulation which is nonzero only on triangles meeting $\partial\Omega$. Then for any $1 \le q \le 2$,*

$$\|\rho\|_{L^q} \le Ch^{1/q-1/2}\|\rho\|_{L^2},$$

where the constant C depends only on the polynomial degree and the shape regularity of the triangulation.

Proof. By scaling and equivalence of norms on a finite-dimensional space, we have

$$\|\rho\|_{L^q(T)} \le Ch^{2/q-1}\|\rho\|_{L^2(T)}, \quad \rho \in \mathcal{P}_r(T),$$

where the constant C depends only on the polynomial degree r and the shape constant for the triangle T. Now, let $\mathcal{T}_h^\partial$ denote the set of triangles meeting $\partial\Omega$. Then

$$\|\rho\|_{L^q(\Omega)}^q = \sum_{T\in\mathcal{T}_h^\partial}\|\rho\|_{L^q(T)}^q \le Ch^{2-q}\sum_{T\in\mathcal{T}_h^\partial}\|\rho\|_{L^2(T)}^q.$$

Applying Hölder's inequality we have

$$\sum_{T\in\mathcal{T}_h^\partial} \|\rho\|_{L^2(T)}^q \le (\#\mathcal{T}_h^\partial)^{(2-q)/2} \left(\sum_{T\in\mathcal{T}_h^\partial} \|\rho\|_{L^2(T)}^2 \right)^{q/2}$$

and $\#\mathcal{T}_h^\partial \le Ch^{-1}$ by the assumption of shape regularity. Combining these results gives the lemma. □

Theorem 3.5. *Let* $2 \le p \le \infty$. *Then*

$$\big(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}, \operatorname{curl}\tau\big) \le Ch^{-1/2-1/p}\big\|\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\big\|_{L^p}\|\tau\|, \quad \tau \in \Sigma_h, \quad \mathbf{v} \in \mathring{H}(\operatorname{div}) \cap L^p. \tag{3.25}$$

Proof. Define $\mathring{\tau} \in \mathring{\Sigma}_h$ by taking the Lagrange degrees of freedom to be the same as those for τ, except setting equal to zero those associated to vertices or edges in $\partial\Omega$. Then $\|\mathring{\tau}\| \le C\|\tau\|$ and $\tau - \mathring{\tau}$ is nonzero only on triangles meeting $\partial\Omega$. By (3.17),

$$\big(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}, \operatorname{curl}\tau\big) = \big(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}, \operatorname{curl}[\tau - \mathring{\tau}]\big).$$

Let $q = p/(p-1)$, so $1 \le q \le 2$. Applying Hölder's inequality, the lemma, and an inverse inequality, we obtain

$$\begin{aligned}\big(\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}, \operatorname{curl}(\tau - \mathring{\tau})\big) &\le \big\|\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\big\|_{L^p}\|\operatorname{curl}(\tau - \mathring{\tau})\|_{L^q} \\ &\le C\big\|\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\big\|_{L^p} h^{1/2-1/p}\|\operatorname{curl}(\tau - \mathring{\tau})\|_{L^2} \\ &\le C\big\|\mathbf{v} - P_{\mathring{V}_h}\mathbf{v}\big\|_{L^p} h^{-1/2-1/p}\|\tau - \mathring{\tau}\|_{L^2},\end{aligned}$$

from which the result follows. □

3.5. *Error estimates by an energy argument*

Using the projection operators defined in the last subsection and the stability result of the preceding section, we now obtain a basic error estimate (which is not, however, of optimal order).

Theorem 3.6. *Let* $r \ge 1$ *denote the polynomial degree. There exists a constant* C *independent of the mesh size* h *and of* $p \in [2, \infty)$, *for which*

$$\begin{aligned}&\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{div})} \\ &\quad \le C \begin{cases} h^{l-1/2-1/p}\big(p\|\mathbf{u}\|_{W_p^l} + \|\mathbf{u}\|_{l+1/2-1/p}\big), & 2 \le l \le r, \quad \textit{if } r \ge 2, \\ h^{1/2-1/p}\big(p\|\mathbf{u}\|_{W_p^1} + h^{1/2+1/p}\|\mathbf{u}\|_2\big), & \textit{if } r = 1, \end{cases}\end{aligned}$$

whenever the norms on the right-hand side are finite.

Proof. We divide the errors into the projection and the remainder:

$$\sigma - \sigma_h = (\sigma - P_{\Sigma_h}\sigma) + (P_{\Sigma_h}\sigma - \sigma_h), \quad \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}) + (P_{\mathring{V}_h}\mathbf{u} - \mathbf{u}_h).$$

Since,

$$\|\sigma - P_{\Sigma_h}\sigma\| + h\|\sigma - P_{\Sigma_h}\sigma\|_1 \le Ch^t\|\sigma\|_t \le Ch^t\|\mathbf{u}\|_{t+1}, \quad 1 \le t \le r+1,$$

and, by Theorem 3.4,

$$\|\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}\|_{H(\mathrm{div})} \le Ch^t\|\mathbf{u}\|_{t+1}, \quad 1 \le t \le r,$$

the projection error satisfies the necessary bounds (without the $p\|\mathbf{u}\|_{W_p^l}$ term on the right-hand side).

Therefore, setting $\rho = \sigma_h - P_{\Sigma_h}\sigma$ and $\mathbf{w} = \mathbf{u}_h - P_{\mathring{V}_h}\mathbf{u}$, it suffices to show that for $2 \le p < \infty$,

$$\|\rho\| + \|\mathbf{w}\|_{H(\mathrm{div})} \le C(\|\sigma - P_{\Sigma_h}\sigma\| + h\|\sigma - P_{\Sigma_h}\sigma\|_1 + h^{-1/2-1/p}\|\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}\|_{L^p}). \tag{3.26}$$

Indeed, both cases of the theorem follow from (3.26), Theorem 3.4, and the inverse inequality $Ch\|\rho\|_1 \le \|\rho\|$. By the stability result of Theorem 3.3, there exists $(\tau, \mathbf{v}) \in \Sigma_h \times \mathring{V}_h$ satisfying

$$B(\rho, \mathbf{w}; \tau, \mathbf{v}) \ge c(\|\rho\|_{\Sigma_h}^2 + \|\mathbf{w}\|_{H(\mathrm{div})}^2),$$
$$\|\tau\|_{\Sigma_h} + \|\mathbf{v}\|_{H(\mathrm{div})} \le C(\|\rho\|_{\Sigma_h} + \|\mathbf{w}\|_{H(\mathrm{div})}).$$

By Galerkin orthogonality,

$$\begin{aligned} B(\rho, \mathbf{w}; \tau, \mathbf{v}) &= B(\sigma - P_{\Sigma_h}\sigma, \mathbf{u} - P_{\mathring{V}_h}\mathbf{u}; \tau, \mathbf{v}) \\ &= (\sigma - P_{\Sigma_h}\sigma, \tau) - (\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}, \mathrm{curl}\,\tau) + (\mathrm{curl}(\sigma - P_{\Sigma_h}\sigma), \mathbf{v}), \end{aligned}$$

where we used the definition of B and (3.19) in the last step. Applying the Cauchy–Schwarz inequality, Theorem 3.5, and (3.15), we then obtain

$$\begin{aligned} B(\rho, \mathbf{w}; \tau, \mathbf{v}) \le C\big(&\|\sigma - P_{\Sigma_h}\sigma\|^2 + h^2\|\mathrm{curl}(\sigma - P_{\Sigma_h}\sigma)\|^2 \\ &+ h^{2(-1/2-1/p)}\|\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}\|_{L^p}^2\big)^{1/2}(\|\tau\|^2 + \|\mathbf{v}\|_{H(\mathrm{div})}^2)^{1/2}. \end{aligned}$$

Together, these imply (3.26) and so the proof of the theorem is complete. □

Choosing $p = |\ln h|$ in the theorem gives a limiting estimate.

Corollary 3.1. *The following estimates hold whenever the right-hand side norm is finite*:

$$\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div})}$$
$$\le C \begin{cases} h^{l-1/2}(|\ln h|\|\mathbf{u}\|_{W_\infty^l} + \|\mathbf{u}\|_{l+1/2}), & 2 \le l \le r, \quad \textit{if } r \ge 2, \\ h^{1/2}(|\ln h|\|\mathbf{u}\|_{W_\infty^1} + h^{1/2}\|\mathbf{u}\|_2), & \qquad\qquad \textit{if } r = 1. \end{cases}$$

For smooth solutions, choosing the maximum value of $l = r$ in the corollary gives suboptimal approximation of σ by order $h^{3/2}$, and suboptimal approximation of $\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ by order $h^{1/2}$ (ignoring logarithms). In the next section, we show how to improve the L^2 error estimate for $\mathbf{u}$ to optimal order. The other estimates are essentially sharp, as demonstrated by the numerical experiments already presented.

3.6. *Improved estimates for* $\mathbf{u} - \mathbf{u}_h$

Using duality, we can prove the following estimate for $\mathbf{u} - \mathbf{u}_h$ in L^2, which is of optimal order (modulo logarithms for $r = 1$).

Theorem 3.7. *These estimates hold whenever the right-hand side norm is finite*:

$$\|\mathbf{u} - \mathbf{u}_h\| \le C \begin{cases} h^l \|\mathbf{u}\|_l, \quad 2 \le l \le r, & \text{if } r \ge 2, \\ h\big(|\ln h|^{5/2} \|\mathbf{u}\|_{W^1_\infty} + \|\mathbf{u}\|_2\big), & \text{if } r = 1. \end{cases}$$

Proof. Define $\phi \in \Sigma$, $\mathbf{w} \in \mathring{H}(\operatorname{div})$ by

$$B(\tau, \mathbf{v}; \phi, \mathbf{w}) = (\mathbf{v}, \mathbf{u} - \mathbf{u}_h), \quad \tau \in \Sigma, \quad \mathbf{v} \in \mathring{H}(\operatorname{div}).$$

Thus $\mathbf{w}$ solves the Poisson equation $-\Delta \mathbf{w} = \mathbf{u} - \mathbf{u}_h$ in Ω with homogeneous Dirichlet boundary conditions, and $\phi = -\operatorname{rot}\mathbf{w}$. Under our assumption that Ω is a convex polygon, we know that $\mathbf{w} \in H^2$, $\phi \in H^1$, and $\|\phi\|_1 + \|\mathbf{w}\|_2 \le C\|\mathbf{u} - \mathbf{u}_h\|$.

Choosing $\tau = \sigma - \sigma_h$ and $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ and then using Galerkin orthogonality, we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|^2 = B(\sigma - \sigma_h, \mathbf{u} - \mathbf{u}_h; \phi, \mathbf{w}) = B(\sigma - \sigma_h, \mathbf{u} - \mathbf{u}_h; \phi - P_{\Sigma_h}\phi, \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}).$$

The right-hand side is the sum of following four terms:

$$T_1 = (\sigma - \sigma_h, \phi - P_{\Sigma_h}\phi), \qquad T_2 = -(\mathbf{u} - \mathbf{u}_h, \operatorname{curl}[\phi - P_{\Sigma_h}\phi]),$$
$$T_3 = \big(\operatorname{curl}[\sigma - \sigma_h], \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big), \quad T_4 = \big(\operatorname{div}[\mathbf{u} - \mathbf{u}_h], \operatorname{div}\big[\mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big]\big).$$

We have replaced $\langle \cdot, \cdot \rangle$ by the L^2-inner products because $\phi \in H^1$ and $\sigma = \operatorname{rot}\mathbf{u}$ is in H^1 whenever the right-hand side norm in the theorem is finite. For T_1, we use the Cauchy–Schwarz inequality, the bound $\|\phi - P_{\Sigma_h}\phi\| \le Ch\|\phi\|_1 \le Ch\|\mathbf{u} - \mathbf{u}_h\|$ for the elliptic projection, and the estimate of Theorem 3.6 with $p = 2$ to obtain

$$|T_1| \le C \begin{cases} h^l \|\mathbf{u}\|_l \|\mathbf{u} - \mathbf{u}_h\|, \quad 2 \le l \le r, & \text{if } r \ge 2, \\ h(\|\mathbf{u}\|_1 + h\|\mathbf{u}\|_2)\|\mathbf{u} - \mathbf{u}_h\|, & \text{if } r = 1. \end{cases}$$

Similar considerations give the same bound for T_4.

To bound T_2, we split it as $\big(P_{\mathring{V}_h}\mathbf{u} - \mathbf{u}, \operatorname{curl}[\phi - P_{\Sigma_h}\phi]\big)$ and $T_2' = \big(\mathbf{u}_h - P_{\mathring{V}_h}\mathbf{u}, \operatorname{curl}[\phi - P_{\Sigma_h}\phi]\big)$. The first term is clearly bounded by $Ch^l\|\mathbf{u}\|_l\|\mathbf{u} - \mathbf{u}_h\|$, while, for the second, we use (3.15) to find that

$$|T_2'| \le Ch\|\mathrm{curl}(\phi - P_{\Sigma_h}\phi)\|\big\|\mathrm{div}\big(\mathbf{u}_h - P_{\mathring{V}_h}\mathbf{u}\big)\big\|.$$

Bounding $\mathrm{div}\big(\mathbf{u}_h - P_{\mathring{V}_h}\mathbf{u}\big)$ via Theorem 3.6 and (3.21), we get

$$|T_2'| \le Ch^l\|\mathbf{u}\|_l\|\mathbf{u} - \mathbf{u}_h\|, \quad 2 \le l \le r,$$

$$|T_2'| \le Ch(\|\mathbf{u}\|_1 + h\|\mathbf{u}\|_2)\|\mathbf{u} - \mathbf{u}_h\|, \quad r = 1.$$

Finally, we bound T_3. If $r \ge 2$, then we simply use the Cauchy–Schwarz inequality, the bound

$$\big\|\mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big\| \le Ch^2\|\mathbf{w}\|_2 \le Ch^2\|\mathbf{u} - \mathbf{u}_h\|, \tag{3.27}$$

and the $p = 2$ case of Theorem 3.6 to obtain

$$|T_3| \le Ch^l\|\mathbf{u}\|_l\|\mathbf{u} - \mathbf{u}_h\|, \quad 2 \le l \le r.$$

If $r = 1$, then (3.27) does not hold. Instead we split T_3 as $\big(\mathrm{curl}[\sigma - P_{\Sigma_h}\sigma], \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big) + \big(\mathrm{curl}[P_{\Sigma_h}\sigma - \sigma_h], \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big)$. Since $\big\|\mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big\| \le Ch\|\mathbf{w}\|_1 \le Ch\|\mathbf{u} - \mathbf{u}_h\|$, the first term is bounded by $Ch\|\sigma\|_1\|\mathbf{u} - \mathbf{u}_h\| \le Ch\|\mathbf{u}\|_2\|\mathbf{u} - \mathbf{u}_h\|$. For the second, we apply Theorem 3.5 and (3.20) to obtain

$$\begin{aligned}\big|\big(\mathrm{curl}[P_{\Sigma_h}\sigma - \sigma_h], \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big)\big| &\le Ch^{-1/2-1/p}\big\|\mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big\|_{L^p}\|P_{\Sigma_h}\sigma - \sigma_h\| \\ &\le Ch^{1/2-1/p}p\|\mathbf{w}\|_{W_p^1}\|P_{\Sigma_h}\sigma - \sigma_h\|, \quad 2 \le p < \infty.\end{aligned}$$

By the Sobolev inequality, $\|\mathbf{w}\|_{W_p^1} \le K_p\|\mathbf{w}\|_{W_q^2}$, where $q = 2p/(2+p) < 2$. Moreover, from Ref. 18 and a simple extension argument the constant $K_p \le Cp^{1/2}$. Since $\|\mathbf{w}\|_{W_q^2} \le C\|\mathbf{w}\|_2$ with C depending only on the area of the domain, we obtain

$$\big|\big(\mathrm{curl}[P_{\Sigma_h}\sigma - \sigma_h], \mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big)\big| \le Ch^{1/2-1/p}p^{3/2}\|P_{\Sigma_h}\sigma - \sigma_h\|\|\mathbf{u} - \mathbf{u}_h\|, \quad 2 \le p < \infty.$$

By (3.14) and Theorem 3.6 with $r = 1$,

$$\|P_{\Sigma_h}\sigma - \sigma_h\| \le \|\sigma - P_{\Sigma_h}\sigma\| + \|\sigma - \sigma_h\| \le C(h^{1/2-1/p}p\|\mathbf{u}\|_{W_p^1} + h\|\mathbf{u}\|_2).$$

Thus we obtain

$$|T_3| \le C\big(h^{1-2/p}p^{5/2}\|\mathbf{u}\|_{W_p^1} + h^{3/2-1/p}p^{3/2}\|\mathbf{u}\|_2\big)\|\mathbf{u} - \mathbf{u}_h\|, \quad 2 \le p < \infty,$$

and, by choosing $p = |\ln H|$ and noting that $h^{1/2}|\ln h|^{3/2}$ is bounded,

$$|T_3| \le Ch\big(|\ln h|^{5/2}\|\mathbf{u}\|_{W_\infty^1} + \|\mathbf{u}\|_2\big)\|\mathbf{u} - \mathbf{u}_h\|.$$

The theorem follows easily from these estimates. □

4. The Ciarlet–Raviart Mixed Method for the Biharmonic

In this section, we show that the above analysis immediately gives estimates for the Ciarlet–Raviart mixed method for the biharmonic, including some new estimates which improve on those available in the literature.

Given $g \in H^{-2}(\Omega) = (\mathring{H}^2(\Omega))'$, the standard weak formulation of the Dirichlet problem for the biharmonic seeks $U \in \mathring{H}^2$ such that

$$(\Delta U, \Delta V) = (g, V), \quad V \in \mathring{H}^2.$$

Letting $\sigma := -\Delta U \in L^2$, we have $\Delta\sigma = -g$. Assuming that $g \in H^{-1}(\Omega)$, as we henceforth shall, for Ω a convex polygon, we have that $U \in H^3(\Omega)$, $\sigma \in H^1(\Omega)$ and

$$\|U\|_3 + \|\sigma\|_1 \le C\|g\|_{-1}.$$

Hence $(\sigma, U) \in H^1 \times \mathring{H}^1$ satisfy

$$\begin{aligned} (\sigma, \tau) - (\operatorname{curl} U, \operatorname{curl} \tau) &= 0, \qquad \tau \in H^1, \\ (\operatorname{curl} \sigma, \operatorname{curl} V) &= (g, V), \quad V \in \mathring{H}^1. \end{aligned}$$

We note that a mixed formulation in these variables, but with spaces that are less regular, can also be given for this problem,[4] but we shall not pursue this approach here.

The Ciarlet–Raviart mixed method[6] for the approximation of the Dirichlet problem for the biharmonic equation using Lagrange elements of degree r, seeks $\sigma_h \in \Sigma_h$, $U_h \in \mathring{\Sigma}_h$ such that

$$\begin{aligned} (\sigma_h, \tau) - (\operatorname{curl} U_h, \operatorname{curl} \tau) &= 0, \qquad \tau \in \Sigma_h, \\ (\operatorname{curl} \sigma_h, \operatorname{curl} V) &= (g, V), \quad V \in \mathring{\Sigma}_h. \end{aligned}$$

This discretization has been analyzed in many papers under the assumption that Ω is a convex polygon. It has been proven[3,12] that for $r \ge 2$,

$$\|U - U_h\|_1 \le Ch^r \|U\|_{r+1}, \quad \|\sigma - \sigma_h\| \le Ch^{r-1}\|U\|_{r+1}.$$

The former estimate is optimal, while the estimate for $\|\sigma - \sigma_h\|$ is two orders suboptimal. In the case $r = 1$, it has been proven[17] that

$$\|U - U_h\|_1 \le Ch^{3/4}|\ln h|^{3/2}\|U\|_4, \quad \|\sigma - \sigma_h\| \le Ch^{1/2}|\ln h|\|U\|_4.$$

These estimates are suboptimal by 1/4 and 3/2 orders respectively (modulo logarithms) and require H^4 regularity of U. (As has been noted,[17] the same technique could be applied for $r \ge 2$ to obtain a 3/2 suboptimal estimate on $\|\sigma - \sigma_h\|$.) Below we improve the estimate on $\|U - U_h\|_1$ for $r = 1$ to an optimal order estimate (modulo logarithms), with decreased assumptions on the regularity of the solution U.

We now show how to obtain all of these results from the analysis of the previous section, with only minor modifications. Let $\mathbf{u} = \operatorname{curl} U$. Then

$$B(\sigma, \mathbf{u}; \tau, \operatorname{curl} V) = (g, V), \quad (\tau, V) \in H^1 \times \mathring{H}^1.$$

Similarly, with $\mathbf{u}_h = \operatorname{curl} U_h$,

$$B(\sigma_h, \mathbf{u}_h; \tau, \operatorname{curl} V) = (g, V), \quad (\tau, V) \in \Sigma_h \times \mathring{\Sigma}_h.$$

As above, set $\rho = \sigma_h - P_{\Sigma_h}\sigma \in \Sigma_h$, $\mathbf{w} = \mathbf{u}_h - P_{\mathring{V}_h}\mathbf{u} \in \mathring{V}_h$. Note that $\mathbf{w} = \operatorname{curl} U_h - \operatorname{curl} P_{\mathring{\Sigma}_h} U \in \operatorname{curl} \mathring{\Sigma}_h$. Subtracting the above equations and writing $\mathbf{v}$ for $\operatorname{curl} V$, we have

$$B(\rho, \mathbf{w}; \tau, \mathbf{v}) = B\big(\sigma - P_{\Sigma_h}\sigma, \mathbf{u} - P_{\mathring{V}_h}\mathbf{u}; \tau, \mathbf{v}\big), \quad (\tau, \mathbf{v}) \in \Sigma_h \times \operatorname{curl} \mathring{\Sigma}_h.$$

Since the stability result of Theorem 3.3 holds over the space $\Sigma_h \times \operatorname{curl} \mathring{\Sigma}_h$, as stated in the last sentence of the theorem, we can argue exactly as in proof of Theorem 3.6 and conclude that the estimates proved in that theorem for the Hodge Laplacian hold as well in this context with one improvement. To estimate the term $\big\|\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}\big\|_{L^p}$ in (3.26), instead of using (3.20), we note that $\big\|\mathbf{u} - P_{\mathring{V}_h}\mathbf{u}\big\|_{L^p} = \big\|\operatorname{curl}(U - P_{\mathring{\Sigma}_h} U)\big\|_{L^p}$ and invoke (3.16). In this way we avoid a factor of p. The improved estimates of Theorem 3.7 also translate to this problem, with essentially the same proof and a similar improvement. The dual problem is, of course, now taken to be: Find $\phi \in \Sigma, \mathbf{w} \in \operatorname{curl} \mathring{H}^1$ such that

$$B(\tau, \mathbf{v}; \phi, \mathbf{w}) = (\mathbf{v}, \mathbf{u} - \mathbf{u}_h), \quad \tau \in \Sigma, \quad \mathbf{v} \in \operatorname{curl} \mathring{H}^1.$$

Thus $\mathbf{w} = \operatorname{curl} W$, where W solves the biharmonic problem $\Delta^2 W = \operatorname{rot}(\mathbf{u} - \mathbf{u}_h) \in H^{-1}$ with Dirichlet boundary conditions, and $\phi = \Delta W$. The relevant regularity result, valid on a convex domain, is

$$\|\mathbf{w}\|_2 + \|\phi\|_1 \le C\|W\|_3 \le C\|\operatorname{rot}(\mathbf{u} - \mathbf{u}_h)\|_{-1} \le C\|\mathbf{u} - \mathbf{u}_h\|.$$

The remainder of the proof goes through as before, with the simplification that now the terms T_4 and T_2' are zero, and the term $\big\|\mathbf{w} - P_{\mathring{V}_h}\mathbf{w}\big\|_{L^p}$ can be bounded without introducing a factor of p as just described. The suppressed factors of p lead to fewer logarithms in the final result. Stating this result in terms of the original variable U instead of $\mathbf{u} = \operatorname{curl} U$, we have the following theorem.

Theorem 4.1. *Let U solve the Dirichlet problem for the biharmonic equation, $\sigma = -\Delta U$, and let $U_h \in \mathring{\Sigma}_h$, $\sigma_h \in \Sigma_h$ denote the discrete solution obtained by the Ciarlet–Raviart mixed method with Lagrange elements of degree $r \ge 1$. If $r \ge 2$ and $2 \le l \le r$, then the following estimates, requiring differing amounts of regularity,* *hold whenever the norms on the right-hand side are finite*:

$$\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 \le C \begin{cases} h^{l-1}\|U\|_{l+1} \\ h^{l-1/2}\big(\|U\|_{W^{l+1}_\infty} + \|U\|_{l+3/2}\big), \end{cases}$$

$$\|U - U_h\|_1 \le Ch^l\|U\|_{l+1}.$$

If $r = 1$, *the estimates are*:

$$\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 \le C \begin{cases} (\|U\|_2 + h\|U\|_3), \\ h^{1/2}(\|U\|_{W^2_\infty} + h^{1/2}\|U\|_3), \end{cases}$$

$$\|U - U_h\|_1 \le Ch\big(|\ln h|^{1/2}\|U\|_{W^2_\infty} + \|U\|_3\big).$$

5. Stationary Stokes Equations

Another application in which the vector Laplacian with Dirichlet boundary conditions arises is the stationary Stokes equations, in which the vector field represents the velocity, subject to no-slip conditions on the boundary. A standard weak formulation (with viscosity equal to one) seeks $\mathbf{u} \in \mathring{H}^1(\Omega, \mathbb{R}^2)$ and $p \in \hat{L}^2$ such that

$$\begin{aligned} (\operatorname{grad}\mathbf{u}, \operatorname{grad}\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \quad && \mathbf{v} \in \mathring{H}^1(\Omega, \mathbb{R}^2), \\ (\operatorname{div}\mathbf{u}, q) &= 0, && q \in L^2. \end{aligned}$$

Mixed methods, such as we have considered, have been used to approximate this problem, based on the vorticity-velocity-pressure formulation. For example, using the spaces defined in Sec. 3, the following weak formulation has been discussed.[10] Find $\sigma \in \Sigma$, $\mathbf{u} \in \mathring{H}(\operatorname{div})$, $p \in \hat{L}^2$ such that

$$\begin{aligned} (\sigma, \tau) - \langle \operatorname{curl}\tau, \mathbf{u}\rangle &= 0, && \tau \in \Sigma, \\ \langle \operatorname{curl}\sigma, \mathbf{v}\rangle - (p, \operatorname{div}\mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \quad && \mathbf{v} \in \mathring{H}(\operatorname{div}), \\ (\operatorname{div}\mathbf{u}, q) &= 0, && q \in L^2. \end{aligned}$$

This formulation is obtained just as for the vector Laplacian, by writing

$$(\operatorname{grad}\mathbf{u}, \operatorname{grad}\mathbf{v}) = (\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v}) + (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})$$

and introducing the variable $\sigma = \operatorname{rot}\mathbf{u}$. When $\mathbf{f} \in L^2(\Omega; \mathbb{R}^2)$ and Ω is a convex polygon, $\mathbf{u} \in H^2(\Omega; \mathbb{R}^2)$, $p \in \hat{H}^1(\Omega)$, and $\sigma = \operatorname{rot}\mathbf{u} \in H^1(\Omega)$. Assuming this extra regularity, and setting $\mathbf{u} = \operatorname{curl} U$, and $\mathbf{v} = \operatorname{curl} V$, $(\sigma, U) \in H^1 \times \mathring{H}^1$ satisfy the stream function-vorticity equations:

$$\begin{aligned} (\sigma, \tau) - (\operatorname{curl} U, \operatorname{curl}\tau) &= 0, && \tau \in H^1, \\ (\operatorname{curl}\sigma, \operatorname{curl} V) &= (\mathbf{f}, \operatorname{curl} V), \quad && V \in \mathring{H}^1. \end{aligned}$$

Taking $g = \operatorname{rot}\mathbf{f}$, this formulation coincides with the mixed formulation of the biharmonic problem discussed in the previous section.

We consider here the finite element approximation which seeks $\sigma_h \in \Sigma_h$, $\mathbf{u}_h \in \mathring{V}_h$, $p_h \in \hat{S}_h$ such that

$$\begin{aligned} (\sigma_h, \tau) - (\mathbf{u}_h, \operatorname{curl} \tau) &= 0, & \tau &\in \Sigma_h, \\ (\operatorname{curl} \sigma_h, \mathbf{v}) - (p_h, \operatorname{div} \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), & \mathbf{v} &\in \mathring{V}_h, \\ (\operatorname{div} \mathbf{u}, q) &= 0, & q &\in \hat{S}_h, \end{aligned}$$

where the spaces Σ_h, $\mathring{V}_h$ and $\hat{S}_h$ are defined as above. The existence and uniqueness of the solution is easily established by standard methods. When $\mathbf{f} = 0$, we get by choosing $\tau = \sigma_h$, $\mathbf{v} = \mathbf{u}_h$, $q = p_h$ and adding the equations that $\sigma_h = 0$ and $\operatorname{div} \mathbf{u}_h = 0$. Hence $\mathbf{u}_h = \operatorname{curl} U_h$, $U_h \in \mathring{\Sigma}_h$, and choosing $\tau = U_h$, we see that $\operatorname{curl} U_h = 0$. Since $\operatorname{div} \mathring{V}_h = \hat{S}_h$, we also get $p_h = 0$.

Error estimates for $\|\mathbf{u} - \mathbf{u}_h\|$ and $\|\sigma - \sigma_h\|$ are easily obtained by reducing the problem to its stream function-vorticity form and using the estimates obtained in the previous section. Letting $\mathbf{u}_h = \operatorname{curl} U_h$, and choosing $\mathbf{v} = \operatorname{curl} V$, $V \in \mathring{\Sigma}_h$, we see that (σ_h, U_h) is the unique solution of the Ciarlet–Raviart formulation of the biharmonic with $g = \operatorname{rot} \mathbf{f}$. Hence, the estimates for $\sigma - \sigma_h$ in Theorem 4.1 remain unchanged, except that we can replace $\|U\|_s$ by $\|\mathbf{u}\|_{s-1}$. In particular, we have the following theorem.

Theorem 5.1. *Let* $(\mathbf{u}, p)$ *solve the Dirichlet problem for the Stokes equation*, $\sigma = \operatorname{rot} \mathbf{u}$, *and let* $\mathbf{u}_h \in \mathring{V}_h$, $\sigma_h \in \Sigma_h$ *and* $p_h \in \hat{S}_h$ *denote the discrete solution obtained by the vorticity-velocity-pressure mixed method with* $r \geq 1$ *the polynomial degree. If* $r \geq 2$ *and* $2 \leq l \leq r$, *then the following estimates, requiring different amounts of regularity, hold whenever the norms on the right-hand side are finite*:

$$\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 \leq C \begin{cases} h^{l-1}\|\mathbf{u}\|_l, \\ h^{l-1/2}(\|\mathbf{u}\|_{W^l_\infty} + \|\mathbf{u}\|_{l+1/2}), \end{cases}$$

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^l\|\mathbf{u}\|_l.$$

If $r = 1$, *the estimates are*:

$$\|\sigma - \sigma_h\| + h\|\sigma - \sigma_h\|_1 \leq C \begin{cases} \|\mathbf{u}\|_1 + h\|\mathbf{u}\|_2, \\ h^{1/2}\big(\|\mathbf{u}\|_{W^1_\infty} + h^{1/2}\|\mathbf{u}\|_2\big), \end{cases}$$

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch\big(|\ln h|^{1/2}\|\mathbf{u}\|_{W^1_\infty} + \|\mathbf{u}\|_2\big).$$

The only item remaining is to derive error bounds for the approximation of the pressure. We obtain the following result, which gives error bounds that are suboptimal by $O(h^{1/2})$.

Theorem 5.2. *If* $r \geq 2$ *and* $2 \leq l \leq r$, *then*

$$\|p - p_h\| \leq C \begin{cases} h^{l-1}(\|\mathbf{u}\|_l + \|p\|_{l-1}), \\ h^{l-1/2}\big(\|\mathbf{u}\|_{W^l_\infty} + \|\mathbf{u}\|_{l+1/2} + \|p\|_{l-1/2}\big). \end{cases}$$

If $r = 1$, the estimates are

$$\|p - p_h\| \le C \begin{cases} \|\mathbf{u}\|_1 + h\|\mathbf{u}\|_2 + \|p\|, \\ h^{1/2}\big(\|\mathbf{u}\|_{W^1_\infty} + h^{1/2}\|\mathbf{u}\|_2 + \|p\|_{1/2}\big). \end{cases}$$

Proof. From the variational formulation, we get the error equation

$$(p_h - P_{S_h}p, \operatorname{div}\mathbf{v}_h) = (p - P_{S_h}p, \operatorname{div}\mathbf{v}_h) + (\operatorname{curl}[\sigma_h - \sigma], \mathbf{v}_h), \quad \mathbf{v}_h \in \mathring{V}_h.$$

We choose $\mathbf{v} \in \mathring{H}^1(\Omega;\mathbb{R}^2)$ such that $\operatorname{div}\mathbf{v} = p_h - P_{S_h}p$ and $\|\mathbf{v}\|_1 \le C\|p_h - P_{S_h}p\|$, and take $\mathbf{v}_h = \Pi_h^V\mathbf{v}$. We have that $\operatorname{div}\mathbf{v} = \operatorname{div}\Pi_h^V\mathbf{v}$ and $\|\Pi_h^V\mathbf{v}\|_{H(\mathrm{div})} \le C\|\mathbf{v}\|_1 \le C\|p_h - P_{S_h}p\|$, so

$$\begin{aligned}
\|p_h - P_{S_h}p\|^2 &= \big(p_h - P_{S_h}p, \operatorname{div}\Pi_h^V\mathbf{v}\big) \\
&= \big(p - P_{S_h}p, \operatorname{div}\Pi_h^V\mathbf{v}\big) + \big(\operatorname{curl}[\sigma_h - \sigma], \Pi_h^V\mathbf{v}\big), \\
&= (p - P_{S_h}p, p_h - P_{S_h}p) + \big(\operatorname{curl}[\sigma_h - \sigma], \Pi_h^V\mathbf{v} - \mathbf{v}\big) + (\sigma_h - \sigma, \operatorname{rot}\mathbf{v}) \\
&\le C(\|p - P_{S_h}p\| + h\|\operatorname{curl}(\sigma_h - \sigma)\| + \|\sigma_h - \sigma\|)\|p_h - P_{S_h}p\|.
\end{aligned}$$

It easily follows that

$$\|p - p_h\| \le C(\|p - P_{S_h}p\| + \|\sigma_h - \sigma\| + h\|\operatorname{curl}(\sigma_h - \sigma)\|).$$

The theorem follows directly by applying Theorem 5.1 and standard estimates for the error in the L^2 projection. □

A number of papers have been devoted to finite element approximation schemes of either the vorticity-velocity-pressure or stream-function-vorticity formulation of the Stokes problem. In particular, the lowest-order ($r = 1$) case of the method analyzed here was discussed in Ref. 9 (in which additional references can also be found). In the case of the magnetic boundary conditions, $\sigma = \mathbf{u}\cdot\mathbf{n} = 0$, the authors established stability and first-order convergence, which is optimal, for all variables. But for the no-slip boundary conditions $\mathbf{u} = 0$, with which we are concerned and which arise much more commonly in Stokes flow, they observe in numerical experiments stability problems and reduced rates of convergence which are in agreement with the theory presented above.

We close with a simple numerical example in the case $r = 2$ that demonstrates that the suboptimal convergence orders obtained above are sharp even for very smooth solutions. Our discretization of the vorticity-velocity-pressure mixed formulation of the Stokes problem then approximates the velocity $\mathbf{u}$ by the second lowest-order Raviart–Thomas elements, the vorticity σ by continuous piecewise quadratic functions, and the pressure p by discontinuous piecewise linear functions. We take Ω to be the unit square and compute $\mathbf{f}$ corresponding to the polynomial solution velocity field $\mathbf{u} = (-2x^2(x-1)^2y(2y-1)(y-1),\ 2y^2(y-1)^2x(2x-1)(x-1))$, and pressure $p = (x-1/2)^5 + (y-1/2)^5$. The computations, summarized in Table 3, indeed confirm the convergence rates established above, i.e. $\mathbf{u}_h$ converges with optimal

Table 3. L^2 errors and convergence rates for the mixed finite element approximation of the Stokes problem for the vector Laplacian with boundary conditions $\mathbf{u}\cdot\mathbf{n} = 0$, $\mathbf{u}\cdot\mathbf{s} = 0$ on the unit square.

$\|\mathbf{u}-\mathbf{u}_h\|$	Rate	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|$	Rate	$\|\sigma-\sigma_h\|$	Rate	$\|\mathrm{curl}(\sigma-\sigma_h)\|$	Rate
3.26e-04	1.9	2.34e-03	1.3	2.70e-03	1.3	1.67e-01	0.2
8.35e-05	2.0	8.05e-04	1.5	9.70e-04	1.5	1.24e-01	0.4
2.10e-05	2.0	2.74e-04	1.6	3.47e-04	1.5	8.96e-02	0.5
5.27e-06	2.0	9.39e-05	1.6	1.24e-04	1.5	6.42e-02	0.5

order 2 to $\mathbf{u}$ in L^2, while the approximations to σ and $\mathrm{curl}\,\sigma$ are both suboptimal by 3/2 order and the approximation to the pressure p is suboptimal by 1/2 order.

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