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ANALYSIS OF A ONE-DIMENSIONAL VARIATIONAL MODEL OF THE EQUILIBRIUM SHAPEL OF A DEFORMABLE CRYSTAL *

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Abstract. The equilibrium configurations of a one-dimensional variational model that combines terms expressing the bulk energy of a deformable crystal and its surface energy are studied. After elimination of the displacement, the problem reduces to the minimization of a nonconvex and nonlocal functional of a single function, the thickness. Depending on a parameter which strengthens one of the terms comprising the energy at the expense of the other, it is shown that this functional may have a stable absolute minimum or only a minimizing sequence in which the term corresponding to the bulk energy is forced to zero by the production of a crack in the material.

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1. Introduction

The morphological instabilities of interfaces is a topic of primary interest in physics (*e.g.*, see [4]). Currently, many branches of the natural sciences, including low temperature physics, fracture, crystal growth, epitaxy of nano-scale films, metallurgy, geology, and materials science show a rapidly growing interest in the so called stress driven rearrangement instabilities (SDRI) of surfaces and interfaces in solids. Several examples of the SDRI have been predicted on the basis of Gibbs thermodynamics [5] of heterogeneous systems by studying the positive definiteness of the second energy variations [7] of relevant functionals. At present, some of the predicted instabilities have been confirmed experimentally and found applications in the above mentioned areas.

The thermodynamics of deformable solids with rearrangement leads to certain multi-dimensional variational problems with unknown boundaries and with different specific constraints. Despite its quite simple formulation, the problem in all its entirety is quite complex, and the study of its different features with the help of simpler examples seems quite desirable. Many mathematical aspects of the general problem of thermodynamics of solids with rearrangement can be studied in the framework of the problem of equilibrium shape of deformable crystals formulated in [7, 8]. This problem is of a certain physical interest on its own in the theory of nano-scale solid crystals [10]. Probably, it is the simplest mathematical problem possessing all of the crucial features of the most general situation. From a mathematical point of view, the problem of the equilibrium shape of a deformable

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crystal is the natural synthesis of two classical problems of mathematical physics: (i) the problem of equilibrium shape of a rigid crystal of fixed total volume [11,12] and (ii) the problem of the equilibrium of an elastic solid with fixed geometry. The symbiosis, however, gives some qualitatively new features absent in the ingredients. Some valuable analytical facts for this problem can be established with the help of Nozieres's results [13]. Because of the existing difficulties of the general 3D-problem, it is expedient to analyze first its simpler one-dimensional version which is studied in this and in a forthcoming paper. The one-dimensional problem has been formulated in [9] and it allows us to describe some phenomena in elastic shells and strips with movable defects. Below, we present without derivation some facts relating to this problem. Mathematically it is formulated as the minimization of the functional $\mathcal{E}$ depending on two unknown functions: an elastic displacement $u(x)$ and a strip thickness $h(x)$ of one variable x:

$$\mathcal{E} = \int_0^L \left[(G/2)h(x)[u'(x)]^2 + \sigma\sqrt{1+[h'(x)]^2}\right] \mathrm{d}x$$

where $G > 0$ is the elastic modulus, $\sigma > 0$ is the surface energy, $u'(x)$ is the elastic deformation, and $\sqrt{1+[h'(x)]^2}\,\mathrm{d}x$ is the length element of an outer boundary of the strip.

We assume that the elastic displacements $u(x)$ and the thickness $h(x)$ are fixed at the end-points, *i.e.*,

$$u(0) = U_0, \qquad u(L) = U_L, \qquad h(0) = h_0, \qquad h(L) = h_L$$

and that the total volume of the strip is also fixed, *i.e.*,

$$\int_0^L h(x)\,\mathrm{d}x = A.$$

For simplicity, we consider the case when

$$L = 1, \quad U_0 = 0, \quad U_L = 1, \quad h_0 = 1, \quad h_L = 1, \quad A = 1, \quad \sigma = 1, \quad G = 2N.$$

We are thus led to the minimization problem: Find

$$\inf_{u\in V, h\in H} \mathcal{E}(u,h) = \inf_{u\in V, h\in H} N\int_0^1 h(x)[u'(x)]^2\,\mathrm{d}x + \int_0^1 \sqrt{1+[h'(x)]^2}\,\mathrm{d}x. \tag{1.1}$$

The set of admissible displacements is $V = \{u \in H^1(0,1) : u(0) = 0, u(1) = 1\}$, and the admissible thicknesses lie in the set H of piecewise $\mathcal{C}^1$ functions on $[0,1]$ satisfying

$$h(x) > 0 \text{ in } [0,1], \qquad h(0) = h(1) = 1, \qquad \int_0^1 h(x)\,\mathrm{d}x = 1. \tag{1.2}$$

For a given thickness $h \in H$, one can easily check that

$$u_h'(x) = \left(\int_0^1 [h(x)]^{-1}\,\mathrm{d}x\right)^{-1} \frac{1}{h(x)}$$

minimizes $\mathcal{E}(u,h)$ in V. Thus the displacement can be eliminated in (1.1) and the original problem reduces to minimizing over $h \in H$ the functional

$$\mathcal{I}(h) = \frac{N}{\int_0^1 [h(x)]^{-1}\,\mathrm{d}x} + \int_0^1 \sqrt{1+[h'(x)]^2}\,\mathrm{d}x. \tag{1.3}$$

It is a standard feature in such problems of the calculus of variations, that $\mathcal{I}$ may not attain its infimum on the space of $\mathcal{C}^1$ functions. Generally, minimizing sequences may develop oscillations if $\mathcal{I}$ does not have the right properties of convexity. In the case at hand, the second term of $\mathcal{I}$ is convex since

$$\frac{\partial^2}{\partial f^2}\sqrt{1+f^2} = \left(1+f^2\right)^{-3/2} > 0,$$

but the first term is concave, so the standard direct method is not applicable. Minimizing sequences may also tend to functions which lie outside the initial set of candidates and which are usually less regular. To ensure well-posedness, the problem must be relaxed: a larger class of admissible designs must be allowed and the functional must be extended accordingly [3].

The uniform thickness $h_0 \equiv 1$ will be called the trivial solution. The value of its energy is $N+1$. One readily checks that h_0 satisfies the Euler-Lagrange equation associated to (1.3) (however, this is not a sufficient condition for h_0 to be the absolute minimum!). Many other examples of variational problems whose minimizers do not satisfy the Euler-Lagrange equation can be found in [1]. Because of the nonlocal nature of the term corresponding to the bulk energy in the functional $\mathcal{I}(h)$, the problem discussed here falls outside of the classical theory.

The main results of the paper are the following. In the next section, we consider the standard linearized stability analysis and show that the second variation of the energy for smooth perturbations about the thickness $h_0 \equiv 1$ is positive for $N \leq 2\pi^2$. However, this does not guarantee that $h_0 \equiv 1$ is a minimizer even for N in this range. In Section 3, we show that there exists an $N_0 > 0$ (≈ 1.159) such that for all $N \leq N_0$, $h_0 \equiv 1$ is an absolute minimizer of the functional $\mathcal{I}$. In Section 4, we prove that for $N \geq 2$, $\inf_{h\in H} \mathcal{I}(h) \geq 2 + \pi/4$. Then, in the following section, we explicitly construct a minimizing sequence $h_\epsilon \in H$ such that $\mathcal{I}(h_\epsilon) \to 2+\pi/4$ as $\epsilon \to 0$, which proves that $\inf_{h\in H}\mathcal{I}(h) = 2+\pi/4$. For this minimizing sequence, the term corresponding to the bulk elastic energy tends to 0, and the functional reduces to a measure of the length of the curve defined by h_ϵ. The disappearance of the bulk energy term is achieved by the production of a crack in the specimen and the energy cost for this is equal to twice the extra length induced by the crack. This is shown explicitly by the construction of a non-parametric curve $\mathcal{H}_*$, the length of which equals $2+\pi/4$, such that h_ϵ converges to $\mathcal{H}_*$ a.e. Finally, Section 6 states a relaxation result: since minimizing sequences for $\mathcal{I}$ satisfy natural bounds in the space BV of functions of bounded variation, we define an extension $\mathcal{J}$ of $\mathcal{I}$ on a compact set of BV functions and show that this extension is lower semi-continuous with respect to BV.

2. Stability for the linearized problem

In this section, we consider the standard linearized stability analysis for the trivial solution $h_0 \equiv 1$ and establish the following result.

Lemma 2.1. *If $N \leq 2\pi^2$, and k is a smooth function satisfying $\int_0^1 k\,\mathrm{d}x = 0$, then $D^2\mathcal{I}(h_0)k\otimes k > 0$.*

Before proving this result, we note that we shall show in Section 4 that $h_0 \equiv 1$ is not a minimum for values of N which are much lower than $2\pi^2$. This is not contradictory with the lemma, since the linearized analysis only gives information about smooth perturbations.

Proof. If an admissible function h is smooth, bounded away from 0, and if k is a smooth function such that $\int_0^1 k\,\mathrm{d}x = 0$, then

$$\begin{aligned}\mathcal{I}(h+\epsilon k) &= \mathcal{I}(h) + \epsilon \int_0^1 \frac{h'(x)k'(x)}{\sqrt{1+[h'(x)]^2}}\,\mathrm{d}x + N\epsilon \int_0^1 \frac{k(x)}{[h(x)]^2}\,\mathrm{d}x \left(\int_0^1 [h(x)]^{-1}\,\mathrm{d}x\right)^{-2} \\ &+ N\epsilon^2 \left(\int_0^1 \frac{k(x)}{[h(x)]^2}\,\mathrm{d}x\right)^2 \left(\int_0^1 [h(x)]^{-1}\,\mathrm{d}x\right)^{-3} - N\epsilon^2 \int_0^1 \frac{[k(x)]^2}{[h(x)]^3}\,\mathrm{d}x \left(\int_0^1 [h(x)]^{-1}\,\mathrm{d}x\right)^{-2} \\ &+ \frac{\epsilon^2}{2} \int_0^1 \frac{[k'(x)]^2}{(1+[h'(x)]^2)^{3/2}}\,\mathrm{d}x + O(\epsilon^3).\end{aligned}$$

In particular, for the function $h_0 \equiv 1$, the above becomes

$$\mathcal{I}(h_0+\epsilon k) = \mathcal{I}(h_0) + \frac{\epsilon^2}{2} \int_0^1 \left([k'(x)]^2 - 2N[k(x)]^2\right)\,\mathrm{d}x + O(\epsilon^3).$$

Hence $h_0 \equiv 1$ has a lower energy than a smooth perturbation, provided that

$$\int_0^1 \left([k'(x)]^2 - 2N[k(x)]^2\right)\,\mathrm{d}x \geq 0 \quad \forall\, k \in H_0^1(0,1) \text{ such that } \int_0^1 k\,\mathrm{d}x = 0. \tag{2.1}$$

Now the functions $e_n(x) = \sin(n\pi x)$, $n \geq 1$, form a basis of $H_0^1(0,1)$ and satisfy

$$\begin{gathered}\int_0^1 (e_n')^2(x)\,\mathrm{d}x = n^2\pi^2 \int_0^1 e_n^2(x)\,\mathrm{d}x = n^2\pi^2/2, \\ \int_0^1 e_{2n}(x)\,\mathrm{d}x = 0, \qquad \int_0^1 e_{2n+1}(x)\,\mathrm{d}x = \frac{1}{2n+1}\frac{2}{\pi}\cdot\end{gathered}$$

Let $k(x) = \sum_{n\geq 1} a_n e_n(x)$. The condition that the average of k vanishes yields

$$\begin{aligned}a_1^2 &= \left(\pi/2 \int_0^1 [k(x) - a_1 e_1(x)]\,\mathrm{d}x\right)^2 = \pi^2/4 \left(\int_0^1 \sum_{p\geq 1} a_{2p+1} e_{2p+1}(x)\,\mathrm{d}x\right)^2 \\ &= \pi^2/4 \left(\sum_{p\geq 1} a_{2p+1} \frac{1}{2p+1}\frac{2}{\pi}\right)^2 \leq \sum_{p\geq 1} a_{2p+1}^2 \sum_{p\geq 1} \frac{1}{(2p+1)^2} \\ &= \left(\frac{\pi^2}{8} - 1\right) \sum_{p\geq 1} a_{2p+1}^2.\end{aligned}$$

Condition (2.1) reduces to

$$\sum_{n\geq 1} a_n^2 (n^2\pi^2 - 2N) \geq 0.$$

Obviously, this condition is fulfilled if $2N \leq \pi^2$. Using (2.1) directly, we see that N must be smaller than $2\pi^2$, since the second eigenfunction e_2 has a zero average. However, if $\pi^2 - 2N < 0$, the estimate on a_1 yields

$$\sum_{n\geq 1} a_n^2(n^2\pi^2 - 2N) \geq \sum_{p\geq 1} a_{2p+1}^2[(2p+1)^2\pi^2 - 2N + (\pi^2 - 2N)(\pi^2/8 - 1)] + \sum_{p\geq 1} a_{2p}^2(4p^2\pi^2 - 2N).$$

Since the factor in the first sum of the expression on the right hand side is positive for $N \leq 2\pi^2$, we conclude that $D^2\mathcal{I}(h_0)k \otimes k$ is positive for N in this range. ☐

3. Stability of the trivial solution

In this section, the trivial solution $h_0 \equiv 1$ is shown to be the unique minimum of $\mathcal{I}$, if N is sufficiently small. Specifically, we prove the following.

Theorem 3.1. *The trivial solution $h_0 \equiv 1$ is a stable minimum with respect to perturbations of magnitude $k < 1$, provided that $N \leq \psi(k) \equiv (\sqrt{1+4k^2} - 1)(1 - k + k^2)/k^2$. Also, h_0 is an absolute minimum if $N \leq N_0 \equiv \inf_{0<k\leq 1} \psi(k) \approx 1.16$.*

Proof. We begin by seeking a lower bound for the elastic energy that is quadratic in terms of the maximal and minimal values of h, for any admissible thickness $h \in H$. Since I is translation invariant, we can always assume that

$$h(x) = 1 + K(x) \geq 1 \quad \text{on } [0,\alpha], \qquad h(x) = 1 - k(x) \leq 1 \quad \text{on } [\alpha, 1].$$

The volume constraint on h becomes

$$\int_0^\alpha K(x)\,\mathrm{d}x - \int_\alpha^1 k(x)\,\mathrm{d}x = 0. \tag{3.1}$$

Let $1 + K_0$ and $1 - k_0$ denote the maximum and minimum of h, $0 \leq K_0$, $0 \leq k_0 < 1$. Straightforward computations show that if $\lambda_0 = (1 - k_0)^{-1}$,

$$\frac{1}{1+K} \leq 1 - K + K^2 \qquad \forall\, 0 \leq K \leq K_0, \qquad \frac{1}{1-k} \leq 1 + k + \lambda_0 k^2 \qquad \forall\, 0 \leq k \leq k_0.$$

Using (3.1), it follows that

$$\begin{aligned}\int_0^1 [h(x)]^{-1}\,\mathrm{d}x &= \int_0^\alpha [1 + K(x)]^{-1}\,\mathrm{d}x + \int_\alpha^1 [1 - k(x)]^{-1}\,\mathrm{d}x \\ &\leq \int_0^\alpha [1 - K(x) + K^2(x)]\,\mathrm{d}x + \int_\alpha^1 [1 + k(x) + \lambda_0 k^2(x)]\,\mathrm{d}x \\ &\leq 1 + \alpha K_0^2 + (1-\alpha)\lambda_0 k_0^2.\end{aligned}$$

Thus, the elastic part of the energy can be estimated by

$$\frac{N}{\int_0^1 [h(x)]^{-1}\,\mathrm{d}x} \geq \frac{N}{1 + \alpha K_0^2 + (1-\alpha)\lambda_0 k_0^2}\,. \tag{3.2}$$

On the other hand, a term such as $\int_0^\alpha \sqrt{1+(h')^2}\,\mathrm{d}x$ is the length of a curve that joins the points $(0,1)$ to $(\alpha,1)$, and that rises up to the level $1+K_0$. Suppose that $h(\gamma\alpha)=1+K_0$ for some $0<\gamma<1$. The Jensen inequality applied to the convex function $\sqrt{1+x^2}$ yields

$$\int_0^{\gamma\alpha} \sqrt{1+(h')^2}\,\mathrm{d}x \geq \gamma\alpha\left(1+\left[\int_0^{\gamma\alpha} h'(x)\frac{\mathrm{d}x}{\gamma\alpha}\right]^2\right)^{1/2} = \sqrt{\gamma^2\alpha^2+K_0^2}.$$

Similarly, on the piece $[\gamma\alpha,\alpha]$, we have

$$\int_{\gamma\alpha}^{\alpha} \sqrt{1+(h')^2}\,\mathrm{d}x \geq \sqrt{(1-\gamma)^2\alpha^2+K_0^2}.$$

Using the convexity of $\sqrt{\alpha^2+x^2}$,

$$\begin{aligned}\sqrt{\gamma^2\alpha^2+K_0^2}+\sqrt{(1-\gamma)^2\alpha^2+K_0^2} &= \gamma\sqrt{\alpha^2+\left(\frac{K_0}{\gamma}\right)^2}+(1-\gamma)\sqrt{\alpha^2+\left(\frac{K_0}{(1-\gamma)}\right)^2}\\ &\geq \sqrt{\alpha^2+4K_0^2}.\end{aligned}$$

Hence we obtain

$$\int_0^{\alpha} \sqrt{1+(h')^2}\,\mathrm{d}x \geq \sqrt{\alpha^2+4K_0^2}.$$

A similar estimate holds on the portion $[\alpha,1]$, with a lower bound $\sqrt{(1-\alpha)^2+4k_0^2}$. Again, by convexity, adding these two estimates yields

$$\int_0^{1} \sqrt{1+(h')^2}\,\mathrm{d}x \geq \sqrt{1+4(K_0+k_0)^2}. \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\mathcal{I}(h) \geq \frac{N}{1+\alpha K_0^2+(1-\alpha)\lambda_0 k_0^2}+\sqrt{1+4(K_0+k_0)^2}.$$

As a function of α, the first term on the right hand side is increasing if $K_0^2<\lambda_0 k_0^2$. In this case, the lowest value corresponds to $\alpha=0$ so that

$$\mathcal{I}(h) \geq \frac{N(1-k_0)}{1-k_0+k_0^2}+\sqrt{1+4k_0^2}.$$

Thus, $\mathcal{I}(h)\geq\mathcal{I}(h_0)$, provided N is less than

$$\psi_m(k_0)=\frac{\sqrt{1+4k_0^2}-1}{k_0^2}(1-k_0+k_0^2).$$

If, on the other hand, $K_0^2\geq\lambda_0 k_0^2$, then the lowest value of the bound corresponds to $\alpha=1$, and then

$$\mathcal{I}(h) \geq \frac{N}{1+K_0^2}+\sqrt{1+4K_0^2}.$$

The trivial solution achieves the smallest bound, provided N is smaller than

$$\psi_M(K_0) = \frac{\sqrt{1+4K_0^2}-1}{K_0^2}(1+K_0^2).$$

The first statement of the theorem then follows from the observation that $\psi_m(k) < \psi_M(k)$ for $k \in (0,1)$. This together with some straightforward computations which show that ψ_M is an increasing function of k and that $\inf_{0<k\leq 1} \psi(k) \approx 1.16$ establish the second statement. □

4. A generalized minimizer for $N \geq 2$

In this section, we compute the infimum of (1.3) for values of $N \geq 2$ and show that it corresponds to the length of a parametric curve representing a generalized thickness.

Theorem 4.1. *If $N \geq 2$, then $\inf_{h\in H} \mathcal{I}(h) \geq 2 + \pi/4$. In addition, if $\mathcal{H}_*$ is the parametric curve defined by the functions*

$$h_*(x) = 1 - \pi/8 + \sqrt{(x+x_*)(1-x-x_*)} \qquad \text{if } 0 \leq x < 1 - x_*,$$
$$h_*(x) = 1 - \pi/8 + \sqrt{(x+x_*-1)(2-x-x_*)} \qquad \text{if } 1 - x_* \leq x \leq 1,$$

and the segment $x = 1 - x_*$, $0 < y < 1 - \pi/8$, with $x_* = (4 - \sqrt{16-\pi^2})/8$, then the infimum of $\mathcal{I}$ corresponds to the length of $\mathcal{H}_*$, where the length of the vertical part of $\mathcal{H}_*$ is counted twice.

Proof. To establish this result, we rewrite the minimization problem in the following form.

$$\inf_{h\in H} \mathcal{I}(h) = \inf_{0<\epsilon\leq 1} \left(\inf_{h\in H_\epsilon} \mathcal{I}(h) \right), \tag{4.1}$$

where H_ϵ is the set of piecewise $\mathcal{C}^1$ functions satisfying the constraints (1.2) and

$$\min_{x\in[0,1]} h(x) = \epsilon.$$

For $h \in H_\epsilon$, $1/h \leq 1/\epsilon$, so the first term in $\mathcal{I}$ is bounded from below by $N\epsilon$. Thus, we get

$$\inf_{h\in H_\epsilon} \mathcal{I}(h) \geq N\epsilon + \inf_{h\in H_\epsilon} \mathcal{L}(h), \tag{4.2}$$

where $\mathcal{L}(h) \equiv \int_0^1 \sqrt{1+[h'(x)]^2}\, \mathrm{d}x$ is the length of the curve h. The second term in the above expression is the minimal length of a curve that takes the value 1 at its end points, reaches the value ϵ as its minimum, and bounds an area equal to 1.

Let F_ϵ be the set of piecewise $\mathcal{C}^1$ curves satisfying

$$f(x) \geq \epsilon \text{ in } [0,1], \qquad f(0) = f(1) = \epsilon, \qquad \int_0^1 f(x)\, \mathrm{d}x = 1.$$

To each element f of F_ϵ, we associate an element h of H_ϵ in the following way. If $0 < \epsilon < 1$, the area constraint forces f to take the value 1. Let x_1 be the first point where $f = 1$. Set

$$h(x) = f(x + x_1) \qquad \text{for } 0 \leq x \leq 1 - x_1, \qquad h(x) = f(x - 1 + x_1) \qquad \text{for } 1 - x_1 < x \leq 1.$$

Since the volume constraint and the length of the curve are translation invariant, the function h lies in H_ϵ. In a similar fashion, we can associate to a function $h \in H_\epsilon$, a function $f \in F_\epsilon$: if x_ϵ is the first point where h achieves the value ϵ, we set

$$f(x) = h(x + x_\epsilon) \qquad \text{for } 0 \le x \le 1 - x_\epsilon, \qquad f(x) = h(x - 1 + x_\epsilon) \qquad \text{for } 1 - x_\epsilon \le x \le 1.$$

It follows that the infimum of $\mathcal{L}$ can be computed either on H_ϵ or on F_ϵ. The latter is a case of the isoperimetric problem. Its solution is described in the next proposition, the proof of which is given in the Appendix.

Proposition 4.2. *If $(1 - \pi/8) \le \epsilon \le 1$, the curve of minimal length, with value ϵ at its end points, lying above the value ϵ, and bounding an area equal to 1, is the arc of circle of radius R_ϵ given by*

$$1 - \epsilon = -\sqrt{4R_\epsilon^2 - 1}/4 + R_\epsilon^2 \arcsin(1/[2R_\epsilon]). \tag{4.3}$$

Moreover, the corresponding length is $\inf_{f \in F_\epsilon} \mathcal{L}(f) = 2R_\epsilon \arcsin(1/[2R_\epsilon])$.

If $0 \le \epsilon < 1 - \pi/8$, the infimum of $\mathcal{L}(f)$ is attained by the curve consisting of the vertical straight lines $[0, y]$, $\epsilon \le y \le 1 - \pi/8$ and $[1, y]$, $\epsilon \le y \le 1 - \pi/8$, and the half-circle of radius $1/2$ joining the point $(0, 1 - \pi/8)$ to the point $(1, 1 - \pi/8)$. The minimal length is then

$$\inf_{f \in F_\epsilon} \mathcal{L}(f) = 2(1 - \epsilon) + \pi/4.$$

Returning to (4.2), we can bound the energy from below by

$$\begin{aligned} I_1(\epsilon) &= N\epsilon + 2R_\epsilon \arcsin(1/[2R_\epsilon]) && \text{if } 1 - \pi/8 \le \epsilon \le 1, \\ I_2(\epsilon) &= (N-2)\epsilon + 2 + \pi/4 && \text{if } 0 \le \epsilon < 1 - \pi/8, \end{aligned}$$

and it follows from (4.1) that

$$\inf_{h \in H} \mathcal{I}(h) \ge \min\left(\inf_{1-\pi/8 \le \epsilon \le 1} I_1(\epsilon), \inf_{0 < \epsilon < 1 - \pi/8} I_2(\epsilon) \right). \tag{4.4}$$

We next show that for $N \ge 2$, the infimum in (4.4) is attained at $\epsilon = 0$. Differentiating I_1 with respect to ϵ, we get

$$\frac{\partial I_1}{\partial \epsilon} = N + 2\left(\arcsin(1/[2R_\epsilon]) - \frac{1}{\sqrt{4R_\epsilon^2 - 1}} \right) \frac{\partial R_\epsilon}{\partial \epsilon} \cdot$$

On the other hand, the definition (4.3) of R_ϵ yields

$$1 = 2\left(\frac{R_\epsilon}{\sqrt{4R_\epsilon^2 - 1}} - R_\epsilon \arcsin(1/[2R_\epsilon]) \right) \frac{\partial R_\epsilon}{\partial \epsilon} \cdot$$

Eliminating $\partial R_\epsilon / \partial \epsilon$ between these two relations shows that

$$\frac{\partial I_1}{\partial \epsilon} = N - \frac{1}{R_\epsilon} \ge N - 2,$$

since $R_\epsilon \ge 1/2$. Thus, for $N \ge 2$, I_1 is an increasing function of ϵ. On the other hand, I_2 is also increasing in this case, which establishes the result.

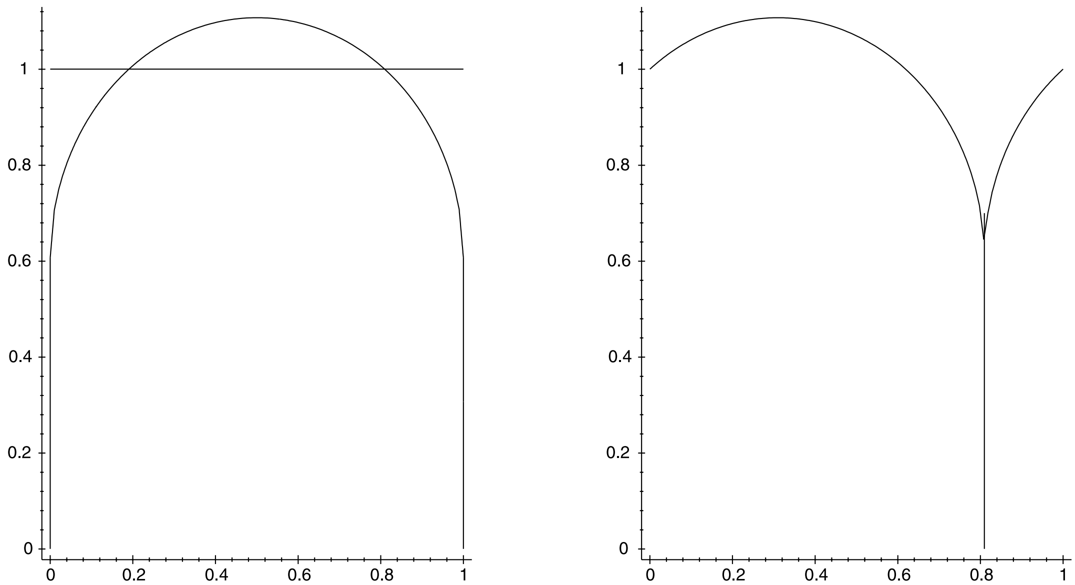

FIGURE 1. Plot of $\mathcal{F}_*(x)$ along with $y = 1$ and corresponding plot of $\mathcal{H}_*(x)$.

It is then easily checked that $I_2(0) = 2 + \pi/4$ is the length of the curve $\mathcal{F}_*$ defined by the function

$$f_*(x) = 1 - \pi/8 + \sqrt{x(1-x)} \qquad \text{if } 0 < x < 1$$

and by the two vertical lines

$$x = 0, \quad 0 \leq y \leq 1 - \pi/8, \qquad x = 1, \quad 0 \leq y \leq 1 - \pi/8.$$

To go back to the original boundary conditions, let $x_* = (4 - \sqrt{16 - \pi^2})/8$, let

$$h_*(x) = f_*(x + x_*) \ \text{ if } 0 \leq x \leq 1 - x_*, \quad h_*(x) = f_*(x + x_* - 1) \ \text{ if } 1 - x_* < x \leq 1, \tag{4.5}$$

and let $\mathcal{H}_*(x)$ be the curve defined by h_* and the segment $x = x_*$, $0 \leq y \leq 1 - \pi/8$. Then $\mathcal{H}_*(x)$ satisfies the conclusion of the theorem. □

The curve $\mathcal{F}_*(x)$ and corresponding "generalized thickness" $\mathcal{H}_*(x)$ are shown in Figure 1. As is easily seen, h_* is obtained as a rearrangement of f_* by first taking the part of f_* lying above $y = 1$ and then appending the part lying below $y = 1$.

The theorem shows that to minimize $\mathcal{I}$, it is advantageous to cancel the bulk elastic energy term, which is achieved by breaking the specimen. However, the length of the crack has to be accounted for in the remaining surface energy term.

5. Approximation of the generalized thickness

By constructing a minimizing sequence, we now show that the value $2+\pi/4$, given in the previous section as a lower bound for $\inf_{h\in H}\mathcal{I}(h)$, is in fact the value of this quantity.

For $0<\epsilon<1-\pi/8,\ \delta>0,\ \rho>0$, we consider the function $f_{\epsilon,\delta}$ which is continuous on $[0,1]$, linear on $[0,\delta]\cup[1-\delta,1]$, with value ϵ at $x=0,1$ and slope $s_{\epsilon,\delta}=\pm(1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)})/\delta$, and for $x\in[\delta,1-\delta]$,

$$f_{\epsilon,\delta}(x)=1-\pi/8+\rho+\sqrt{x(1-x)}.$$

The constant ρ is selected so that $f_{\epsilon,\delta}$ satisfies the volume constraint

$$\begin{aligned}\int_0^1 f_{\epsilon,\delta} &= 2\int_0^\delta [s_{\epsilon,\delta}x+\epsilon]\,\mathrm{d}x+\int_\delta^{1-\delta}\left[1-\pi/8+\rho+\sqrt{x(1-x)}\right]\,\mathrm{d}x\\ &=\delta\left[1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}\right]+(1-2\delta)(1-\pi/8+\rho)\\ &\quad+\sqrt{\delta(1-\delta)}(1-2\delta)/2+\arcsin(1-2\delta)/4+2\delta\epsilon.\end{aligned}$$

The volume constraint yields

$$\rho=\frac{\delta(1-\epsilon)+\pi/8(1-\delta)-\sqrt{\delta(1-\delta)}/2-\arcsin(1-2\delta)/4}{1-\delta}\cdot$$

Expanding ρ as a series in δ yields $\rho=(1-\pi/8-\epsilon)\delta+O(\delta^{3/2})$, so that ρ is positive and tends to 0 as $\delta\to 0$. Thus, when δ is small enough, $f_{\epsilon,\delta}$ is an admissible function.

Let us now compute the energy $\mathcal{I}(f_{\epsilon,\delta})$. For the surface energy, we have

$$\int_0^1\sqrt{1+(f'_{\epsilon,\delta})^2}\,\mathrm{d}x=K_1+K_2,$$

where K_1 is the length of the linear part, *i.e.*,

$$K_1=2\int_0^\delta\sqrt{1+s^2_{\epsilon,\delta}}\,\mathrm{d}x=2\sqrt{\delta^2+\left[1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}\right]^2}$$

and K_2 is the length of the arc of the circle, *i.e.*,

$$K_2=\int_\delta^{1-\delta}\sqrt{1+(f'_{\epsilon,\delta})^2}\,\mathrm{d}x=\pi/2-\arccos(1-2\delta).$$

For the elastic part, let

$$\int_0^1\frac{1}{f_{\epsilon,\delta}}=J_1+J_2,$$

where J_1 corresponds to the linear part, *i.e.*,

$$\begin{aligned}J_1&=2\int_0^\delta\frac{1}{s_{\epsilon,\delta}x+\epsilon}\,\mathrm{d}x\\ &=\frac{2\delta}{1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}}\log\left(1+\frac{1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}}{\epsilon}\right).\end{aligned}$$

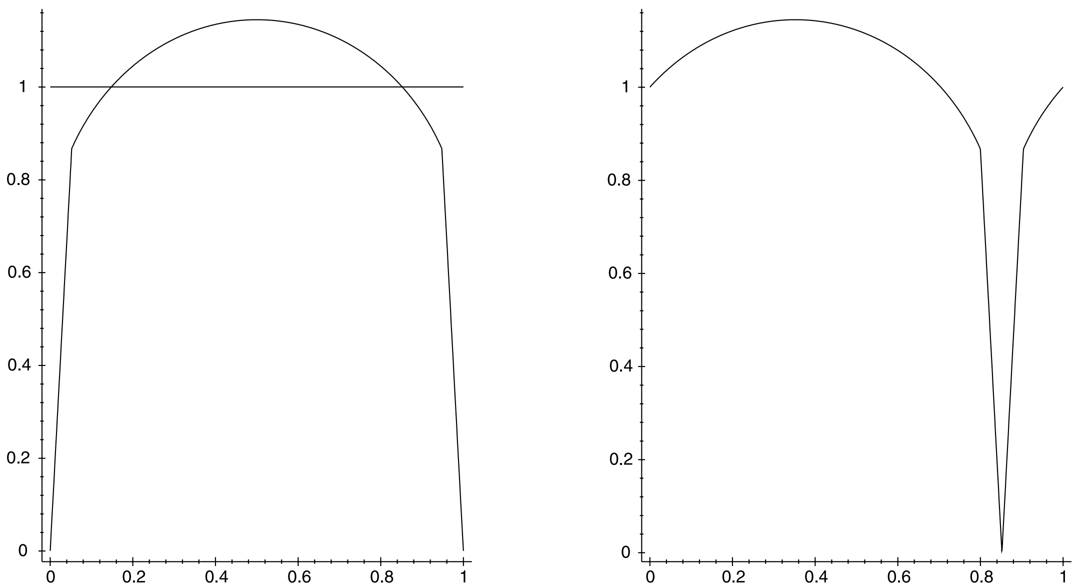

FIGURE 2. Plot of $y = f_{\epsilon,\delta}(x)$ along with $y = 1$ and corresponding plot of $y = h_{\epsilon,\delta}(x)$.

The term J_2 is the contribution of the arc of the circle

$$0 \le J_2 = \int_{\delta}^{1-\delta} \frac{1}{f_{\epsilon,\delta}}\,\mathrm{d}x \le \frac{1}{1-\pi/8}\,.$$

Thus, the total energy is

$$\begin{aligned}\mathcal{I}(f_{\epsilon,\delta}) &= \pi/2 - \arccos(1-2\delta) + 2\sqrt{\delta^2 + \left[1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}\right]^2} \\ &\quad + N\left[J_2 + \frac{2\delta}{1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}}\log\left(1+\frac{1-\pi/8+\rho-\epsilon+\sqrt{\delta(1-\delta)}}{\epsilon}\right)\right]^{-1}\end{aligned}$$

where J_2 is bounded. When ρ, ϵ, and δ tend to 0, this quantity behaves like

$$\mathcal{I}(f_{\epsilon,\delta}) \sim \pi/2 + 2(1-\pi/8) + N\left[J_2 + \frac{2\delta}{1-\pi/8}\log\left(\frac{1-\pi/8}{\epsilon}\right)\right]^{-1}.$$

The choice $\delta = [\log(1/\epsilon)]^{-1/2}$ shows that $I(f_{\epsilon,\delta}) \to 2 + \pi/4 = I_2(0)$, the length of $\mathcal{F}_*(x)$, when $\epsilon \to 0$. On the other hand, the sequence $f_{\epsilon,\delta}$ converges pointwise to f_*. Therefore, it follows from Theorem 4.1 that $f_{\epsilon,\delta}$ is a minimizing sequence, when $N \ge 2$.

From $(f_{\epsilon,\delta})$, it is then straightforward to construct a minimizing sequence $h_{\epsilon,\delta}$ that satisfies the boundary conditions. Let $h_{\epsilon,\delta}$ be defined by

$$\begin{aligned} h_{\epsilon,\delta}(x) &= f_{\epsilon,\delta}(x + x_*) \quad \text{if } 0 \le x \le 1 - x_*, \\ h_{\epsilon,\delta}(x) &= f_{\epsilon,\delta}(x + x_* - 1) \quad \text{if } 1 - x_* \le x \le 1. \end{aligned}$$

The function $(f_{\epsilon,\delta})$ and corresponding function $h_{\epsilon,\delta}$ are shown in Figure 2 for the choice $\epsilon = 10^{-160}$ and $\delta = \log(1/\epsilon)^{-1/2} \approx 0.05$.

For $N \geq 2$, the curve $\mathcal{H}_*$ (*i.e.*, h_* defined by (4.5) and a crack) is thus a "generalized minimizer" for our original problem.

6. A relaxed form of the energy

In the result of stability for values of $N \geq 2$, we constructed a sequence of piecewise C^1 functions h_n, the energies of which converge to the value of the infimum $2 + \pi/4$. This number is also the length of the non-parametric curve $\mathcal{H}_*$, defined in Theorem 4.1. The sequence $\{h_n\}$, satisfies the estimates

$$\|h_n\|_{L^1} = 1, \qquad \int_0^1 |h_n'| \, \mathrm{d}x \le \int_0^1 \sqrt{1 + (h_n')^2} \, \mathrm{d}x \le M$$

for some constant M. In other words, $\{h_n\}$ is a sequence in the space BV of functions of bounded variation [6], which is bounded in the norm in BV. It follows that $\{h_n\}$ is precompact in BV [6], *i.e.*, that upon extracting a subsequence, $\{h_n\}$ converges to the BV function h_* defined by (4.5):

$$\begin{aligned} h_*(x) &= 1 - \pi/8 + \sqrt{(x + x_*)(1 - x - x_*)} \qquad \text{for } 0 \le x \le 1 - x_* = (4 + \sqrt{16 - \pi^2})/8, \\ h_*(x) &= 1 - \pi/8 + \sqrt{(x + x_* - 1)(2 - x - x_*)} \qquad \text{for } 1 - x_* < x \le 1. \end{aligned}$$

The convergence holds in the following sense:

$$\begin{aligned} & h_n \longrightarrow h_* \qquad \text{strongly in} \quad L^1(0,1), \\ & \liminf_{n \to \infty} \int_0^1 \sqrt{1 + (h_n')^2} \, \mathrm{d}x \ge \int_0^1 \sqrt{1 + (h_*')^2} \, \mathrm{d}x. \end{aligned}$$

We would like to cast the problem of minimizing (1.3) in a setting that ensures well-posedness. In other words, we would like to consider a functional, which is lower semi-continuous in the natural norm, and which is defined on a compact set of admissible thicknesses.

The space BV seems to be the natural space and for $h \in BV$, strictly positive, the definition of $\mathcal{I}(h)$ in (1.3) makes sense. The closure of this subset of BV functions however, contains functions that vanish, for which we need to extend the definition of $\mathcal{I}$. Clearly, the trouble comes from the term $(\int_0^1 [h(x)]^{-1} \, \mathrm{d}x)^{-1}$ that reflects the fact that no uniform coercive estimates on the displacements are available in the original minimization problem (1.1).

Let H_* denote the set of positive BV functions, satisfying the boundary conditions and the volume constraint of (1.2). In H_*, we define

$$\begin{aligned} \mathcal{J}(h) &= \min\left(2 \min(h), \frac{N}{\int_0^1 [h(x)]^{-1} \, \mathrm{d}x}\right) + \int_0^1 \sqrt{1 + (h')^2} \, \mathrm{d}x, \qquad \text{if } \min(h) > 0, \\ \mathcal{J}(h) &= \int_0^1 \sqrt{1 + (h')^2} \, \mathrm{d}x, \qquad \text{otherwise.} \end{aligned}$$

Proposition 6.1. *The functional $\mathcal{J}$ extends the functional $\mathcal{I}$ in the following sense:*
(i) If $h \in H_*$ is bounded away from 0, i.e., $h(x) \geq \alpha > 0$ a.e. in $[0,1]$, then $\mathcal{I}(h) \geq \mathcal{J}(h)$.
(ii) If $\{h_n\}$ is a sequence of functions in H_* that converges to $h \in H_*$ in $L^1(0,1)$, such that each h_n is bounded away from 0, then $\liminf_{n\to\infty} \mathcal{I}(h_n) \geq \mathcal{J}(h)$.

Proof. The first statement is a trivial consequence of the definition of $\mathcal{J}$. To prove the second point, we consider a sequence $\{h_n\} \subset H_*$, such that for each n, $\min(h_n) = m_n > 0$, and $h_n(x) \to h$ in $L^1(0,1)$. By density, we can always assume that the functions h_n are C^1 on $[0,1]$ [6].

Case 1. If $\min(h) = 0$, then

$$\liminf_{n\to\infty} \mathcal{I}(h_n) \geq \liminf_{n\to\infty} \int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x \geq \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x = \mathcal{J}(h),$$

where the last inequality follows from the lower semi-continuity of $\int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x$ (*i.e.*, the length of h) in BV [6].

Case 2. If $\min(h) = m > 0$ and $\liminf_{n\to\infty} m_n > 0$, then, $h^{-1} \in L^1(0,1)$ and for a subsequence

$$h_n \to h \text{ a.e.}, \qquad h_n^{-1} \to h^{-1} \text{ a.e.}$$

From the Lebesgue Dominated Convergence Theorem, it follows that

$$\frac{N}{\int_0^1 h_n^{-1}\,\mathrm{d}x} \to \frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x}\,\cdot$$

Thus, using again the lower semi-continuity of the length in BV, we obtain

$$\liminf_{n\to\infty} \mathcal{I}(h_n) \geq \frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x} + \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x \geq \mathcal{J}(h).$$

Case 3. If $\min(h) = m > 0$ and $\liminf_{n\to\infty} m_n = 0$, then we can always assume that the whole sequence h_n tends to h a.e. and that

$$m_n \to 0. \tag{6.1}$$

Let $\epsilon > 0$ be such that $m - \epsilon > m/2 > 0$. For n larger than some N_0, $m_n + \epsilon < m - \epsilon$. Let $\phi_n(x) = \sup(m-\epsilon, h_n(x))$. Since $h_n \leq \phi_n \leq \sup(h, h_n)$ a.e.,

$$\phi_n \to h \quad \text{a.e.}$$

Moreover, since h_n is continuous, for $n > N_0$ there exists an interval $[x_n, y_n]$, of length d_n, such that

$$h_n(x_n) = h_n(y_n) = m - \epsilon, \qquad \min(h_n) = m_n \text{ in } [x_n, y_n], \qquad h_n \leq m - \epsilon \text{ in } [x_n, y_n].$$

The length of h_n on $[x_n, y_n]$ is greater than the length of two straight lines connecting the points $(x_n, m - \epsilon)$, $([x_n + y_n]/2, m_n)$, $(y_n, m - \epsilon)$. Hence,

$$\int_{x_n}^{y_n} \sqrt{1+(h_n')^2}\,\mathrm{d}x \geq \sqrt{d_n^2 + 4(m - \epsilon - m_n)^2}.$$

On the other hand, since the length of ϕ_n on that segment is simply d_n, we get

$$\int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x \geq \int_0^1 \sqrt{1+(\phi_n')^2}\,\mathrm{d}x + \sqrt{d_n^2+4(m-\epsilon-m_n)^2} - d_n.$$

We claim that

$$d_n \to 0 \quad \text{as } n \to \infty. \tag{6.2}$$

Indeed, if $d_n \to \alpha > 0$, we could find a subsequence such that $x_n \to x$, $y_n \to y$, and for n large enough,

$$x_n - \alpha/5 \;<\; x \;<\; x_n + \alpha/5 \;<\; y_n - \alpha/5 \;<\; y \;<\; y_n + \alpha/5,$$

so that we would have $h_n \leq m - \epsilon$ on $[x + \alpha/5, y - \alpha/5]$ for n large enough. This contradicts the fact that $h_n \to h$ a.e.

Finally, using (6.1–6.2) and the semi-continuity of the length in BV, we have

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{I}(h_n) &\geq \liminf_{n\to\infty} \int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x \\
&\geq \liminf_{n\to\infty} \left[\int_0^1 \sqrt{1+(\phi_n')^2}\,\mathrm{d}x + \sqrt{d_n^2+4(m-\epsilon-m_n)^2} - d_n\right] \\
&\geq \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x + 2(m-\epsilon).
\end{aligned}$$

Letting ϵ tend to 0, we obtain $\liminf_{n\to\infty} \mathcal{I}(h_n) \geq \mathcal{J}(h)$. □

Proposition 6.2. *The functional $\mathcal{J}$ is lower semi-continuous on H_*.*

Proof. Let $\{h_n\} \subset H_*$ be such that $h_n \to h$ in $L^1(0,1)$ and $\int \sqrt{1+(h_n')^2}\,\mathrm{d}x$ is bounded. We want to show that

$$\liminf_{n\to\infty} \mathcal{J}(h_n) \geq \mathcal{J}(h). \tag{6.3}$$

We can always assume that the functions h_n are C^1 [6].

If $\min(h) = 0$, then (6.3) is satisfied trivially. If $\min(h) = m > 0$ and $\min(h_n) = m_n$ tends to some value $m^* \geq m$, then

$$\min\left(2m_n, \frac{N}{\int_0^1 [h_n]^{-1}\,\mathrm{d}x}\right) \to \min\left(2m^*, \frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x}\right) \geq \min\left(2m, \frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x}\right),$$

and (6.3) follows from the lower semi-continuity of the length.

If $m^* < m$, then let $\epsilon > 0$ be such that $m^* + \epsilon < m - \epsilon$ and let $\phi_n(x) = \sup(m-\epsilon, h_n(x))$. Then, by the same arguments as those of Proposition 6.1,

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x &\geq \liminf_{n\to\infty} \left[\int_0^1 \sqrt{1+(\phi_n')^2}\,\mathrm{d}x + \sqrt{d_n^2+4(m-\epsilon-m_n)^2} - d_n\right] \\
&\geq \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x + 2(m-\epsilon-m^*).
\end{aligned}$$

If $m^* = 0$, since $\liminf_{n\to\infty} \mathcal{J}(h_n)$ is larger than the right-hand side of the above inequality, (6.3) is obtained by letting $\epsilon \to 0$.

If $m^* > 0$, then we get

$$\liminf_{n\to\infty} \left[2m_n + \int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x\right] \geq \mathcal{J}(h) - 2\epsilon.$$

On the other hand, the Dominated Convergence Theorem and the lower semi-continuity of the length yield

$$\liminf_{n\to\infty} \left[\frac{N}{\int_0^1 [h_n]^{-1}\,\mathrm{d}x} + \int_0^1 \sqrt{1+(h_n')^2}\,\mathrm{d}x\right] \geq \frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x} + \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x \geq \mathcal{J}(h),$$

and (6.3) follows from these last two inequalities by letting ϵ tend to 0. □

Proposition 6.2 and the precompactness of sequences in H_* in the norm in BV imply that $\mathcal{J}$ achieves a minimum in H_*. For $N \geq 2$,

$$\frac{N}{\int_0^1 h^{-1}\,\mathrm{d}x} \geq \frac{N}{\int_0^1 [\min(h)]^{-1}\,\mathrm{d}x} \geq 2\min(h)$$

and thus,

$$\forall h \in H_*, \qquad \mathcal{J}(h) = 2\,\min(h) + \int_0^1 \sqrt{1+(h')^2}\,\mathrm{d}x.$$

The arguments of Section 4 show not only that $\mathcal{I}(h) \geq 2 + \pi/4$, but also that $\mathcal{J}(h) \geq 2 + \pi/4$. On the other hand, the *function* h_*, defined by (4.5) satisfies

$$\mathcal{J}(h_*) = 2 + \pi/4 = \min\,\mathcal{J}.$$

7. Appendix

Proof of Proposition 4.2. Assume $f \in F_{e_0}$, *i.e.*, that f is a piecewise C^1 function such that

$$f(0) = f(1) = e_0, \qquad f(x) \geq e_0, \quad x \in [0,1], \qquad \int_0^1 f(x)\,\mathrm{d}x = 1.$$

Since the minimal length of $f \in F_{e_0}$ depends on the value of e_0, we shall consider two cases.
Case 1. $1 - \pi/8 \leq e_0 \leq 1$.
In this case, we first show that there is an arc of a circle which is an admissible curve. To this effect, we seek y_0 and R_{e_0}, such that

$$f_0(x) = y_0 + \sqrt{R_{e_0}^2 - (x-1/2)^2}$$

defines an arc of a circle that connects the point $(0, e_0)$ to $(1, e_0)$, that encloses an area equal to 1, and that lies above the level e_0. Expressing these conditions, we get

$$(e_0 - y_0) = \sqrt{R_{e_0}^2 - 1/4},$$

$$1 = \int_0^1 \left[y_0 + \sqrt{R_{e_0}^2 - (x-1/2)^2}\right]\,\mathrm{d}x = y_0 + \frac{\sqrt{4\,R_{e_0}^2 - 1}}{4} + R_{e_0}^2 \arcsin(1/[2\,R_{e_0}]).$$

It follows that

$$e_0 = 1 + (1/4)\sqrt{4\,R_{e_0}^2 - 1} - R_{e_0}^2 \arcsin(1/[2\,R_{e_0}]). \tag{7.1}$$

For $r \in (1/2, \infty)$, let

$$\phi(r) = 1 + (1/4)\sqrt{4\,r^2 - 1} - r^2 \arcsin(1/[2\,r]).$$

Then $\phi'(r) = 2r\rho(r)$ where $\rho(r) = (4r^2-1)^{-1/2} - \arcsin(1/[2\,r])$. Since $\rho'(r) = -1/[r(4r^2-1)^{3/2}] < 0$ for $r > 1/2$, ρ is a decreasing function. Since ρ tends to zero as r tends to infinity, it follows that ρ and hence $\phi'(r)$ are positive, which implies that ϕ is a strictly increasing function. It is easy to check that ϕ maps $(1/2, \infty)$ onto $(1-\pi/8, 1)$. Thus, for each $1 - \pi/8 < e_0 < 1$, there is a unique R_{e_0} solution of (7.1). Furthermore, R_{e_0} tends to $1/2$ (resp. ∞) as e_0 tends to $1 - \pi/8$ (resp. 1).

Let D_0 denote the upper half of the disc of radius R_{e_0}, centered at $(1/2, y_0)$, and let Γ_0 denote the part of its boundary that lies below e_0 and above y_0. The domain D enclosed by Γ_0, the line $y = y_0$, and the curve defined by f has the same area as D_0. The classical isoperimetric inequality [2] implies that the length of the boundary of D is greater or equal to the length of the boundary of D_0. Thus

$$\int_0^1 \sqrt{1+(f')^2}\,\mathrm{d}x \geq \int_0^1 \sqrt{1+(f_0')^2}\,\mathrm{d}x = 2R_{e_0}\arcsin(1/[2R_{e_0}]).$$

Case 2. $0 \leq e_0 < 1 - \pi/8$.
We can no longer draw an arc of a circle bounding an area of 1 through the points $(0, e_0)$ and $(1, e_0)$. The proof of this case is divided into two steps. In the first one, we replace f by another function f_* that has length less than or equal to the length of f. Then in the second step, we obtain a lower bound on the length of f_*.

Step 1:
In addition to the previous hypotheses, assume that f is piecewise linear. We shall subsequently extend the results obtained by a density argument. Let e_M be the maximum of f. For $e \in [e_0, e_M]$, we define

$$\Omega_e = \{0 \leq x \leq 1 : f(x) \geq e\}, \qquad g(e) = \int_{\Omega_e} (f-e)\,\mathrm{d}x, \qquad h(e) = \pi/8\,|\Omega_e|^2.$$

The function h is the area of a half circle of diameter $|\Omega_e|$. It is a right-continuous, decreasing function. The function g measures the area enclosed by f above the level e. One can readily check that g is decreasing and continuous: if $e < e'$,

$$\begin{aligned} g(e) &= \int_{\Omega_e'} (f-e)\,\mathrm{d}x + \int_{\Omega_e \setminus \Omega_e'} (f-e)\,\mathrm{d}x \\ &\leq \int_{\Omega_e'} (f-e')\,\mathrm{d}x + \int_{\Omega_e'} (e'-e)\,\mathrm{d}x + (e'-e)\,|\Omega_e \setminus \Omega_e'| \\ &\leq g(e') + (e'-e). \end{aligned}$$

Also, we have

$$g(e_0) = 1 - e_0 > \pi/8 = h(e_0). \tag{7.2}$$

Again, we consider several cases.

Case a: $h(e) \geq g(e)$ for some value of $e \in (e_0, e_M)$ or $h(e_M) > g(e_M)$.
The motonicity and continuity properties of g and h, together with (7.2), imply that $h(e_*) = g(e_*)$, for some value $e_* \in (e_0, e_M)$.

Since the area enclosed by f and the length of f are translation invariant, Ω_{e_*} can be assumed to be connected and centered at some point x_*. Then, the function f_*, given by

$$f_*(x) = f(x) \qquad \text{if } x \in [0,1] \setminus \Omega_{e_*},$$
$$f_*(x) = e_* + \sqrt{|\Omega_{e_*}^2|/4 - (x - x_*)^2} \qquad \text{if } x \in \Omega_{e_*}$$

also encloses an area equal to 1. It is a consequence of the standard isoperimetric inequality on Ω_{e_*}, that f_* has a smaller length than f.

Case b: $h(e) < g(e)$ for all $e \in (e_0, e_M)$ and $h(e_M) \leq g(e_M)$.
First, observe that if Ω_{e_M} contains a subset ω where f is flat, then $g(e_M) = 0$, while $h(e_M) \geq \pi/8\,|\omega|^2 > 0$. Since this cannot occur under the hypothesis of Case b, we conclude that $f' \neq 0$ a.e. in Ω_{e_M}. Hence, for e close enough to e_M, Ω_e consists of a finite number of intervals

$$\Omega_e = \cup_{1 \leq i \leq N} [x_i - r_i^-, x_i + r_i^+],$$

such that f is increasing on $[x_i - r_i^-, x_i]$ from $f(e)$ to $f(e_M)$ and decreasing on $[x_i, x_i + r_i^+]$ from $f(e_M)$ to $f(e)$. Again, by translation invariance, Ω_e can be assumed to be connected (*i.e.*, $x_i + r_i^+ = x_{i+1} - r_{i+1}^-$) and centered at some point x_*. Since f is piecewise linear and has a saw-tooth profile in Ω_e, we have

$$\int_{\Omega_e} (f - e)\,\mathrm{d}x = |\Omega_e|(e_M - e)/2. \tag{7.3}$$

Thus, $h(e) < g(e)$ implies that $(e_M - e)/2 - \pi/8\,|\Omega_e|$ is positive. Hence,

$$e_* \equiv e + (e_M - e)/2 - \pi/8\,|\Omega_e| = (e_M + e)/2 - \pi/8\,|\Omega_e| > e.$$

Clearly, we also have $e_* < e_M$. Let $f_*(x)$ be the function defined by

$$f_*(x) = f(x) \qquad \text{if } x \in [0,1] \setminus \Omega_e,$$
$$f_*(x) = e_* + \sqrt{|\Omega_e|^2/4 - (x - x_*)^2} \qquad \text{if } x \in \Omega_e,\ \textit{i.e.},\ |x - x_*| \leq |\Omega_e|/2,$$

and let C_* be the curve defined by the union of the half circle $(x, f_*(x))$, $|x - x_*| \leq |\Omega_e|/2$, and the two vertical segments $[x_* \pm |\Omega_e|/2, y]$, $e \leq y \leq e_*$.

According to the definition of e_*,

$$\int_{\Omega_e} (f_* - e)\,\mathrm{d}x = \int_{\Omega_e} \left[(e_* - e) + \sqrt{|\Omega_e|^2/4 - (x - x_*)^2}\right] \mathrm{d}x = (e_* - e)|\Omega_e| + \pi|\Omega_e|^2/8$$
$$= [(e_M - e)/2 - \pi|\Omega_e|/8]\,|\Omega_e| + \pi|\Omega_e|^2/8 = |\Omega_e|(e_M - e)/2 = \int_{\Omega_e} (f - e)\,\mathrm{d}x.$$

Besides, (7.3) gives the following estimate of $|\Omega_e|$.

$$h(e) = \pi|\Omega_e|^2/8 < g(e) = |\Omega_e|(e_M - e)/2 \qquad \textit{i.e.}, \quad \pi|\Omega_e|/4 < e_M - e. \tag{7.4}$$

Now the length of f on Ω_e is given by

$$\sum_{i=1}^{N}\left[\sqrt{(r_i^-)^2+(e_M-e)^2}+\sqrt{(r_i^+)^2+(e_M-e)^2}\right]$$

$$=\sum_{i=1}^{N}\left(\frac{r_i^-}{|\Omega_e|}\sqrt{|\Omega_e|^2+\left[\frac{|\Omega_e|}{r_i^-}(e_M-e)\right]^2}+\frac{r_i^+}{|\Omega_e|}\sqrt{|\Omega_e|^2+\left[\frac{|\Omega_e|}{r_i^+}(e_M-e)\right]^2}\right)$$

$$\geq\sqrt{|\Omega_e|^2+4N^2(e_M-e)^2}\geq\sqrt{|\Omega_e|^2+4(e_M-e)^2},$$

by the convexity of the function $\sqrt{a^2+x^2}$. On the other hand, using (7.4), the length of C_* is

$$2(e_*-e)+\pi|\Omega_e|/2=(e_M-e)+\pi|\Omega_e|/4\leq 2(e_M-e)<\sqrt{|\Omega_e|^2+4(e_M-e)^2},$$

and thus is smaller then the length of f.

Step 2:
So far, given a piecewise linear admissible function f, we have constructed another admissible function f_*, which may have jumps, but whose length, $l(f_*)$, is less than or equal to the length of f. In particular, the constraint on the area yields

$$1=\int_{\{f_*\geq e_*\}}f_*\,\mathrm{d}x+\int_{[0,1]\setminus\{f_*\geq e_*\}}f_*\,\mathrm{d}x.$$

In Case a, Ω_{e_*} is also the set where $f_*\geq e_*$ and so it follows easily from the above that

$$1\leq\int_{\Omega_{e_*}}f_*\,\mathrm{d}x+e_*(1-|\Omega_{e_*}|)=(\pi/8)|\Omega_{e_*}|^2+e_*\,|\Omega_{e_*}|+e_*(1-|\Omega_{e_*}|)\leq(\pi/8)+e_*.$$

In Case b, Ω_e is the set where $f_*\geq e_*$ and so

$$1\leq\int_{\Omega_e}f_*\,\mathrm{d}x+e_*(1-|\Omega_e|)=(\pi/8)|\Omega_e|^2+e_*\,|\Omega_e|+e_*(1-|\Omega_e|)\leq(\pi/8)+e_*.$$

Hence, in both cases,

$$e_*\geq 1-\pi/8.$$

Let D be the domain that consists of the area enclosed by f_* above the level e_0 and its symmetric image about the line $y=e_0$. Its area is $\mathcal{A}(D)=2(1-e_0)$, and its length is $l(D)=2l(f_*)$. Further, by construction, D contains two discs of radius $|\Omega_{e_*}|/2$, whose centers are separated by a distance $d=2(e_*-e_0)$. In this situation, the following isoperimetric inequality holds (see p. 7 in [2]).

$$l(D)^2\geq 4\pi\,\mathcal{A}(D)+4d^2,$$

i.e.,

$$l(f_*)^2\geq 2\pi(1-e_0)+4(e_*-e_0)^2\geq 2\pi(1-e_0)+4(1-\pi/8-e_0)^2=[2(1-e_0)+\pi/4]^2.$$

By density, it follows that for all $f\in F_{e_0}$

$$\int_0^1\sqrt{1+(f')^2}\,\mathrm{d}x\geq 2(1-e_0)+\pi/4.$$

It is easy to check that this value is attained by the curve consisting of a half circle of radius $1/2$ centered at $(1/2, e_*)$ and the two vertical segments $[0, y]$, and $[1, y]$, with $e_0 \leq y \leq e_*$, where $e_* = 1 - \pi/8$. Note that this is precisely the curve C_* in the case when $e = e_0$ (and $\Omega_e = [0, 1]$).

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