CORRIGENDUM TO "ENERGY GAP FOR YANG–MILLS CONNECTIONS, II: ARBITRARY CLOSED RIEMANNIAN MANIFOLDS"

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Abstract. In [8], we proved an $L^{d/2}$ energy gap for Yang–Mills connections on principal G-bundles P over arbitrary, closed, Riemannian, smooth manifolds of dimension $d \geq 2$. Our proof relied in part on a result [26, Corollary 4.3] due to Uhlenbeck (we had attempted to reprove this as [8, Theorem 5.1]) which asserted that the $W^{1,p}$-distance between the gauge-equivalence class of a connection A and the moduli subspace of flat connections $M(P)$ on a principal G-bundle P over a closed Riemannian manifold X of dimension $d \geq 2$ is bounded by a constant times the L^p norm of the curvature, $\|F_A\|_{L^p(X)}$, when G is a compact Lie group, F_A is L^p-small, and $p > d/2$. In [5], we proved that this estimate holds when the Yang–Mills energy function on the quotient space of Sobolev connections is Morse–Bott along the moduli subspace $M(P)$ of flat connections, but that it need not hold when the Yang–Mills energy function fails to be Morse–Bott, such as at the product connection in the moduli space of flat SU(2) connections over a real two-dimensional torus. However, in [5], we also proved that a useful modification of Uhlenbeck's estimate always holds provided one replaces $\|F_A\|_{L^p(X)}$ by a suitable power $\|F_A\|_{L^p(X)}^{\lambda}$, where the positive exponent λ reflects the structure of non-regular points in $M(P)$. As we explain in this corrigendum, our $L^{d/2}$ energy gap for Yang–Mills connections still follows from this modification of Uhlenbeck's estimate.

Contents

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1. Introduction

The purpose of this corrigendum is to correct the proof that we gave in [8] to the following

Theorem 1 ($L^{d/2}$-energy gap for Yang–Mills connections)**.** *(See Feehan* [8, Theorem 1]*.) Let G be a compact Lie group and P be a principal G-bundle over a closed, smooth Riemannian manifold (X, g) of dimension $d \geq 2$. Then there is a positive constant*[1] $\varepsilon = \varepsilon(g, G) \in (0, 1]$ *with the following significance. If A is a smooth Yang–Mills connection on P with respect to the metric g and its curvature F_A obeys*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{1.1}$$

then A is a flat connection.

Our proof of Theorem 1 in [8] relied on a result [26, Corollary 4.3] due to Uhlenbeck, which we had attempted to reprove as [8, Theorem 5.1] and which we quote in this corrigendum as "Theorem" 3.1. This result asserts the existence of a flat connection Γ on P, given a connection A on P with curvature F_A obeying $\|F_A\|_{L^p(X)} \leq \varepsilon$ for some $p > d/2$ and small enough $\varepsilon = \varepsilon(g, G, p) \in (0, 1]$, and a gauge transformation $u \in \operatorname{Aut}(P)$ such that $u(A)$ is in Coulomb gauge with respect to Γ and

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)}, \tag{1.2}$$

for some constant $C = C(g, G) \in [1, \infty)$. The argument provided by Uhlenbeck in [26] was very brief and that had prompted us to attempt a more detailed justification in [8]. In [5, Theorems 1 and 9], we proved that the estimate (1.2) holds when the Yang–Mills energy function on the quotient space of Sobolev connections is Morse–Bott along the moduli subspace $M(P)$ of flat connections. We also noted that the estimate (1.2) need not hold when the Yang–Mills energy function fails to be Morse–Bott, such as at the product connection Θ, when X is the two-dimensional torus, $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and $G = \mathrm{SU}(2)$ and $P = \mathbb{T}^2 \times \mathrm{SU}(2)$ — an example suggested to the author by Mrowka [19]. Nishinou gave similar examples in [20]. When X is the unit ball in $\mathbb{R}^d$, then the estimate (1.2) does, of course, hold by the local Coulomb gauge-fixing result due to Uhlenbeck [24], quoted in this corrigendum as Theorem 2.1. The fact that (1.2) could be false when X is not a simply-connected manifold was noticed by Fukaya in [13] and later by Nishinou in [20], although neither Fukaya nor Nishinou appear to have been aware of [26, Corollary 4.3].

Fukaya proved a version [13, Proposition 3.1] of [26, Corollary 4.3], when $d = 4$ and A is anti-self-dual, that essentially replaces $\|F_A\|_{L^p(X)}$ by $\|F_A\|_{L^p(X)}^\lambda$, where $\lambda = \lambda(g, G, \Gamma) \in (0, 1]$ is a constant that depends on the geometry of the moduli space of flat connections near Γ. (Fukaya uses a different system of norms.) He explained to us [12] that his proof should extend to allow arbitrary dimensions $d \geq 2$, connections A on P of Sobolev class $W^{1,p}$, and the system of Sobolev norms in (1.2).

1.1. Outline.

In Section 2, we give minor corrections of our statements in [8] of results due to Uhlenbeck [25, 24]. Those corrections primarily concern the case of base manifolds of dimension $d = 2$ and the fact that when $p = 1$, standard *a priori* L^p elliptic estimates do not hold. In Section 3, we recall our version [8, Theorem 5.1] of the statement of [26, Corollary 4.3] due to Uhlenbeck as "Theorem" 3.1 and recall our correction [5, Theorems 1 and 9] to that result as Theorem 3.2. in Section 4, we describe a counterexample to the estimates stated in [8, Theorem 5.1] and [26, Corollary 4.3], based on an observation due to Mrowka [19]. In Section 5, we describe the error

[1]The indicated dependence of the constant $\varepsilon(g, G)$ on G includes the dependence on a choice of G-invariant inner product on the Lie algebra $\mathfrak{g}$ of G; see Feehan [8, Section 2, second paragraph] for further discussion.

in our proof of "Theorem" 3.1 (our version of [26, Corollary 4.3]) that we provided in [8, Section 6]. In Section 6, we describe a result, Lemma 6.1, that provides an *a priori* estimate which is adequate for the purposes of our proof of Theorem 1 in the case $d = 2$, when Theorem 2.7 does not apply. In Section 7, we correct the proof of Theorem 1 that we had provided in [8, Section 7]. The changes are minor and involve treatment of the exceptional case $d = 2$ and a replacement of the role of "Theorem" 3.1 by that of Theorem 3.2. Lastly, we take the opportunity to correct other minor typographical errors in [8]. We refer to our published article [8] for the necessary background, conventions, and notation.

1.2. **Acknowledgments.** I am very grateful to Tom Mrowka for his description of a counterexample [19] to exponential convergence (implied by our [6, Theorem 2 and Remark 1.8], where our [6] is a preliminary version of our published article [10]) for Yang–Mills gradient flow near the moduli space of flat connections and that serves also as a counterexample to the estimates claimed in [8, Theorem 5.1] and [26, Corollary 4.3]. I thank Changyou Wang for alerting me to subtleties particular to dimension two. Last but not least, I am extraordinarily grateful to the referee for providing corrections and helpful suggestions. The author was partially supported by National Science Foundation grants DMS-1510064 and DMS-2104865 during the preparation of this article.

2. Connections with $L^{d/2}$-small curvature and a priori estimates for Yang–Mills connections

In this section, we give minor corrections to our statements in [8] of results due to Uhlenbeck [25, 24]. These corrections primarily concern the case of base manifolds of dimension $d = 2$ and the fact that when $p = 1$, standard *a priori* L^p elliptic estimates, such as those in Gilbarg and Trudinger [14, Chapter 9] for the Laplace operator, do not hold.

2.1. Connections with $L^{d/2}$-small curvature.

We recall a statement of Uhlenbeck's Theorem [24] on existence of local Coulomb gauges (with a clarification due to Wehrheim [27]), together with two extensions proved by the author in [10].

Theorem 2.1 (Existence of a local Coulomb gauge and *a priori* estimate for a Sobolev connection with $L^{d/2}$-small curvature)**.** *(Correction to our quotation* [8, Theorem 4.1] *of Uhlenbeck's* [24, Theorem 1.3 or Theorem 2.1 and Corollary 2.2]*; compare Wehrheim* [27, Theorem 6.1]*.) Let $d \geq 2$, and G be a compact Lie group, and $p \in (1, \infty)$ be a constant obeying $d/2 \leq p < d$. Choose $s_0 = d/2$ when $d \geq 3$ and $s_0 > 1$ when $d = 2$. Then there are constants, $\varepsilon = \varepsilon(d, G, p, s_0) \in (0, 1]$ and $C = C(d, G, p, s_0) \in [1, \infty)$, with the following significance. For $q \in [p, \infty)$, let A be a $W^{1,q}$ connection on $B \times G$ such that*

$$\|F_A\|_{L^{s_0}(B)} \leq \varepsilon, \tag{2.1}$$

where $B \subset \mathbb{R}^d$ is the unit ball with center at the origin. Then there is a $W^{2,q}$ gauge transformation, $u : B \to G$, such that the following holds. If $A = \Theta + a$, where Θ is the product connection on $B \times G$, and $u(A) = \Theta + u^{-1}au + u^{-1}du$, then

$$d^*(u(A) - \Theta) = 0 \quad \text{a.e. on } B, \tag{2.2}$$

$$(u(A) - \Theta)(\vec{n}) = 0 \quad \text{on } \partial B, \tag{2.3}$$

where $\vec{n}$ is the outward-pointing unit normal vector field on ∂B, and

$$\|u(A) - \Theta\|_{W^{1,p}(B)} \leq C\|F_A\|_{L^p(B)}. \tag{2.4}$$

Remark 2.2 (Restriction of p to the range $1 < p < \infty$). (See Feehan [10, Remark 2.4].) The restriction $p \in (1, \infty)$ should be included in the statements of [24, Theorem 1.3 or Theorem 2.1 and Corollary 2.2] since the bound (2.4) ultimately follows from an *a priori* L^p estimate for an elliptic system that is apparently only valid when $1 < p < \infty$. Wehrheim makes a similar observation in her [27, Remark 6.2 (d)]. This is also the reason that when $d = 2$, we require $s_0 > 1$ in (2.1).

Remark 2.3 (Construction of a $W^{k+1,q}$ transformation to Coulomb gauge). (See Feehan [8, Remark 4.3].) We note that if A is of class $W^{k,q}$, for an integer $k \geq 1$ and $q \geq 2$, then the gauge transformation, u, in Theorem 2.1 is of class $W^{k+1,q}$; see [24, page 32], the proof of [24, Lemma 2.7] via the Implicit Function Theorem for smooth functions on Banach spaces, and our proof of [7, Theorem 1.1] — a global version of Theorem 2.1.

Remark 2.4 (Non-flat Riemannian metrics). (See Feehan [10, Remark 2.7].) Theorem 2.1 continues to hold for geodesic unit balls in a manifold X endowed with a non-flat Riemannian metric g. The only difference in this more general situation is that the constants C and ε will depend on bounds on the Riemann curvature tensor, Riem. See Wehrheim [27, Theorem 6.1].

We now recall an extension (and corollary of the proof) of Theorem 2.1 to include the range $1 < p < d/2$ when $d \geq 3$.

Corollary 2.5 (Existence of a local Coulomb gauge and *a priori* $W^{1,p}$ estimate for a Sobolev connection with $L^{d/2}$-small curvature when $p < d/2$). *(See Feehan* [10, Corollary 2.8]*.) Assume the hypotheses of Theorem 2.1, but assume $d \geq 3$ and allow $1 < p < d/2$. Then the estimate* (2.4) *holds.*

For completeness, we also recall the following extension (and corollary of the proof) of Theorem 2.1 (and slight improvement of our [8, Corollary 4.4]) to include the range $d \leq p < \infty$.

Corollary 2.6 (Existence of a local Coulomb gauge and *a priori* $W^{1,p}$ estimate for a Sobolev connection one-form with $L^{\bar{p}}$-small curvature when $p \geq d$). *(See Feehan* [10, Corollary 2.9]*.) Assume the hypotheses of Theorem 2.1, but consider $d \leq p < \infty$ and strengthen* (2.1) *to*[2]

$$\|F_A\|_{L^{\bar{p}}(B)} \leq \varepsilon, \tag{2.5}$$

where $\bar{p} = dp/(d + p)$ when $p > d$ and $\bar{p} > d/2$ when $p = d$. Then the estimate (2.4) *holds for $d \leq p < \infty$ and constant $C = C(d, p, \bar{p}, G) \in [1, \infty)$.*

Taken together, Corollaries 2.5 and 2.6 correct and replace our our [8, Corollary 4.4] (which should have included the restriction $p > 1$ when $d = 2$). However, neither Theorem 2.1 nor Corollaries 2.5 and 2.6 play a direct role in our corrected proof of Theorem 1 in this corrigendum.

2.2. **A priori estimate for the curvature of a Yang–Mills connection.** The forthcoming Theorem 2.7 corrects our quotation [8, Theorem 4.5] of Uhlenbeck's [25, Theorem 3.5] by explicitly adding the restriction $d \geq 3$ that is implicit in her proof. (See, for example, her proofs of [25, Lemma 3.3 and 3.4], results that she uses to prove [25, Theorem 3.5]. The following result corrects our quotation [8, Theorem 4.5] of Uhlenbeck's [25, Theorem 3.5].

Theorem 2.7 (*A priori* interior estimate for the curvature of a Yang–Mills connection). *(See Uhlenbeck* [25, Theorem 3.5]*.) If $d \geq 3$ is an integer, then there are constants, $K_0 = K_0(d) \in [1, \infty)$ and $\varepsilon_0 = \varepsilon_0(d) \in (0, 1]$, with the following significance. Let G be a compact Lie group, $\rho > 0$ be a constant, and A be a Yang–Mills connection with respect to the standard Euclidean*

[2] In [8, Corollary 4.4], we assumed the still stronger condition, $\|F_A\|_{L^p(B)} \leq \varepsilon$.

metric on $B_{2\rho}(0) \times G$, where $B_r(x_0) \subset \mathbb{R}^d$ is the open ball with center at $x_0 \in \mathbb{R}^d$ and radius $r > 0$. If

$$\|F_A\|_{L^{d/2}(B_{2\rho}(0))} \leq \varepsilon_0, \tag{2.6}$$

then, for all $B_r(x_0) \subset B_\rho(0)$,

$$|F_A|(x_0) \leq K_0 r^{-d/2} \|F_A\|_{L^2(B_r(x_0))}. \tag{2.7}$$

Theorem 2.7 may be improved with the aid of the monotonicity formula due to Price [21]. The following global version of Theorem 2.7 corrects our [8, Corollary 4.6] by adding the restriction $d \geq 3$ inherited from Theorem 2.7.

Corollary 2.8 (*A priori* estimate for the curvature of a Yang–Mills connection over a closed manifold)**.** *(Correction to Feehan* [8, Corollary 4.6]*.) Let X be a closed, smooth manifold of dimension $d \geq 3$ and endowed with a Riemannian metric, g. Then there are constants, $K = K(g) \in [1, \infty)$ and $\varepsilon = \varepsilon(g) \in (0, 1]$, with the following significance. Let G be a compact Lie group and A be a smooth Yang–Mills connection with respect to the metric, g, on a smooth principal G-bundle P over X. If*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon, \tag{2.8}$$

then

$$\|F_A\|_{L^\infty(X)} \leq K \|F_A\|_{L^2(X)}. \tag{2.9}$$

As noted earlier, the restriction $d \geq 3$ in Theorem 2.7 (and hence Corollary 2.8) was not explicitly stated by Uhlenbeck in her [25, Theorem 3.5] (although it does appear in her [25, Corollary 2.9]). However, the condition $d \geq 3$ can be inferred from Uhlenbeck's proof of [25, Theorem 3.5], in particular through her proof of the required [25, Lemma 3.3], where the exponent $\nu = 2d/(d-2)$ is undefined when $d = 2$. The restriction $d \geq 3$ also appears in Sibner's proof of her *a priori* L^∞ estimate for $|F_A|$ in [22, Proposition 1.1], where the necessity of the condition appears in her definition [22, p. 94] of the positive constant $\gamma_1 := (2d-4)/(d^2 C_d)$, with C_d denoting a Sobolev embedding constant in dimension d. When $d = 2$, a simple argument based on the Kato Inequality yields stronger conclusions than those of Theorem 2.7 and Corollary 2.8: see Lemma 6.1 and Corollary 6.2.

3. Global existence of a flat connection and a Sobolev distance estimate

In this section, we quote our version [8, Theorem 5.1] of the statement of [26, Corollary 4.3] due to Uhlenbeck as the forthcoming “Theorem” 3.1 below. The estimates in Items (1) and (3) do *not* hold in general — they are contradicted by the example discussed in Section 4. In Section 3.2, we quote our correction [5, Theorems 1 and 9] as the forthcoming Theorem 3.2.

3.1. Uhlenbeck's Corollary 4.3.

We recall from [8] the following version of [26, Corollary 4.3]:

“Theorem” 3.1 (Existence of a nearby $W^{1,p}$ flat connection on a principal bundle supporting a $W^{1,p}$ connection with L^p-small curvature)**.** *(See Feehan* [8, Theorem 5.1] *and Uhlenbeck* [26, Corollary 4.3]*.) Let X be a closed, smooth manifold of dimension $d \geq 2$ and endowed with a Riemannian metric, g, and G be a compact Lie group, and $p \in (d/2, \infty)$. Then there are constants, $\varepsilon = \varepsilon(d, g, G, p) \in (0, 1]$ and $C = C(d, g, G, p) \in [1, \infty)$, with the following significance. Let A be a $W^{1,p}$ connection on a principal G-bundle P over X. If*

$$\|F_A\|_{L^p(X)} \leq \varepsilon, \tag{3.1}$$

then the following hold:

(1) (Existence of a flat connection) *There exists a $W^{1,p}$ flat connection, Γ, on P obeying*

$$\|A-\Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)},$$
$$\|A-\Gamma\|_{W^{1,d/2}_\Gamma(X)} \leq C\|F_A\|_{L^{d/2}(X)};$$

(2) (Existence of a global Coulomb gauge transformation) *There exists a $W^{2,p}$ gauge transformation, $u \in \operatorname{Aut}(P)$, such that*

$$d_\Gamma^*(u(A)-\Gamma) = 0 \quad \text{a.e. on } X; \tag{3.2}$$

(3) (Estimate of Sobolev distance to the flat connection) *One has*

$$\|u(A)-\Gamma\|_{W^{1,p}_\Gamma(X)} \leq C\|F_A\|_{L^p(X)}, \tag{3.3a}$$
$$\|u(A)-\Gamma\|_{W^{1,d/2}_\Gamma(X)} \leq C\|F_A\|_{L^{d/2}(X)}. \tag{3.3b}$$

Our statement of "Theorem" 3.1 slightly modified that of [26, Corollary 4.3]. First, Item (2) was implied by Uhlenbeck's proof of [26, Corollary 4.3], but was not explicitly stated. Second, Uhlenbeck did not draw the distinction that we do here between the estimates obeyed by A in Item (1) and that obeyed by $u(A)$ in Item (3). Third, Uhlenbeck did not assert the $W^{1,d/2}$ estimates obeyed by A in Item (1) and by $u(A)$ in Item (3).

3.2. **A corrected Sobolev distance estimate.** In the forthcoming Theorem 3.2, we quote from [5] a corrected statement of "Theorem" 3.1 which effectively replaces the term $\|F_A\|_{L^p(X)}$ on the right-hand side with $\|F_A\|^\nu_{L^p(X)}$ for some $\nu = \nu(g,G,[\Gamma]) \in (0,1]$.

Theorem 3.2 (Existence of a nearby $W^{1,p}$ flat connection on a principal bundle supporting a $W^{1,p}$ connection with L^p-small curvature)**.** *(See Feehan* [5, Theorems 1 and 9] *for a more general statement.) Let (X,g) be a closed, smooth Riemannian manifold of dimension $d \geq 2$, and G be a compact Lie group, and $p \in (d/2,\infty)$ be a constant. Then there are a constant $\varepsilon = \varepsilon(g,G,p) \in (0,1]$ and, for any $r \in (1,p]$, a constant $C = C(g,G,r) \in [1,\infty)$ with the following significance. Let A be a $W^{1,p}$ connection on a principal G-bundle P over X. If*

$$\|F_A\|_{L^p(X)} \leq \varepsilon, \tag{3.4}$$

then there are a $W^{1,p}$ flat connection Γ on P, a constant $\nu = \nu(g,G,[\Gamma]) \in (0,1]$, and a $W^{2,p}$ gauge transformation $u \in \operatorname{Aut}(P)$ such that

$$d_\Gamma^*(u(A)-\Gamma) = 0 \quad \text{a.e. on } X, \tag{3.5}$$
$$\|u(A)-\Gamma\|_{W^{1,r}_\Gamma(X)} \leq C\|F_A\|^\nu_{L^r(X)}. \tag{3.6}$$

Moreover, if $d \geq 3$ or $d = 2$ and $p > 4/3$, then we may assume that Γ is C^∞.

The main difference between Theorem 3.2 and "Theorem" 3.1 is that we only assert that the estimate (3.6) holds for *some* $\nu(g,G,[\Gamma]) \in (0,1]$ and *not* necessarily for $\nu = 1$. In [8, Appendix A.2], we gave a proof that (3.6) holds with $\nu = 1$ in the special case where $\operatorname{Ker}\Delta_\Gamma \cap \Omega^1(X;\operatorname{ad}P) = \{0\}$, where we assumed that Γ was C^∞ for simplicity. More generally (see [5, Theorem 9]), if the Yang–Mills energy function

$$\mathscr{E}(A) := \frac{1}{2}\int_X |F_A|^2\,d\operatorname{vol}_g, \tag{3.7}$$

is *Morse–Bott* at the point $[\Gamma]$ in the moduli space of flat connections $M(P)$ in the sense that

$$\mathbf{U}_\Gamma(\delta) := \Gamma + \left\{a \in \operatorname{Ker} d_\Gamma^* \cap \Omega^1(X;\operatorname{ad}P) : \|a\|_{W^{1,p}_\Gamma(X)} < \delta \text{ and } F_{\Gamma+a} = 0\right\}$$

is a smooth manifold for small enough $\delta = (g, G, p, \Gamma) \in (0, 1]$ and

$$T_\Gamma \mathbf{U}_\Gamma(\delta) = \operatorname{Ker} \mathscr{E}''(\Gamma),$$

where $\mathscr{E}''(\Gamma) = \operatorname{Ker} \Delta_\Gamma$ is the Hessian operator on $\Omega^1(X; \operatorname{ad}P)$, then (3.6) also holds with $\nu = 1$.

Donaldson and Kronheimer [3, Proposition 4.4.11] employ the local Coulomb gauge estimate (2.4) and a patching argument to prove that (3.6) holds with $\nu = 1$ when X is *strongly simply connected* and $p = 2$ and $d = 2, 3$ and $\Gamma = \Theta$ but remark [3, p. 163] that their result extends to $d = 4$ and $p > 2$. In [3, Proposition 4.4.11], it is not claimed that $d_\Theta^*(u(A) - \Theta) = 0$. Recall that X is strongly simply connected [3, p. 161] if it can be covered by smoothly embedded balls $B_1, \dots, B_m$ such that for any $2 \leq r \leq m$, the intersection $B_r \cap (B_1 \cup \cdots \cup B_{r-1})$ is connected; the condition implies that X is simply connected.

Fukaya [13, Proposition 3.1] proved that a version of (3.6) holds when $d = 4$, and X is a compact manifold with boundary, and A is anti-self-dual. Fukaya's proof of [13, Proposition 3.1] uses the local Coulomb gauge estimate (2.4) and difficult patching argument. In [13, Proposition 3.1], it is not claimed that $d_\Gamma^*(u(A) - \Gamma) = 0$. It is likely [12] that his argument extends to allow arbitrary dimensions $d \geq 2$, connections $A \in \mathscr{A}^{1,p}(P)$, and the system of Sobolev norms in (3.6). If X is a compact manifold without boundary and A is anti-self-dual and $\|F_A\|_{L^2(X)}$ is smaller than a constant that depends at most on G, then the Chern–Weil Theorem (see Milnor and Stasheff [18, Appendix C]) would imply that A is necessarily flat.

Nishinou [20] proved that a version of (3.6) holds when $X = \mathbb{T}^2$ (the real two-dimensional torus) and $P = \mathbb{T}^2 \times \mathrm{SU}(2)$ and and Γ is the product connection and $\nu = 1/2$.

In Section 4, we describe an example due to Mrowka [19] which shows that (3.6) cannot hold for $\nu > 1/2$ when A_t, for $t \in (-\delta, \delta)$, is a certain family of smooth connections on $\mathbb{T}^2 \times \mathrm{SU}(2)$ in Coulomb gauge with respect to the product connection Θ. There is no other flat connection Γ on $\mathbb{T}^2 \times \mathrm{SU}(2)$ that is in Coulomb gauge with respect to Θ and closer in the $W^{1,p}$ norm to A_t, for $t \in (-\delta, \delta)$, and that is no gauge transformation $u \in \operatorname{Aut}(P)$ that can be used to improve the estimate (3.6) by replacing A_t by $u(A_t)$.

The argument provided by Uhlenbeck in her proof of [26, Corollary 4.3] was very brief and that had prompted us to attempt a more detailed justification in [8, 4] using the local Coulomb-gauge estimate (2.4) in Theorem 2.1 and a patching argument. We shall explain why our argument was incorrect in Section 5.

The proof of the estimate (3.6) stated in Theorem 3.2 is quite involved and occupies the majority of our article [5]. Readers familiar with the results of Łojasiewicz will notice that the estimate (3.6) resembles a *Łojasiewicz distance inequality*:

Theorem 3.3 (Łojasiewicz distance inequality for analytic functions on Euclidean space)**.** *(See Łojasiewicz*[3] [17, Theorem 2, p. 85 (62)]*.) Let $n \geq 1$ be an integer, $U \subset \mathbb{R}^n$ be an open neighborhood of the origin, and $f : U \to [0, \infty)$ be an analytic function. If $f(0) = 0$ and $f'(0) = 0$, then there are constants $C \in [1, \infty)$, and $\sigma \in (0, 1]$, and $\delta \in (0, \sigma/4]$, and $\beta \in [1, \infty)$ such that*

$$f(x) \geq C \operatorname{dist}(x, B_\sigma \cap \operatorname{Zero} f)^\beta, \quad \forall\, x \in B_\delta, \tag{3.8}$$

where $\operatorname{Zero} f := f^{-1}(0)$ *and* $\operatorname{dist}(x, Z) := \inf\{\|x - z\| : z \in Z\}$ *for any* $Z \subset \mathbb{R}^n$ *and* $B_r := \{x \in \mathbb{R}^n : \|x\| < r\}$ *for* $r \in (0, \infty)$.

We refer the reader to Feehan [9, Theorem 1 and Corollary 4] for a simpler proof of Theorem 3.3 which is partly inspired by that of Bierstone and Milman [2, Theorem 2.8]. This resemblance

[3]The first page number refers to the version of Łojasiewicz's original manuscript mimeographed by IHES while the page number in parentheses refers to the cited LaTeX version of his manuscript prepared by M. Coste and available on the Internet.

between the inequalities (3.6) and (3.8) can be made a little closer when $r = 2$ in (3.6) and f in Theorem 3.3 is replaced by the Yang–Mills energy function $\mathscr{E}$ in (3.7). Of course, this replacement requires us to consider analogues in infinite dimensions of the arguments in finite dimensions that we used in [9] to prove Theorem 3.3 and, in infinite dimensions, those arguments unfortunately become far more involved. As described in our article [9], Theorem 3.3 is proved with the aid of gradient flow for the analytic function f; its analogue for the Yang–Mills energy function $\mathscr{E}$ is also proved using gradient flow, a task that is necessarily far more involved as researchers in gauge theory are aware. We refer readers who seek a better understanding of how Theorem 3.2 is established to read our proof of Theorem 3.3 for the essential idea.

4. Flat SU(2) connections over a torus and Uhlenbeck's Corollary 4.3

In this section, we describe a counterexample to the estimates stated in [8, Theorem 5.1] and [26, Corollary 4.3], based on an observation due to Mrowka [19]. We give a far more detailed analysis of this example in [5, Appendix A] and so we shall only highlight the main ideas here.

Example 4.1 (Estimate for distance to moduli subspace of flat SU(2) connections over a two-dimensional torus)**.** In the notation of "Theorem" 3.1, choose

$$G = \mathrm{SU}(2), \quad X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad P = \mathbb{T}^2 \times \mathrm{SU}(2),$$

identify connection one-forms on P with $\mathfrak{su}(2)$-valued one-forms on $\mathbb{T}^2$, where $\mathfrak{su}(2)$ denotes the Lie algebra of SU(2), and equip $\mathbb{T}^2$ with its flat Riemannian metric. For a pair of matrices $\xi, \eta \in \mathfrak{su}(2)$, consider the connection one-form

$$A = \xi \otimes dx + \eta \otimes dy \in \Omega^1(\mathbb{T}^2; \mathfrak{su}(2)).$$

We have

$$F_A = dA + \frac{1}{2}[A, A] = \frac{1}{2}[\xi, \eta] dx \wedge dy \in \Omega^2(\mathbb{T}^2; \mathfrak{su}(2)),$$

and thus $F_A = 0 \iff [\xi, \eta] = 0$. Using $d^* = (-1)^{d(p+1)+1} \star d\star$ on $\Omega^p(X; \mathbb{R})$, we note that

$$d^* A = - \star d \star A = - \star d(\xi \otimes dy + \eta \otimes dx) = 0,$$

since $d^2 x = 0 = d^2 y$, and thus A is in Coulomb gauge with respect to the product connection Θ on P. Recall that $\mathfrak{su}(2)$ has basis

$$I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.1}$$

with relations $[I, J] = 2K$, and $[J, K] = 2I$, and $[K, I] = 2J$. For the Lie algebra $\mathfrak{su}(2)$, one can take $B(\xi, \eta) = \operatorname{tr}(\xi\eta)$ to be the Killing form, giving $B(I, I) = B(J, J) = B(K, K) = -2$, and choose $\langle \xi, \eta \rangle := -\frac{1}{2} B(\xi, \eta)$ to be an $\operatorname{Ad} \mathrm{SU}(2)$-invariant inner product on $\mathfrak{su}(2)$, with respect to which the basis $\{I, J, K\}$ is orthonormal.

If $\xi = tI$ and $\eta = tJ$, for a constant $t \in \mathbb{R}$, and we write A_t for the resulting one-parameter family of connections, then

$$F_{A_t} = \frac{1}{2} t^2 [I, J] dx \wedge dy = t^2 K dx \wedge dy,$$

and so $|A_t| \propto |t|$ and $|F_{A_t}| \propto |t|^2$. Consequently, for any $p \in (1, \infty)$,

$$\|A_t\|_{W^{1,p}(\mathbb{T}^2)} \leq C \|F_{A_t}\|_{L^p(\mathbb{T}^2)}^{1/2}, \quad \forall\, t \in \mathbb{R}, \tag{4.2}$$

where $C = C(p) \in [1, \infty)$ is a constant. □

For the family A of connections parameterized by $\mathfrak{su}(2)\times\mathfrak{su}(2)$ in Example 4.1, we also have $dA=0$ and so

$$A \in \mathbf{H}^1_\Theta(\mathbb{T}^2;\mathfrak{su}(2)) := \operatorname{Ker}(d+d^*)\cap\Omega^1(\mathbb{T}^2;\mathfrak{su}(2)) \cong H^1(\mathbb{T}^2;\mathbb{R})\otimes\mathfrak{su}(2) \cong \mathbb{R}^2\otimes\mathfrak{su}(2),$$

where $\mathbf{H}^1_\Theta(\mathbb{T}^2;\mathfrak{su}(2))$ is the Zariski tangent space at Θ to $M(\mathbb{T}^2,\mathrm{SU}(2))$, and by dimension-counting every element of $\mathbf{H}^1_\Theta(\mathbb{T}^2;\mathfrak{su}(2))$ has this form. Furthermore, $[\Theta]$ is not a regular point of $M(\mathbb{T}^2,\mathrm{SU}(2)$ because

$$\mathbf{H}^2_\Theta(\mathbb{T}^2;\mathrm{ad}P) := \operatorname{Ker}(d+d^*)\cap\Omega^2(\mathbb{T}^2;\mathrm{ad}P) \cong H^2(\mathbb{T}^2;\mathbb{R})\otimes\mathfrak{su}(2) \cong \mathfrak{su}(2)$$

and, in particular, $\mathbf{H}^2_\Theta(\mathbb{T}^2;\mathrm{ad}P)$ is non-zero.

As an aside, we note that the virtual dimension s of $M(\mathbb{T}^2,\mathrm{SU}(2))$ is equal to zero since

$$\begin{aligned} s &:= \operatorname{Index}\left(d+d^* : \Omega^1(\mathbb{T}^2;\mathfrak{su}(2)) \to \Omega^2(\mathbb{T}^2;\mathfrak{su}(2))\oplus\Omega^0(\mathbb{T}^2;\mathfrak{su}(2))\right) \\ &= \dim\mathbf{H}^1_\Theta(\mathbb{T}^2;\mathrm{ad}P) - \dim\mathbf{H}^2_\Theta(\mathbb{T}^2;\mathrm{ad}P) - \dim\mathbf{H}^0_\Theta(\mathbb{T}^2;\mathrm{ad}P) = 6-3-3=0. \end{aligned}$$

Regarding Example 4.1, one might further ask if the estimate (4.2) for A_t could be improved in the sense of replacing $\|F_{A_t}\|^{1/2}_{L^p(\mathbb{T}^2)}$ by $\|F_{A_t}\|_{L^p(\mathbb{T}^2)}$ by finding a flat connection Γ such that $\|A_t-\Gamma\|_{W^{1,p}(\mathbb{T}^2)} \le C\|A_t\|_{W^{1,p}(\mathbb{T}^2)}$.

However, we shall explain in the following paragraph that such a strategy cannot be used to improve the estimate (4.2). If such an improvement were possible, then one might expect that a relatively minor modification of the statement of “Theorem” 3.1 would continue hold.

Our [5, Theorem A.9] describes the stratified-space structure of the moduli space of flat SU(2) connections over $\mathbb{T}^2$ as the well-known two-dimensional pillowcase (see Hedden, Herald, and Kirk [15, Sections 3.1 and 3.2] and Kirk [16, Section 1.2]), where $[\Theta]$ represents one corner of the pillowcase (see [5, Figure A.3]). The connections A_t in Example 4.1 are closest to Θ, with $\|A_t-\Gamma\|_{W^{1,p}(\mathbb{T}^2)} \ge \|A_t\|_{W^{1,p}(\mathbb{T}^2)}$ for any $[\Gamma]\in M(\mathbb{T}^2,\mathrm{SU}(2))$ obeying $d^*\Gamma=0$, and so (4.2) cannot be improved by the suggested strategy. Indeed, the parameterization due to Kirk [16, Section 1.2] (see also [5, Equation (A.18)]) of the pillowcase $\operatorname{Hom}(\pi_1(\mathbb{T}^2),\mathrm{SU}(2))/\mathrm{SU}(2)$ and [5, Theorem A.9] show that the family of flat connections

$$\begin{aligned} \Gamma(\alpha,\beta) &:= \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} dx + \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} dy \\ &= \alpha K dx + \beta K dy \in \operatorname{Ker} d^* \cap \Omega^1(\mathbb{T}^2;\mathfrak{su}(2)), \quad \forall\,(\alpha,\beta)\in[0,\pi]\times[0,2\pi] \end{aligned}$$

is a parameterization of $M(\mathbb{T}^2,\mathrm{SU}(2))$. But then

$$\begin{aligned} |A_t-\Gamma(\alpha,\beta)| &= |(tI-\alpha K)dx + (tJ-\beta K)dy| \\ &= \left(t^2+\alpha^2+\beta^2\right)^{1/2}, \end{aligned}$$

and so $|A_t-\Gamma(\alpha,\beta)| \ge |A_t|$, with equality if and only $(\alpha,\beta)=(0,0)$ and $\Gamma(0,0)=\Theta$.

5. Error in the proof of the estimates in “Theorem” 3.1

In this section, we describe a subtle error in our proof of “Theorem” 3.1 (our version of [26, Corollary 4.3]) that we provided in [8, Section 6], referring the reader to that article for a further explanation of notation. In [8, Section 6.3], we assumed that the local gauge transformations $\rho_\alpha : U_\alpha \to G$ defined in [8, Equation (6.14)] (and provided by Theorem 2.1) that take local sections $\sigma^0_\alpha : U_\alpha \to P$ (with respect to which $(\sigma^0_\alpha)^*\Gamma = 0$ on U_α) to local sections $\sigma_\alpha : U_\alpha \to P$ (with respect to which $d\sigma^*_\alpha A = 0$ on U_α) necessarily obey the estimates in [8, Equation (6.4)] on $V_\alpha \subset U_\alpha$ provided by [8, Corollary 6.4].

We proved [8, Corollary 6.4] by explicitly constructing of a collection of maps $\tilde{\rho}_\alpha : U_\alpha \to G$, given two collections of transition functions (denoted by $g_{\alpha\beta}$ and $h_{\alpha\beta}$ in the statement of [8, Corollary 6.4]) that are C^0-close. Our proof of [8, Corollary 6.4] assumed that one could choose $\rho_1 = \mathbf{1} \in G$ on $V_1 = U_1$; however, that assumption will not necessarily hold for the collection of maps $\rho_\alpha : U_\alpha \to G$ provided by Theorem 2.1. We explain this issue and its consequences further in the remainder of this section.

In Uhlenbeck's local Coulomb gauge-fixing result, Theorem 2.1, the Neumann boundary condition (2.3) for $u(A) - \Theta$ on ∂B is invariant under the replacement of $\rho : B \to G$ by $\rho g : B \to G$, for any $g \in G$[4]. The crucial estimate (2.4) for the $W^{1,p}$ norm $u(A) - \Theta$ in terms of the L^p norm of F_A is a consequence of the *a priori* estimate (with $p \in (1, \infty)$ and $C = C(d, G, p) \in [1, \infty)$)

$$\|a\|_{W^{1,p}(B)} \leq C \|(d + d^*)a\|_{L^p(B)},$$

for the first-order elliptic operator

$$d + d^* : \Omega^1(B; \mathfrak{g}) \to \Omega^2(B; \mathfrak{g}) \otimes \Omega^0(B; \mathfrak{g})$$

and Neumann boundary condition $a(\vec{n}) = 0$ on ∂B. With this boundary condition, the operator $d + d^*$ has trivial kernel. See Uhlenbeck [24, Lemma 2.5] or Wehrheim [27, Theorem 5.1] for details. If the operator $d + d^*$ had nontrivial kernel, an additional term $\|a\|_{L^p(B)}$ would be present on the right-hand side of the preceding estimate. Moreover, unless $d^* a = 0$ on B (as one has by the Coulomb gauge-fixing condition (2.2) for the choice $u(A) - \Theta$), one cannot expect an *a priori* estimate of the form

$$\|a\|_{W^{1,p}(B)} \leq C \|da\|_{L^p(B)},$$

due to the nontrivial (in fact, infinite-dimensional) kernel of $d : \Omega^1(B; \mathfrak{g}) \to \Omega^2(B; \mathfrak{g})$ in the absence of a suitable boundary condition for a on ∂B. Hence, for $a_\alpha^0 = (\sigma_\alpha^0)^*(A - \Gamma) = (\sigma_\alpha^0)^* A$ and $(\sigma_\alpha^0)^* F_A = da_\alpha^0 + \frac{1}{2}[a_\alpha^0, a_\alpha^0]$, one cannot expect the estimate

$$\|a_\alpha^0\|_{W^{1,p}(V_\alpha)} \leq C \|F_A\|_{L^p(U_\alpha)}$$

to hold as we had stated on [8, p. 575].

We recall from our proofs of [5, Theorems 1 and 9] that the need for Łojasiewicz exponents $\nu \in (0, 1]$ rather than the optimal $\nu = 1$ in [26, Corollary 4.3] can arise when the kernel $\operatorname{Ker}(d_\Gamma + d_\Gamma^*) \cap \Omega^1(X; \operatorname{ad} P)$ is nontrivial. That issue can be mitigated when $\operatorname{Crit} \mathscr{E} \cap \{\Gamma + \operatorname{Ker} d_\Gamma^* \cap \Omega^1(X; \operatorname{ad} P)\}$ is a smooth manifold near Γ of dimension equal to that of $\operatorname{Ker}(d_\Gamma + d_\Gamma^*) \cap \Omega^1(X; \operatorname{ad} P)$, that is, when the Yang–Mills energy function $\mathscr{E}$ is Morse–Bott at $[\Gamma] \in M(P)$ in the sense of [5, Definition 7.6]. However, at the product connection $[\Theta] \in M(\mathbb{T}^2, \mathrm{SU}(2))$, the Yang–Mills energy function $\mathscr{E}$ is *not* Morse–Bott.

We can use the example of $[\Theta] \in M(\mathbb{T}^2, \mathrm{SU}(2))$ and the family A_t for $t \in (-\delta, \delta)$ to further clarify our error in [8, Section 6.3]. The local gauge transformations $\rho_\alpha : U_\alpha \to \mathrm{SU}(2)$ take the $a_\alpha^0 = (\sigma_\alpha^0)^* A_t$ to Coulomb-gauge local connection one-forms $a_\alpha = \rho_\alpha^{-1} a_\alpha^0 \rho_\alpha + \rho_\alpha^{-1} d\rho_\alpha$ that obey the Neumann boundary condition $a_\alpha(\vec{n}) = 0$ on ∂U_α. Now A_t already obeys $d^* A_t = 0$ and thus $d^* a_\alpha^0 = 0$, so the main purpose of the ρ_α here will be to ensure that the Neumann boundary conditions are obeyed over each U_α and so

$$\|a_\alpha\|_{W^{1,p}(U_\alpha)} \leq C \|da_\alpha\|_{L^p(U_\alpha)}.$$

We do not know and cannot expect that any of these ρ_α will be constant. In particular, we cannot assume, as we did at the beginning of our proof of [8, Corollary 6.4], that $\rho_1 = \mathbf{1}$ on the

[4]Each element $g \in G$ can be viewed as a constant gauge transformation and element of the stabilizer in $\operatorname{Aut}(B, P)$ of the product connection Θ on P.

geodesic ball $V_1 = U_1$ and hence that the estimates in [8, Equation (6.4)] for the $L^p(V_\alpha)$ norms of $\nabla\rho_\alpha$ and $\nabla^2\rho_\alpha$ are all bounded by a constant times $\eta \in (0,1]$. Their bounds in terms of a constant times $\|F_A\|_{L^p(U_\alpha)}$ come from the $L^p(U_\alpha \cap U_\beta)$ estimates for $dg_{\alpha\beta}$ in [8, Equation (6.3)] in terms of a constant times $\eta \in (0,1]$, where the transition functions $g_{\alpha\beta}$ intertwine σ_α and σ_β, and from the $L^p(U_\alpha \cap U_\beta)$ estimates in [8, Equation (6.11a)] for $dg_{\alpha\beta}$ in terms of a constant times $\|F_A\|_{L^p(U_\alpha\cap U_\beta)}$, and from the choice $\eta = \|F_A\|_{L^p(U_\alpha\cap U_\beta)}$.

6. The exceptional case of two-dimensional manifolds

As we noted in Section 2.2, Theorem 2.7 does not cover the case $d = 2$, but the forthcoming Lemma 6.1 provides an *a priori* estimate that is adequate for the purposes of our proof of Theorem 1 in the case $d = 2$. Recall from Uhlenbeck [24, p. 33] or Wehrheim [27, Theorem 9.4 (i)] that if A is a $W^{1,p}$ Yang–Mills connection (for $p \in (d/2,\infty)$ and $p \geq 2$ if $d = 2, 3$), then A is gauge-equivalent to a smooth Yang–Mills connection.

Lemma 6.1 (*A priori* estimate for the curvature of a Yang–Mills connection in dimension two)**.** *Let G be a compact Lie group and A be a smooth Yang–Mills connection with respect to the standard Euclidean metric on $B_r \times G$, where $B_r \subset \mathbb{R}^2$ is the open ball with center at the origin in $\mathbb{R}^2$ and radius $r > 0$. If $F_A \in L^1(B_r;\Lambda^2 \otimes \mathfrak{g})$, then*

$$\|F_A\|_{L^\infty(B_r)} = \mathrm{Vol}(B_r)^{-1}\|F_A\|_{L^1(B_r)}. \tag{6.1}$$

Proof. We adapt the proof of [23, Theorem 4.1] due to P. Smith. Noting that $*F_A \in \Omega^0(B_r;\mathfrak{g})$ when $d = 2$, the Kato Inequality [11, Equation (6.20)] and the Yang–Mills equation for A imply that

$$|d|F_A|| = |d| * F_A|| \leq |d_A * F_A| = |d_A^* F_A| = 0 \quad \text{on } B_r. \tag{6.2}$$

Thus, $|F_A|$ is constant on B_r and the conclusion follows. □

Lemma 6.1 serves as a replacement for Theorem 2.7 when $d = 2$ and in our proof of Theorem 1 in that case, we use the following immediate corollary of the proof of Lemma 6.1 (just replace the ball $B_r \subset \mathbb{R}^2$ by a two-dimensional manifold X) and analogue of Corollary 2.8.

Corollary 6.2 (*A priori* estimate for the curvature of a Yang–Mills connection over a closed two-dimensional manifold)**.** *Let X be a closed, smooth, two-dimensional manifold endowed with a Riemannian metric, g. Let G be a compact Lie group and A be a smooth Yang–Mills connection with respect to the metric, g, on a smooth principal G-bundle P over X. Then*

$$\|F_A\|_{L^\infty(X)} = \mathrm{Vol}_g(X)^{-1}\|F_A\|_{L^1(X)}. \tag{6.3}$$

7. Corrections to the proof of Theorem 1

In this section, we correct the proof of Theorem 1 that we had provided in [8, Section 7]. The changes are minor but involve special handling for the exceptional case $d = 2$ and a replacement of the role of "Theorem" 3.1 by that of Theorem 3.2 and an application of the following version of the Łojasiewicz gradient inequality:

Corollary 7.1 (Łojasiewicz–Simon gradient inequality for the Yang–Mills energy function near flat connections)**.** *(See Feehan* [8, Corollary 3.3]*.) Let X be a closed, smooth manifold of dimension d and endowed with a Riemannian metric, g, and G be a compact Lie group. Assume that $d \geq 2$ and $p \in [2,\infty)$ obeys $p \geq d/2$. Then there are positive constants c, σ, and $\theta \in [1/2, 1)$,* *depending on g, G, and p with the following significance. If A is a connection of class $W^{1,q}$ on a principal G-bundle, P, over X, with $q \in [2,\infty)$ obeying $q > d/2$ and $q \geq p$, such that*

$$\|A - \Gamma\|_{W^{1,p}_\Gamma(X)} < \sigma, \tag{7.1}$$

for some flat connection, Γ, of class $W^{1,q}$ on P, then

$$\|d_A^* F_A\|_{W^{-1,p}_\Gamma(X)} \geq c|\mathscr{E}(A)|^\theta. \tag{7.2}$$

In particular, if A in Corollary 7.1 is a Yang–Mills connection, then (7.2) implies that A is necessarily flat. Rather than attempt to indicate the line-by-line changes to [8, Section 7], we give the modifications here in full. We begin with the

Corollary 7.2 (Existence of a nearby flat connection on a principal bundle supporting a C^∞ Yang–Mills connection with $L^{d/2}$-small curvature)**.** *(Correction to Feehan* [8, Corollary 7.1]*.) Let X be a closed, smooth manifold of dimension $d \geq 2$ and endowed with a Riemannian metric, g, and G be a compact Lie group. Then there are a constant $\varepsilon = \varepsilon(g,G) \in (0,1]$ and, for any $r \in (1,\infty)$, a constant $C = C(g,G,r) \in [1,\infty)$ with the following significance. Let A be a C^∞ Yang–Mills connection on a C^∞ principal G-bundle P over X. If the curvature F_A obeys* (1.1)*, that is,*

$$\|F_A\|_{L^{d/2}(X)} \leq \varepsilon,$$

then there are a C^∞ flat connection Γ on P, a constant $\nu = \nu(g,G,[\Gamma]) \in (0,1]$, and a C^∞ gauge transformation $u \in \mathrm{Aut}(P)$ such that

$$d_\Gamma^*(u(A) - \Gamma) = 0 \quad \text{a.e. on } X, \tag{7.3}$$

$$\|u(A) - \Gamma\|_{W^{1,r}_\Gamma(X)} \leq C\|F_A\|_{L^r(X)}^\nu. \tag{7.4}$$

Proof. The estimate (2.9) in Corollary 2.8, for any $d \geq 3$ and $p \in [1,\infty]$, and the equality (6.3) in Corollary 6.2, for $d = 2$, yield

$$\|F_A\|_{L^p(X)} \leq (\mathrm{Vol}_g(X))^{1/p}\,\|F_A\|_{L^\infty(X)} \leq K\,(\mathrm{Vol}_g(X))^{1/p}\,\|F_A\|_{L^2(X)} \quad (d \geq 3), \tag{7.5a}$$

$$\|F_A\|_{L^\infty(X)} = (\mathrm{Vol}_g(X))^{-1}\,\|F_A\|_{L^1(X)} = (\mathrm{Vol}_g(X))^{-1}\,\|F_A\|_{L^{d/2}(X)} \quad (d = 2), \tag{7.5b}$$

for $K = K(g) \in [1,\infty)$. If $d \geq 4$, then (writing $1/2 = (d-4)/(2d) + 2/d$)

$$\|F_A\|_{L^2(X)} \leq (\mathrm{Vol}_g(X))^{(d-4)/2d}\,\|F_A\|_{L^{d/2}(X)}, \quad \forall\, d \geq 4. \tag{7.6}$$

If $d = 3$, then interpolation [14, Equation (7.9)] implies that

$$\|F_A\|_{L^2(X)} \leq \|F_A\|_{L^{3/2}(X)}^\lambda \|F_A\|_{L^t(X)}^{1-\lambda},$$

where the exponent t obeys $2 < t \leq \infty$ and the constant $\lambda \in (0,1)$ is defined by $1/2 = \lambda/(3/2) + (1-\lambda)/t$. We may choose $t = \infty$ and thus $\lambda = 3/4$ to give

$$\begin{aligned}\|F_A\|_{L^2(X)} &\leq \|F_A\|_{L^{3/2}(X)}^{3/4}\|F_A\|_{L^\infty(X)}^{1/4} \\ &\leq \|F_A\|_{L^{3/2}(X)}^{3/4}\left(K\|F_A\|_{L^2(X)}\right)^{1/4} \quad \text{(by Corollary 2.8)},\end{aligned}$$

and thus

$$\|F_A\|_{L^2(X)} \leq K^{(4-d)/d}\|F_A\|_{L^{d/2}(X)} \quad (d = 3). \tag{7.7}$$

Therefore, by combining (7.5) (for $d \geq 2$), (7.6) (for $d \geq 4$), and (7.7) (for $d = 3$), we obtain

$$\|F_A\|_{L^p(X)} \leq C_1\|F_A\|_{L^{d/2}(X)}, \quad \forall\, d \geq 2 \text{ and } p \geq 1, \tag{7.8}$$

for $C_1 = C_1(g,p) \in [1,\infty)$. Hence, the preceding inequality and the hypothesis (1.1), namely $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$, of Corollary (7.2) ensure that the hypothesis (3.4) of Theorem 3.2 applies for small enough $\varepsilon = \varepsilon(g,G) \in (0,1]$ by taking $p = (d+1)/2$ in (3.4). The conclusions now follow from Theorem 3.2 and Remark 2.3 for smoothness of u. □

We can now finally give the corrections to our proof of Theorem 1. The changes to [8, Proof of Theorem 1, p. 577] are minor since they only involve a replacement of the role of [8, Corollary 7.1] by its corrected version, Corollary 7.2 (allowing an exponent $\nu \in (0,1]$ rather than assuming $\nu = 1$ in the third line of [8, Proof of Theorem 1, p. 577]) and a slight adjustment for the case $d = 2$ (tenth line of [8, Proof of Theorem 1, p. 577]), but for clarity we give the proof in full.

Proof of Theorem 1. For small enough $\varepsilon = \varepsilon(g,G) \in (0,1]$, Corollary 7.2 provides a smooth flat connection Γ on P, a constant $\nu = \nu(g,G,[\Gamma]) \in (0,1]$, and a smooth gauge transformation $u \in \mathrm{Aut}(P)$, and the estimate

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} \leq C_0 \|F_A\|^\nu_{L^p(X)},$$

for $p \in (d/2,\infty)$ obeying $p \geq 2$ and $C_0 = C_0(g,G) \in [1,\infty)$. The preceding inequality ensures that the hypothesis (7.1) holds for the Łojasiewicz gradient inequality (7.2) for the Yang–Mills energy function (3.7) near the flat connection Γ,

$$\|u(A) - \Gamma\|_{W^{1,p}_\Gamma(X)} < \sigma,$$

provided, for example, $\|F_A\|_{L^p(X)} \leq (\sigma/(2C_0))^{1/\nu}$. The latter condition is ensured in turn by the hypothesis (1.1), namely $\|F_A\|_{L^{d/2}(X)} \leq \varepsilon$, of Theorem 1 for small enough $\varepsilon = \varepsilon(g,G) \in (0,1]$, since (7.5b) and (7.7) give

$$\|F_A\|_{L^2(X)} \leq C_1 \|F_A\|_{L^{d/2}(X)}, \quad \text{for } d = 2,3,$$

for $C_1 = C_1(g) \in [1,\infty)$. Indeed, the constant

$$\varepsilon := \begin{cases} \sigma/(2C_0) & \text{for } d \geq 4, \\ \sigma/(2C_0C_1) & \text{for } d = 2,3, \end{cases}$$

will suffice. If $p' = p/(p-1) \in (1,2]$ is the Hölder exponent dual to $p \in [2,\infty)$, then the Sobolev Embedding [1, Theorem 4.12] (for $d \geq 2$) implies that $W^{1,p'}(X) \subset L^r(X)$ is a continuous embedding if (*i*) $1 < p' < d$ and $1 < r = (p')^* := dp'/(d-p') \in (1,\infty)$, or (*ii*) $p' = d$ and $1 < r < \infty$, or (*iii*) $d < p' < \infty$ and $r = \infty$. Since $d \geq 2$ by hypothesis, only the first two cases can occur and by duality and density, we obtain a continuous Sobolev embedding, $L^{r'}(X) \subset W^{-1,p}(X)$, where $r' = r/(r-1) \in (1,\infty)$ is the Hölder exponent dual to $r \in (1,\infty)$. The Kato Inequality [11, Equation (6.20)] implies that the norm of the induced Sobolev embedding, $W^{1,p'}_\Gamma(X;\Lambda^1 \otimes \mathrm{ad}P) \subset L^r(X;\Lambda^1 \otimes \mathrm{ad}P)$, is independent of Γ, and hence the norm, $\kappa = \kappa(g,p) \in [1,\infty)$ of the dual Sobolev embedding, $L^{r'}(X;\Lambda^1 \otimes \mathrm{ad}P) \subset W^{-1,p}_\Gamma(X;\Lambda^1 \otimes \mathrm{ad}P)$, is also independent of Γ. The preceding embedding and the Łojasiewicz–Simon gradient inequality (7.2) applied to $u(A)$, now yield

$$\|d^*_{u(A)} F_{u(A)}\|_{L^{r'}(X)} \geq \kappa^{-1} c |\mathscr{E}(u(A))|^\theta,$$

and thus

$$\|d^*_A F_A\|_{L^{r'}(X)} \geq \kappa^{-1} c |\mathscr{E}(A)|^\theta,$$

noting that each side of the gradient inequality remains unchanged when $u(A)$ is interchanged with A. But A is a Yang–Mills connection, so $d^*_A F_A = 0$ on X and $\mathscr{E}(A) = \frac{1}{2}\|F_A\|_{L^2(X)} = 0$ by (3.7) and thus A must be a flat connection. □

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