SHORT READING COURSE ON YANG–MILLS GAUGE THEORY AND APPLICATIONS TO LOW-DIMENSIONAL TOPOLOGY

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Overview

Classical gauge theory, as understood by geometric analysts, is the study of spaces of connections on a principle G-bundle over a smooth manifold X equippped with a Riemannian metric g, where G is a Lie group, such as the orthogonal, special orthogonal, unitary, or special unitary groups, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, or $\mathrm{SU}(n)$, respectively. Gauge theory has its origins in Theoretical Physics and *Yang–Mills gauge theory* was proposed by C. N. Yang and R. Mills [44] to describe the behavior of elementary particles using non-Abelian Lie groups and is at the core of the unification of the electromagnetic force and weak forces (for example, $\mathrm{U}(1) \times \mathrm{SU}(2)$) as well as quantum chromodynamics, the theory of the strong force (based on $\mathrm{SU}(3)$). It forms the basis of our understanding of the Standard Model of particle physics. Maxwell's theory of electromagnetics provided the first example of a gauge theory and can be viewed as a special case of Yang–Mills gauge theory with the simplest Abelian Lie group, $G = \mathrm{U}(1)$. While Physicists consider quantum Yang–Mills theory, quantum field theories will not be the subject of this reading course, but rather applications of classical Yang–Mills theory and geometric analysis to understand the topology of 3-manifolds and smooth 4-manifolds.

Beginning in the late 1970s, classical gauge theory was increasingly studied by Mathematicians in an effort to give a rigorous foundation to at least some of the theory employed by Physicists. Key articles included those of Atiyah and Bott [1], Atiyah, Hitchin, and Singer [2], Bourguignon and Lawson [4], Taubes [37, 38], and Uhlenbeck [41, 40]. Donaldson [5] gave a dramatic application of Yang–Mills gauge theory with $G = \mathrm{SU}(2)$ to the topology of smooth 4-manifolds using tools from geometric analysis — nonlinear elliptic partial differential equations on vector bundles over Riemannian manifolds. This was one of a series of landmark results due to him in the period 1983–86 that led to his being awarded a Fields Medal in 1986 along with Freedman [17], who proved the Topological Poincaré Conjecture for closed topological 4-manifolds. Freedman showed that every closed, simply-connected, topological 4-manifold X is completely classified by its intersection form (a homotopy invariant), $Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$, and its Kirby–Siebenmann invariant, $\kappa(X) \in H^4(X;\mathbb{Z}/2\mathbb{Z})$, which is zero if X admits a smooth structure and non-zero otherwise. For example, if $H_2(X;\mathbb{Z}) = 0$ and $Q_X = 0$ and $\kappa(X) = 0$, so X is homotopy-equivalent to the 4-sphere, S^4, then Freedman's Theorem implies that X is homeomorphic to S^4, in other words, the Poincaré Conjecture is true for topological 4-manifolds. The intersection form Q_X can be used to define the *signature* of a closed 4-manifold, $\sigma(X) = b_+(X) - b_-(X)$, where $b_\pm(X)$ are the dimensions of the maximal positive (negative) subspaces of $H_2(X;\mathbb{Z})$ defined by Q_X (equivalently, the number of positive (negative) eigenvalues of symmetric, bilinear form $Q_X : H_2(X;\mathbb{R}) \times H_2(X;\mathbb{R}) \to \mathbb{R}$), while $b_2(X) = b_+(X) + b_-(X)$, the second Betti number or rank of $H_2(X;\mathbb{Z})$. Using Yang–Mills gauge theory, Donaldson proved that if X is a closed, simply-connected, *smooth* 4-manifold with

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$b_-(X) = 0$ and $b_+(X) \geq 1$, then $Q_X = \mathrm{diag}(1, \dots, 1)$ and X is homeomorphic to $\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2$ (the connected sum of $b_2 = b_+$ copies of $\mathbb{CP}^2$).

Donaldson greatly extended his ideas in [5] to define his *Donaldson invariants* [6, 8] of smooth 4-manifolds and which he computed explicitly for complex algebraic surfaces and used to give examples of 4-manifolds that are homeomorphic but not diffeomorphic. His theory was further extended by many mathematicians, but especially Kronheimer and Mrowka [22]. Donaldson invariants were also given a quantum Yang–Mills interpretation by Witten [42] and his interpretation, in collaboration with Seiberg [35, 34], ultimately led to to the development of Seiberg–Witten invariants of smooth 4-manifolds [43]. Those invariants can be computed more easily than Donaldson invariants and the underlying geometric analysis is much simpler than that underlying Donaldson's theory. Witten conjectured an explicit relationship that expressed Donaldson invariants in terms of Seiberg–Witten invariants, since proved by the author and Leness in [14, 13, 12, 11] (modulo a gluing theorem [10], in preparation).

By considering cylindrical 4-manifolds $Y \times \mathbb{R}$, where Y is a closed 3-manifold, Floer [15] constructed 3-dimensional analogues of Donaldson invariants for 3-manifolds, called *instanton Floer homology groups* (see [7] for an exposition), and which he hoped would lead to a proof of the *Poincaré Conjecture for 3-manifolds*. (This was since proved by Perelman [30, 32, 31] using an entirely different approach via the Hamilton–Ricci flow of Riemannian metrics on a given 3-manifold.) Floer's ideas were adapted by Kronheimer and Mrowka [23] and Marcolli and Wang [27] to the setting of the Seiberg–Witten monopole equations on $Y \times \mathbb{R}$ and used to define 3-dimensional analogues of Seiberg–Witten invariants for 3-manifolds, called *monopole Floer homology groups*. While research on applications of Donaldson and Seiberg–Witten invariants to 4-manifolds has matured, Floer homology and its applications to 3-manifolds is an extremely active area of research today.

This short reading course will provide an introduction to the methods required to do research in gauge theory and its applications to 3-manifolds and smooth 4-manifolds. The point of departure will depend on the mathematical background of interested students, but can include:

- Tools from differential and algebraic topology, homotopy theory, Riemannian geometry, vector bundles, Lie groups, and principle bundles.
- Non-linear elliptic equations on vector bundles over Riemannian manifolds.
- Atiyah–Singer index theorem and the Dirac operator.
- Yang–Mills connections, anti-self-dual connections, and Seiberg–Witten monopoles.
- Topological classification of 3-manifolds and 4-manifolds.
- Construction of Donaldson and Seiberg–Witten invariants.
- Gradient flows for analytic functions on Banach manifolds.
- Construction of instanton and monopole Floer homologies.

References for basic tools include Aubin [3], Gilkey [20], Lee [25], Jost [21], Li [26], and Struwe [36]. The monographs by Donaldson and Kronheimer [8], Freed and Uhlenbeck [16], Friedman and Morgan [18], Lawson [24], Morgan [28], Nicolaescu [29], Salamon [33], and Taubes [39] provide introductions to gauge theory. The monographs by Donaldson [7], Donaldson and Kronheimer [8], the author [9], Frøyshov [19], and Kronheimer and Mrowka [23] provide introductions to more advanced topics and serve as a bridge to current research in this field.

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