Lecture 1: SO(3) monopoles and relations between Donaldson and Seiberg-Witten invariants

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Outline

Introduction and main results

Introduction I

In his article [54], Witten (1994)

- Gave a formula expressing the Donaldson series in terms of Seiberg-Witten invariants for standard four-manifolds,
- Outlined an argument based on supersymmetric quantum field theory, his previous work [53] on topological quantum field theories (TQFT), and his work with Seiberg [45, 46] explaining how to derive this formula.

We call a four-manifold standard if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

In a later article [37], Moore and Witten

- extended the scope of Witten's previous formula by allowing four-dimensional manifolds with $b^1 \neq 0$ and $b^+ = 1$, and

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- provided more details underlying the derivation of these formulae using supersymmetric quantum field theory.

Using similar supersymmetric quantum field theoretic ideas methods, Marinõ, Moore, and Peradze (1999) also showed that a certain low-degree polynomial part of the Donaldson series always vanishes [33, 34], a consequence of their notion of superconformal simple type.

Marinõ, Moore, and Peradze noted that this vanishing would confirm a conjecture (attributed to Fintushel and Stern) for a lower bound on the number of (Seiberg-Witten) basic classes of a four-dimensional manifold.

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Soon after the Seiberg-Witten invariants were discovered, Pidstrigatch and Tyurin (1994) proposed a method [43] to prove Witten's formula using a classical field theory paradigm via the space of SO(3) monopoles which simultaneously extend the

- Anti-self-dual SO(3) connections, defining Donaldson invariants, and
- U(1) monopoles, defining Seiberg-Witten invariants.

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The Pidstrigatch-Tyurin paradigm is intuitively appealing, but there are also significant technical difficulties in such an approach.

The purpose of our series of lecture series is to describe a proof using SO(3) monopoles that for all standard four-manifolds,

Seiberg-Witten simple type $\Longrightarrow$ *Superconformal simple type,*
Superconformal simple type $\Longrightarrow$ *Witten's Conjecture.*

Taken together, these implications prove

- Marinõ, Moore, and Peradze's Conjecture on superconformal simple type and Fintushel and Stern's Conjecture on the lower bound on the number of basic classes, and

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- Witten's Conjecture on the relation between Donaldson and Seiberg-Witten invariants.

It is unknown whether all four-manifolds have Seiberg-Witten simple type.

More details can be found in our two articles (in review since October 2014):

1. P. M. N. Feehan and T. G. Leness, *The* $\mathrm{SO}(3)$ *monopole cobordism and superconformal simple type*, arXiv:1408.5307.
2. P. M. N. Feehan and T. G. Leness, *Superconformal simple type and Witten's conjecture*, arXiv:1408.5085.

These are in turn based on methods described earlier in our

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1. P. M. N. Feehan and T. G. Leness, *A general* $\mathrm{SO}(3)$*-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, Memoirs of the American Mathematical Society, in press, arXiv:math/0203047.
2. P. M. N. Feehan and T. G. Leness, *Witten's conjecture for many four-manifolds of simple type*, Journal of the European Mathematical Society **17** (2015), 899–923.

Additional useful references include

1. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles. I: Regularity, compactness and transversality*, Journal of Differential Geometry 49 (1998), 265–410.

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2. P. M. N. Feehan, *Generic metrics, irreducible rank-one* PU(2) *monopoles, and transversality*, Communications in Analysis & Geometry 8 (2000), 905–967.
3. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles and links of top-level Seiberg-Witten moduli spaces*, Journal fr die Reine und Angewändte Mathematik 538 (2001), 57–133.
4. P. M. N. Feehan and T. G. Leness, PU(2) *monopoles. II: Top-level Seiberg-Witten moduli spaces and Witten's conjecture in low degrees*, Journal für die Reine und Angewändte Mathematik 538 (2001), 135–212.

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5. P. M. N. Feehan and T. G. Leness, SO(3) *monopoles, level-one Seiberg-Witten moduli spaces, and Witten's conjecture in low degrees*, Topology and its Applications 124 (2002), 221–326.

Our proofs of these conjectures rely on an assumption of certain analytical properties of gluing maps for SO(3) monopoles (see Hypothesis 5.1), analogous to properties proved by Donaldson and Taubes in contexts of gluing maps for SO(3) anti-self-dual connections.

Verification of those analytical gluing map properties is work in progress [9] and appears well within reach.

Statements of main results I

A closed, oriented four-manifold X has an *intersection form*,

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

One lets $b^{\pm}(X)$ denote the dimensions of the maximal positive or negative subspaces of the form Q_X on $H_2(X;\mathbb{Z})$ and

$$e(X) := \sum_{i=0}^{4}(-1)^i b_i(X) \quad \text{and} \quad \sigma(X) := b^+(X) - b^-(X)$$

denote the *Euler characteristic* and *signature* of X, respectively.

Statements of main results II

We define the characteristic numbers,

$$
\begin{aligned}
c_1^2(X) &:= 2e(X) + 3\sigma(X), \\
\chi_h(X) &:= (e(X) + \sigma(X))/4, \qquad (1) \\
c(X) &:= \chi_h(X) - c_1^2(X).
\end{aligned}
$$

We call a four-manifold standard if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

(The methods we will describe allow $b^+(X) = 1$ and $b_1(X) > 0$.)

For a standard four-manifold, the Seiberg-Witten invariants comprise a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

Statements of main results III

on the set of spin^c structures on X.

The set of Seiberg-Witten basic classes, $B(X)$, is the image under $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X, that is

$$B(X) := \{K \in H^2(X;\mathbb{Z}) : K = c_1(\mathfrak{s}) \text{ with } SW_X(\mathfrak{s}) \neq 0\},$$

and is finite.

X has Seiberg-Witten simple type if $K^2 = c_1^2(X)$, $\forall\, K \in B(X)$.

(Here, $c_1(\mathfrak{s})^2 = c_1^2(X) \iff$ the moduli space, $M_{\mathfrak{s}}$, of Seiberg-Witten monopoles has dimension zero.)

In the context of Donaldson invariants [2], there are also concepts of basic class and simple type [27] due to Kronheimer and Mrowka

Statements of main results IV

(1995) which we shall subsequently explain, but prior to Witten, there was no obvious relationship between the Kronheimer-Mrowka and Seiberg-Witten concepts of basic class or simple type.

By virtue of the structure theorem of Kronheimer and Mrowka [27], the Donaldson invariants of a standard four-manifold of simple type (in their sense) can be expressed, for any $w \in H^2(X;\mathbb{Z})$, in the form

$$\mathbf{D}_X^w(h) = e^{Q_X(h)/2} \sum_{K \in H^2(X;\mathbb{Z})} (-1)^{(w^2+K\cdot w)/2} \beta_X(K) e^{\langle K,h \rangle},$$

Statements of main results V

where $\beta_X : H^2(X;\mathbb{Z}) \to \mathbb{Q}$ is a function such that $\beta_X(K) \neq 0$ for at most finitely many classes, K, which are integral lifts of $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ (the Kronheimer-Mrowka basic classes).

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Conjecture 1.1 (Witten's Conjecture)

Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then X has Kronheimer-Mrowka simple type, the Seiberg-Witten and Kronheimer-Mrowka basic classes coincide, and for any $w \in H^2(X;\mathbb{Z})$,

$$(2) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}, \quad \forall\, h \in H_2(X;\mathbb{R}).$$

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As defined by Mariño, Moore, and Peradze, [34, 33], a manifold X has superconformal simple type if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic,

(3) $$\boxed{SW_X^{w,i}(h) = 0 \quad \text{for } i \leq c(X) - 4}$$

and all $h \in H_2(X;\mathbb{R})$, where

$$\boxed{SW_X^{w,i}(h) := \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}), h\rangle^i}$$

From [8], we have the

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Theorem 1.2 (All standard four-manifolds with Seiberg-Witten simple type have superconformal simple type)

(See F and Leness [8, Theorem 1.1].) Assume Hypothesis 5.1. If X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

Hypothesis 5.1 asserts certain analytical properties of local gluing maps for SO(3) monopoles constructed by the authors in [10].

Proofs of these analytical properties, analogous to known properties of local gluing maps for anti-self-dual SO(3) connections and Seiberg-Witten monopoles, are being developed by us [9].

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Global gluing maps are used to describe the topology of neighborhoods of Seiberg-Witten monopoles appearing at all levels of the compactified moduli space of SO(3) monopoles and hence construct "links" of those singularities.

Marinõ, Moore, and Peradze had previously shown [34, Theorem 8.1.1] that if the set of Seiberg-Witten basic classes, $B(X)$, is non-empty and X has superconformal simple type, then

$$|B(X)/\{\pm 1\}| \geq [c(X)/2]. \tag{4}$$

Theorem 1.2 and [34, Theorem 8.1.1] therefore yield a proof of the following result, first conjectured by Fintushel and Stern [18].

Statements of main results X

Corollary 1.3 (Lower bound for the number of basic classes)

(See F and Leness [8, Corollary 1.2]) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume Hypothesis 5.1. If $B(X)$ is non-empty and $c(X) \geq 3$, then the number of basic classes obeys the lower bound (4).

From [11], we have the

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Theorem 1.4 (Superconformal simple type $\implies$ Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [11, Theorem 1.2].) Assume Hypothesis 5.1. If a standard four-manifold has superconformal simple type, then it satisfies Witten's Conjecture 1.1.

Combining Theorems 1.2 and 1.4 thus yields the following

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Corollary 1.5 (Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [11, Corollary 1.3] or [8, Corollary 1.4].) Assume Hypothesis 5.1. If X is a standard four-manifold of Seiberg-Witten simple type then X satisfies Witten's Conjecture 1.1.

Further results and future directions I

We describe some additional results, motivations, and future directions for research.

Kronheimer and Mrowka applied our SO(3)-monopole cobordism formula (Theorem 3.7) to give their first of two proofs of Property P for knots in [29].

Property P asserts that $+1$ surgery on a non-trivial knot K in S^3 yields a three-manifold which is not a homotopy sphere.

In [29, Theorem 6], Kronheimer and Mrowka employed our SO(3) monopole cobordism formula (Theorem 3.7) to prove that Witten's Conjecture holds for a large family of four-manifolds.

Further results and future directions II

They then argue that a counterexample to Property P would allow them to construct a four-manifold with non-trivial Seiberg-Witten invariants but trivial Donaldson invariants.

As such a four-manifold would contradict Theorem 3.7, there can be no counterexample to Property P.

They generalized their result in [28], by similar methods, but provided an entirely new proof of Property P in [31] that does not rely on Theorem 3.7.

Sivek has applied our SO(3) monopole cobordism formula (Theorem 3.7) to show that symplectic four-manifolds with $b_1 = 0$ and odd $b^+ > 1$ have nonvanishing Donaldson invariants, and that

Further results and future directions III

the canonical class is always a Kronheimer-Mrowka basic class [47].

In parallel to their role in confirming the relationship between the Donaldson and Seiberg-Witten invariants of four-manifolds, one could use SO(3) monopoles to explore the relationship between the instanton (Yang-Mills) and monopole (Seiberg-Witten) Floer homologies of three-manifolds.

The SO(3)-monopole cobordism yields a proof of Witten's formula for Donaldson invariants in terms of Seiberg-Witten invariants.

While lengthy, the SO(3)-monopole cobordism approach is mathematically self-contained.

Further results and future directions IV

As far as we can tell, classical field theory and the SO(3)-monopole cobordism yields no insight into Witten's original intuition or his quantum-field theory approach to deriving the relationship between Donaldson and Seiberg-Witten invariants.

Understanding the relationship between these two very different viewpoints (classical and quantum field theory) remains an interesting open problem.

Lecture outline I

1. SO(3) *monopoles and relations between Donaldson and Seiberg-Witten invariants.*

   Introduction to the SO(3)-monopole cobordism formula and its consequences.

2. SO(3)*-monopole cobordism formula and superconformal simple type.*

   Verification using the SO(3)-monopole cobordism formula that all Seiberg-Witten simple type standard four-manifolds have superconformal simple type.

Lecture outline II

3. *Superconformal simple type and Witten's conjecture.*

   Verification using the SO(3)-monopole cobordism formula that all superconformal simple type standard four-manifolds satisfy Witten's formula.

Review of Donaldson and Seiberg-Witten invariants

Seiberg-Witten invariants

Seiberg-Witten invariants I

Detailed expositions of the theory of Seiberg-Witten invariants, introduced by Witten in [54], are provided in [30, 38, 41].

These invariants define an integer-valued map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

on the set of spin^c structures on X.

A spin^c structure, $\mathfrak{s} = (W^\pm, \rho_W)$ on X, consists of a pair of complex rank-two bundles $W^\pm \to X$ and a Clifford multiplication map $\rho = \rho_W : T^*X \to \mathrm{Hom}_\mathbb{C}(W^\pm, W^\mp)$ such that [26, 32, 44]

$$\rho(\alpha)^* = -\rho(\alpha) \quad \text{and} \quad \rho(\alpha)^*\rho(\alpha) = g(\alpha, \alpha)\,\mathrm{id}_W, \tag{5}$$

Seiberg-Witten invariants II

for all $\alpha \in C^\infty(T^*X)$, where $W = W^+ \oplus W^-$ and g denotes the Riemannian metric on T^*X.

The Clifford multiplication ρ induces canonical isomorphisms $\Lambda^\pm \cong \mathfrak{su}(W^\pm)$, where $\Lambda^\pm = \Lambda^\pm(T^*X)$ are the bundles of self-dual and anti-self-dual two-forms, with respect to the Riemannian metric g on T^*X.

Any two spin connections on W differ by an element of $\Omega^1(X; i\mathbb{R})$, since the induced connection on $\mathfrak{su}(W) \cong \Lambda^2$ is determined by the Levi-Civita connection for the metric g on T^*X.

Consider a spin connection, B, on W and section $\Psi \in C^\infty(W^+)$.

Seiberg-Witten invariants III

We call a pair (B, Ψ) a Seiberg-Witten monopole if

(6)
$$\begin{aligned} \operatorname{Tr}(F_B^+) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - \eta &= 0, \\ D_B\Psi + \rho(\vartheta)\Psi &= 0, \end{aligned}$$

where, writing $\mathfrak{u}(W^+) = i\underline{\mathbb{R}} \oplus \mathfrak{su}(W^+)$,

- $F_B^+ \in C^\infty(\Lambda^+ \otimes \mathfrak{u}(W^+))$ is the self-dual component of the curvature F_B of B, and
- $\operatorname{Tr}(F_B^+) \in C^\infty(\Lambda^+ \otimes i\underline{\mathbb{R}})$ is the trace part of F_B^+,
- $D_B = \rho \circ \nabla_B : C^\infty(W^+) \to C^\infty(W^-)$ is the Dirac operator defined by the spin connection B,
- The perturbation terms τ and ϑ are as in our version of the forthcoming SO(3)-monopole equations (17),

Seiberg-Witten invariants IV

- $\eta \in C^\infty(i\Lambda^+)$ is an additional perturbation term,
- The quadratic term $\Psi \otimes \Psi^*$ lies in $C^\infty(i\mathfrak{u}(W^+))$ and $(\Psi \otimes \Psi^*)_0$ denotes the traceless component lying in $C^\infty(i\mathfrak{su}(W^+))$, so $\rho^{-1}(\Psi \otimes \Psi^*)_0 \in C^\infty(i\Lambda^+)$.

In the usual presentation of the Seiberg-Witten equations, one takes $\tau = \mathrm{id}_{\Lambda^+}$ and $\vartheta = 0$, while η is a generic perturbation.

However, in order to identify solutions to the Seiberg-Witten equations (6) with reducible solutions to the forthcoming SO(3)-monopole equations (17), one needs to employ the perturbations given in equation (6) and choose

$$\eta = F^+_{A_\Lambda}, \tag{7}$$

Seiberg-Witten invariants V

where A_Λ is the fixed unitary connection on the line bundle $\det^{\frac{1}{2}}(V^+)$ with Chern class denoted by $c_1(\mathfrak{t}) = \Lambda \in H^2(X;\mathbb{Z})$ and represented by the real two-form $(1/2\pi i)F_{A_\Lambda}$, where $V = W \otimes E$ and $V^\pm = W^\pm \otimes E$.

Here, E is a rank-two, Hermitian bundle over X arising in definitions of anti-self-dual SO(3) connections and SO(3) monopoles.

Given a spin^c structure, $\mathfrak{s}$, one may construct a moduli space, $M_\mathfrak{s}$, of solutions to the Seiberg-Witten monopole equations, modulo gauge equivalence.

Seiberg-Witten invariants VI

The space, $M_{\mathfrak{s}}$, is a compact, finite-dimensional, oriented, smooth manifold (for generic perturbations of the Seiberg-Witten monopole equations) of dimension

$$\dim M_{\mathfrak{s}} = \frac{1}{4}\left(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma\right), \tag{8}$$

and contains no zero-section points $[B, 0]$.

When $M_{\mathfrak{s}}$ has odd dimension, the Seiberg-Witten invariant, $SW_X(\mathfrak{s})$, is defined to be zero.

When $M_{\mathfrak{s}}$ has dimension zero, then $SW_X(\mathfrak{s})$, is defined by counting the number of points in $M_{\mathfrak{s}}$.

Seiberg-Witten invariants VII

When $M_\mathfrak{s}$ has even positive dimension d_s, one defines

$$SW_X(\mathfrak{s}) := \langle \mu_\mathfrak{s}^{d_s/2}, [M_\mathfrak{s}] \rangle,$$

where $\mu_\mathfrak{s} = c_1(\mathbb{L}_\mathfrak{s})$ is the first Chern class of the universal complex line bundle over the configuration space of pairs.

If $\mathfrak{s} \in \mathrm{Spin}^c(X)$, then $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ and $c_1(\mathfrak{s}) \equiv w_2(X) \pmod 2 \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, where $w_2(X)$ is second Stiefel-Whitney class of X.

One calls $c_1(\mathfrak{s})$ a Seiberg-Witten basic class if $SW_X(\mathfrak{s}) \neq 0$.

Define

$$B(X) = \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}. \tag{9}$$

Seiberg-Witten invariants VIII

If $H^2(X;\mathbb{Z})$ has 2-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ is not injective.

Because we will work with functions involving real homology and cohomology, we define

$$(10) \qquad SW_X' : H^2(X;\mathbb{Z}) \ni K \mapsto \sum_{\mathfrak{s}\in c_1^{-1}(K)} SW_X(\mathfrak{s}) \in \mathbb{Z}.$$

With the preceding definition, Witten's Formula (2) is equivalent to

$$(11) \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2+K\cdot w)} SW_X'(K) e^{\langle K,h\rangle}.$$

Seiberg-Witten invariants IX

A four-manifold, X, has Seiberg-Witten simple type if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1^2(\mathfrak{s}) = c_1^2(X)$ (or, in other words, $\dim M_\mathfrak{s} = 0$).

As discussed by Morgan [38, Section 6.8], there is an involution on $\mathrm{Spin}^c(X)$, denoted by $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$ and defined by taking complex conjugates, and having the property that $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

By Morgan [38, Corollary 6.8.4], one has

$$SW_X(\bar{\mathfrak{s}}) = (-1)^{\chi_h(X)} SW_X(\mathfrak{s})$$

and so $B(X)$ is closed under the action of $\{\pm 1\}$ on $H^2(X;\mathbb{Z})$.

Donaldson invariants

Donaldson invariants I

In [27, Section 2], Kronheimer and Mrowka defined the Donaldson series which encodes the Donaldson invariants developed in [2].

For $w \in H^2(X;\mathbb{Z})$, the Donaldson invariant is a linear function,

$$D_X^w : \mathbb{A}(X) \to \mathbb{R},$$

where $\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X;\mathbb{R}))$, the symmetric algebra.

For $h \in H_2(X;\mathbb{R})$ and a generator $x \in H_0(X;\mathbb{Z})$, one defines $D_X^w(h^{\delta-2m}x^m) = 0$ unless

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4. \tag{12}$$

If (12) holds, then $D_X^w(h^{\delta-2m}x^m)$ is (heuristically) defined by pairing

Donaldson invariants II

1. A cohomology class $\mu(z)$ of dimension 2δ on the configuration space of SO(3) connections on $\mathfrak{su}(E)$, corresponding to the degree-δ element $z = h^{\delta-2m}x^m \in \mathbb{A}(X)$, and
2. A fundamental class $[\bar{M}_\kappa^w(X)]$ defined by the Uhlenbeck compactification of a moduli space $M_\kappa^w(X)$ of anti-self-dual SO(3) connections on $\mathfrak{su}(E)$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ and E is a rank-two Hermitian bundle with $w = c_1(E)$.

Donaldson invariants III

See Donaldson [2], Donaldson and Kronheimer [3], Friedman and Morgan [19], Kronheimer and Mrowka [27], and Morgan and Mrowka [39] for detailed accounts of various approaches to the definition of $D_X^w(h^{\delta-2m}x^m)$.

Suppose A is a unitary connection on a Hermitian vector bundle E over X and $\hat{A}$ is the induced SO(3) connection on $\mathfrak{su}(E)$.

One calls $\hat{A}$ *anti-self-dual* (with respect to the metric, g, on X) if

$$F_{\hat{A}}^+ = 0,$$

where $F_{\hat{A}}$ is the curvature of $\hat{A}$ and $F_{\hat{A}}^+$ is its self-dual component with respect to the splitting, $\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$.

Donaldson invariants IV

We denote $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ and $w = c_1(E)$ and write $M_\kappa^w(X)$ for the moduli space of gauge-equivalence classes of anti-self-dual SO(3) connections on $\mathfrak{su}(E)$.

A four-manifold has Kronheimer-Mrowka simple type if for all $w \in H^2(X;\mathbb{Z})$ and all $z \in \mathbb{A}(X)$ one has

(13) $$D_X^w(x^2 z) = 4D_X^w(z).$$

This equality implies that the Donaldson invariants are determined by the Donaldson series, the formal power series

(14) $$\mathbf{D}_X^w(h) := D_X^w((1+\tfrac{1}{2}x)e^h), \quad \forall\, h \in H_2(X;\mathbb{R}).$$

Donaldson invariants V

The following result was established by Kronheimer and Mrowka just prior to the advent of Seiberg-Witten invariants.

Donaldson invariants VI

Theorem 2.1 (Structure of Donaldson invariants)

[27, Theorem 1.7 (a)] Let X be a standard four-manifold with KM-simple type. Suppose that some Donaldson invariant of X is non-zero. Then there is a function,

$$\beta_X : H^2(X;\mathbb{Z}) \to \mathbb{Q}, \tag{15}$$

such that $\beta_X(K) \neq 0$ for at least one and at most finitely many classes, K, which are integral lifts of $w_2(X) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ (the Kronheimer-Mrowka basic classes*), and for any $w \in H^2(X;\mathbb{Z})$, one has the following equality of analytic functions of $h \in H_2(X;\mathbb{R})$:*

$$\mathbf{D}^w_X(h) = e^{Q_X(h)/2} \sum_{K\in H^2(X;\mathbb{Z})} (-1)^{(w^2+K\cdot w)/2}\beta_X(K)e^{\langle K,h\rangle}. \tag{16}$$

Donaldson invariants VII

More generally (see Kronheimer and Mrowka [25]), a four-manifold X has *finite type* or *type* τ if

$$D_X^w((x^2-4)^\tau z) = 0,$$

for some $\tau \in \mathbb{N}$ and all $z \in \mathbb{A}(X)$.

Kronheimer and Mrowka conjectured [25] that all four-manifolds X with $b^+(X) > 1$ have finite type and state an analogous formula for the series $\mathbf{D}_X^w(h)$.

Proofs of different parts of their conjecture have been given by Frøyshov [20, Corollary 1], Muñoz [42, Corollary 7.2 & Proposition 7.6], and Wieczorek [52, Theorem 1.3].

SO(3)-monopole cobordism

SO(3)-monopole equations I

The SO(3)-monopole equations take the form,

$$\begin{aligned} \mathrm{ad}^{-1}(F_{\hat{A}}^{+}) - \tau\rho^{-1}(\Phi \otimes \Phi^{*})_{00} &= 0, \\ D_A\Phi + \rho(\vartheta)\Phi &= 0. \end{aligned} \tag{17}$$

where

- A is a spin connection on $V = W \otimes E$ and E is a Hermitian, rank-two bundle,
- $\Phi \in C^{\infty}(W^{+} \otimes E)$,
- $F_{\hat{A}}^{+} \in C^{\infty}(\Lambda^{+} \otimes \mathfrak{so}(\mathfrak{su}(E)))$ is the self-dual component of the curvature $F_{\hat{A}}$ of the induced SO(3) connection, $\hat{A}$, on $\mathfrak{su}(E)$,
- $\mathrm{ad}^{-1}(F_{\hat{A}}^{+}) \in C^{\infty}(\Lambda^{+} \otimes \mathfrak{su}(E))$,

SO(3)-monopole equations II

- $D_A = \rho \circ \nabla_A : C^\infty(V^+) \to C^\infty(V^-)$ is the Dirac operator,
- $\tau \in C^\infty(\mathrm{GL}(\Lambda^+))$ and $\vartheta \in C^\infty(\Lambda^1 \otimes \mathbb{C})$ are perturbation parameters.

For $\Phi \in C^\infty(V^+)$, we let Φ^* denote its pointwise Hermitian dual and let $(\Phi \otimes \Phi^*)_{00}$ be the component of $\Phi \otimes \Phi^* \in C^\infty(i\mathfrak{u}(V^+))$ which lies in the factor $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ of the decomposition,

$$i\mathfrak{u}(V^+) \cong \underline{\mathbb{R}} \oplus i\mathfrak{su}(V^+).$$

The Clifford multiplication ρ defines an isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism

$$\rho = \rho \otimes \mathrm{id}_{\mathfrak{su}(E)} : \Lambda^+ \otimes \mathfrak{su}(E) \cong \mathfrak{su}(W^+) \otimes \mathfrak{su}(E).$$

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SO(3)-monopole equations III

Note also that

$$\mathrm{ad} : \mathfrak{su}(E) \to \mathfrak{so}(\mathfrak{su}(E))$$

is an isomorphism of real vector bundles.

We fix, once and for all, a smooth, unitary connection A_Λ on the square-root determinant line bundle, $\det^{\frac{1}{2}}(V^+)$, and require that our unitary connections A on $V = V^+ \oplus V^-$ induce the resulting unitary connection on $\det(V^+)$,

$$A^{\det} = 2A_\Lambda \text{ on } \det(V^+), \tag{18}$$

where $A^{\det}$ is the connection on $\det(V^+)$ induced by $A|_{V^+}$.

SO(3)-monopole equations IV

If a unitary connection A on V induces a connection $A^{\det} = 2A_\Lambda$ on $\det(V^+)$, then it induces the connection A_Λ on $\det^{\frac{1}{2}}(V^+)$.

We let $\mathscr{M}_{\mathfrak{t}}$ denote the moduli space of solutions to the SO(3)-monopole equations (17) moduli gauge-equivalence, where $\mathfrak{t} = (\rho, W^\pm, E)$.

The moduli space, $\mathscr{M}_{\mathfrak{t}}$, of SO(3) monopoles contains the

- Moduli subspace of anti-self-dual SO(3) connections, M_κ^w, identified with the subset of equivalence classes of SO(3) monopoles, $[A, 0]$, with $\Phi \equiv 0$, and

SO(3)-monopole equations V

- Moduli subspaces, $M_{\mathfrak{s}}$, of Seiberg-Witten monopoles, identified with subsets of equivalence classes of SO(3) monopoles, $[A_1 \oplus A_2, \Phi_1 \oplus 0]$, where the connections, A on E, become reducible with respect to different splittings, $E = L_1 \oplus L_2$, and A_i is a $U(1)$ connection on L_i, and $\mathfrak{s} = (\rho, W^{\pm} \oplus L_1)$.

We let $\mathscr{M}_{\mathfrak{t}}^{*,0}$ denote the complement in $\mathscr{M}_{\mathfrak{t}}$ of these subspaces of zero-section and reducible SO(3) monopoles.

SO(3)-monopole equations VI

Theorem 3.1 (Transversality for the moduli space of SO(3) monopoles)

(See F [4], F and Leness [12], or Teleman [51].) Let $\mathfrak{t}$ be a spin^u structure on a standard four-manifold, X. For generic perturbations of the SO(3) *monopole equations, the moduli space, $\mathscr{M}_{\mathfrak{t}}^{*,0}$,is a smooth, orientable manifold of dimension*

$$\dim \mathscr{M}_{\mathfrak{t}} = 2d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}),$$

where, for $\chi_h(X)$ and $c_1^2(X)$ as in (1),

$$d_a(\mathfrak{t}) := \frac{1}{2}\dim M_\kappa^w = -p_1(\mathfrak{su}(E)) - 3\chi_h(X), \tag{19a}$$

$$n_a(\mathfrak{t}) := \frac{1}{4}\left(p_1(\mathfrak{su}(E)) + c_1(W^+ \otimes E)^2 - c_1^2(X) + 8\chi_h(X)\right). \tag{19b}$$

SO(3)-monopole equations VII

The space $\mathscr{M}_\mathfrak{t}$ has an Uhlenbeck compactification, $\bar{\mathscr{M}}_\mathfrak{t}$ (see [12]).

For $\mathfrak{t} = (W^\pm \otimes E, \rho)$ and integer $\ell \geq 0$, define $\mathfrak{t}(\ell) := (W^\pm \otimes E_\ell, \rho)$, where

$$c_1(E_\ell) = c_1(E), \quad c_2(E_\ell) = c_2(E) - \ell.$$

We define the space of ideal monopoles by

$$I^N \mathscr{M}_\mathfrak{t} = \bigcup_{\ell=0}^{N} \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X) \right),$$

where $\mathrm{Sym}^\ell(X)$ is the symmetric product of X (ℓ times), $\mathrm{Sym}^0(X)$ is point, and N is a sufficiently large integer (depending at most on topological invariants of X and E).

SO(3)-monopole equations VIII

Because $\dim \mathscr{M}_{\mathfrak{t}(\ell)} = \dim \mathscr{M}_{\mathfrak{t}} - 6\ell$, one has

$$\dim \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X) = \dim \mathscr{M}_{\mathfrak{t}} - 2\ell.$$

For each level, ℓ, in the range $0 \leq \ell \leq N$, the SO(3) monopole moduli space, $\mathscr{M}_{\mathfrak{t}(\ell)}$, may contain Seiberg-Witten moduli subspaces.

In particular, each lower level of $\bar{\mathscr{M}}_{\mathfrak{t}}$,

$$\bar{\mathscr{M}}_{\mathfrak{t}} \cap \left(\mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X)\right), \quad 1 \leq \ell \leq N,$$

may contain additional Seiberg-Witten moduli subspaces,

$$\mathscr{M}_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X).$$

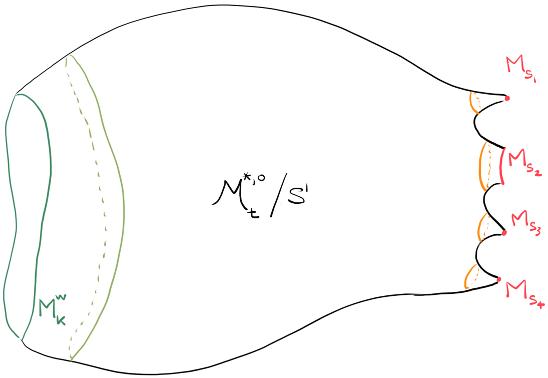
Geometric representatives for cohomology classes I

Our forthcoming SO(3)-monopole cobordism formula (25) is proved by evaluating the pairings of cup products of suitable *cohomology classes* on $\bar{\mathscr{M}}_{\mathfrak{t}}$ with (or intersecting *geometric representatives* of those classes with) the

1. Link of the moduli subspace of anti-self-dual SO(3) connections, $\bar{M}_{\kappa}^{w}$, giving multiples of the Donaldson invariant,
2. Links of the Seiberg-Witten moduli subspaces, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, giving sums of multiples of the Seiberg-Witten invariants.

The following figure illustrates the SO(3)-monopole cobordism between codimension-one links in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ of $\bar{M}_{\kappa}^{w}$ and $M_{\mathfrak{s}_i} \times \mathrm{Sym}^{\ell}(X)$.

Geometric representatives for cohomology classes II



Geometric representatives for cohomology classes III

We recall a definition of a stratified space that will be sufficient for the purposes of defining intersection pairings leading to the general SO(3) monopole cobordism formula.

Definition 3.2 (Smoothly stratified space)

(See Goresky and MacPherson [21], Mather [35], and Morgan, Mrowka, and Ruberman [40, Definition 11.0.1].) A *smoothly stratified space* Z is a topological space with a *smooth stratification* given by a disjoint union, $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_n$, where the *strata* Z_i are smooth manifolds.

There is a partial ordering among the strata, given by $Z_i < Z_j$ if $Z_i \subset \bar{Z}_j$.

There is a unique stratum of highest dimension, Z_0, such that $\bar{Z}_0 = Z$, called the *top stratum*.

Geometric representatives for cohomology classes IV

Definition 3.2 (Smoothly stratified space)

If Y, Z are smoothly stratified spaces, a map $f : Y \to Z$ is *smoothly stratified* if f is a continuous map, there are smooth stratifications of Z and Y such that f preserves strata, and restricted to each stratum f is a smooth map.

A subspace $Y \subset Z$ is *smoothly stratified* if the inclusion is a smoothly stratified map.

If Z is a smoothly stratified space and $f : Z \to \mathbb{R}$ is a smoothly stratified map, that is, f is a continuous map which is smooth on each stratum, then for generic values of ε, the preimage $f^{-1}(\varepsilon)$ is a smoothly stratified subspace of Z.

Geometric representatives for cohomology classes V

We shall use the following definition of a geometric representative for a rational cohomology class.

Definition 3.3 (Geometric representatives for cohomology classes)

(See Donaldson [1], Kronheimer and Mrowka [27, p 588].) Let Z be a smoothly stratified space. A *geometric representative* for a rational cohomology class μ of dimension c on Z is a closed, smoothly stratified subspace $\mathscr{V}$ of Z together with a rational coefficient q, the *multiplicity*, satisfying

1. The intersection $Z_0 \cap \mathscr{V}$ of $\mathscr{V}$ with the top stratum Z_0 of Z has codimension c in Z_0 and has an oriented normal bundle.
2. The intersection of $\mathscr{V}$ with all strata of Z other than the top stratum has codimension 2 or more in $\mathscr{V}$.

Geometric representatives for cohomology classes VI

Definition 3.3 (Geometric representatives for cohomology classes)

3. The pairing of μ with a homology class h of dimension c is obtained by choosing a smooth singular cycle representing h whose intersection with all strata of $\mathscr{V}$ has the codimension $\dim Z_0 - c$ in that stratum of $\mathscr{V}$, and then taking q times the count (with signs) of the intersection points between the cycle and the top stratum of $\mathscr{V}$.

Geometric representatives for cohomology classes VII

Definition 3.4 (Counting intersection of geometric representatives)

Let $\mathscr{V}_1, \ldots, \mathscr{V}_n$ be geometric representatives on a compact, smoothly stratified space Z with multiplicities $q_1, \ldots, q_n$. Assume

1. The sum of the codimensions of the $\mathscr{V}_i$ is equal to the dimension of the top stratum Z_0 of Z.
2. For every smooth stratum Z_s of Z, the smooth submanifolds $\mathscr{V}_i \cap Z_s$ intersect transversely.

Then dimension-counting and the definition of a geometric representative imply that the intersection $\cap_i \mathscr{V}_i$ is a finite collection of points in the top stratum Z_0:

$$\mathscr{V}_1 \cap \cdots \cap \mathscr{V}_n = \{v_1, \ldots, v_N\} \subset Z_0.$$

Geometric representatives for cohomology classes VIII

Definition 3.4 (Counting intersection of geometric representatives)

Let $\varepsilon_j = \pm 1$ be the sign of this intersection at v_j. Then we define the *intersection number of the $\mathscr{V}_i$ in Z* by setting

$$\#\left(\mathscr{V}_1 \cap \cdots \cap \mathscr{V}_n \cap Z\right) = \left(\prod_{i=1}^{n} q_i\right) \sum_{j=1}^{N} \varepsilon_j .$$

A cobordism between two geometric representatives $\mathscr{V}$ and $\mathscr{V}'$ in Z with the same multiplicity is a geometric representative $\mathscr{W} \subset Z \times [0,1]$ which is transverse to the boundary and with $\mathscr{W} \cap Z \times \{0\} = \mathscr{V}$ and $\mathscr{W} \cap Z \times \{1\} = \mathscr{V}'$, with the obvious orientations of normal bundles.

Geometric representatives for cohomology classes IX

The definition of intersection number does not change if $\mathscr{V}_i$ is replaced by $\mathscr{V}_i'$ and there is a cobordism between $\mathscr{V}_i$ and $\mathscr{V}_i'$ whose intersection with the other geometric representatives is transverse in each stratum.

One can see this by observing that the intersection of the cobordism $\mathscr{W}$ with the other geometric representatives will be a collection of one-manifolds contained in Z_0 because the lower strata of Z have codimension two.

The boundaries of these one manifolds are the points in the two intersections

$$\mathscr{V}_1 \cap \cdots \cap \mathscr{V}_n \quad \text{and} \quad \mathscr{V}_1 \cap \ldots \mathscr{V}_{i-1} \cap \mathscr{V}_i' \cap \cdots \cap \mathscr{V}_n,$$

giving the equality of oriented intersection numbers.

Links of strata I

We recall a definition of a link of a stratum in smoothly stratified space, following Mather [35] and Goresky-MacPherson [21].

We need only consider the relatively simple case of a stratified space with two strata since the lower strata in

$$\mathscr{M}_{\mathfrak{t}} \cong \mathscr{M}_{\mathfrak{t}}^{*,0} \cup M_\kappa^w \cup \bigcup_{\mathfrak{s}} M_{\mathfrak{s}} \tag{20}$$

do not intersect when $\mathscr{M}_{\mathfrak{t}}$ *contains no reducible, zero-section solutions*.

The finite union in (20) over $\mathfrak{s}$ is over the subset of all spin^c structures for which $M_{\mathfrak{s}}$ is non-empty and for which there is a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$.

Links of strata II

The space, Z, in the forthcoming Definition 3.5 is a *smoothly stratified space* (with two strata) in the sense of Morgan, Mrowka, and Ruberman [40, Chapter 11].)

Links of strata III

Definition 3.5 (Link of a stratum in a smoothly stratified space)

Let Z be a closed subset of a smooth, Riemannian manifold M, and suppose that $Z = Z_0 \cup Z_1$, where Z_0 and Z_1 are locally closed, smooth submanifolds of M and $Z_1 \subset \bar{Z}_0$.

Let N_{Z_1} be the normal bundle of $Z_1 \subset M$ and let $\mathscr{O}' \subset N_{Z_1}$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$ such that there is a diffeomorphism γ, commuting with the zero section of N_{Z_1} (so $\gamma|_{Z_1} = \mathrm{id}_{Z_1}$), from $\mathscr{O}'$ onto an open neighborhood $\gamma(\mathscr{O}')$ of $Z_1 \subset M$.

Let $\mathscr{O} \Subset \mathscr{O}'$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$, where $\bar{\mathscr{O}} = \mathscr{O} \cup \partial\mathscr{O} \subset \mathscr{O}'$ is a smooth manifold-with-boundary.

Then $L_{Z_1} := Z_0 \cap \gamma(\partial\mathscr{O})$ is a *link of Z_1 in Z_0*.

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General SO(3)-monopole cobordism formula I

It will be more convenient to have Witten's Formula (2) expressed at the level of the Donaldson polynomial invariants rather than the Donaldson power series which they form.

Let $B'(X)$ be a fundamental domain for the action of $\{\pm 1\}$ on $B(X)$.

General SO(3)-monopole cobordism formula II

Lemma 3.6 (Donaldson invariants implied by Witten's formula)

(See F and Leness [16, Lemma 4.2].) Let X be a standard four-manifold. Then X satisfies equation (2) and has Kronheimer-Mrowka simple type if and only if the Donaldson invariants of X satisfy $D^w_X(h^{\delta-2m}x^m) = 0$ for $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$ and for $\delta \equiv -w^2 - 3\chi_h \pmod 4$ satisfy

$$(21)\quad D^w_X(h^{\delta-2m}x^m) = \sum_{\substack{i+2k \\ =\delta-2m}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)}\nu(K) \times \frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-3-m}k!i!}\langle K,h\rangle^i Q_X(h)^k,$$

General SO(3)-monopole cobordism formula III

Lemma 3.6 (Donaldson invariants implied by Witten's formula)

where

$$\varepsilon(w,K) := \frac{1}{2}(w^2 + w\cdot K), \tag{22}$$

and

$$\nu(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases} \tag{23}$$

The SO(3)-monopole cobordism formula given below provides an expression for the Donaldson invariant in terms of the Seiberg-Witten invariants.

General SO(3)-monopole cobordism formula IV

Theorem 3.7 (General SO(3)-monopole cobordism formula)

(See F and Leness [7, Main Theorem].) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume Hypothesis 5.1. Assume further that $w, \Lambda \in H^2(X;\mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfy

$$w - \Lambda \equiv w_2(X) \pmod 2, \tag{24a}$$

$$I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X) > \delta, \tag{24b}$$

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4, \tag{24c}$$

$$\delta - 2m \geq 0. \tag{24d}$$

Then, for any $h \in H_2(X;\mathbb{R})$ and positive generator $x \in H_0(X;\mathbb{Z})$,

General SO(3)-monopole cobordism formula V

Theorem 3.7 (General SO(3)-monopole cobordism formula)

$$(25)\quad D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW_X'(K) \times f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h),$$

where the map,

$$f_{\delta,m}(h) : \mathbb{Z}\times\mathbb{Z}\times H^2(X;\mathbb{Z})\times H^2(X;\mathbb{Z}) \to \mathbb{R}[h],$$

takes values in the ring of polynomials in the variable h with

General SO(3)-monopole cobordism formula VI

Theorem 3.7 (General SO(3)-monopole cobordism formula)

real coefficients, is universal (independent of X) and is given by

$$(26)\quad \begin{aligned} &f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h) \\ &:= \sum_{\substack{i+j+2k \\ =\delta-2m}} a_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m)\,\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k. \end{aligned}$$

For each triple, $i, j, k \in \mathbb{N}$, the coefficients,

$$a_{i,j,k} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N} \to \mathbb{R},$$

are universal (independent of X) real analytic functions of the variables $\chi_h(X)$, $c_1^2(X)$, $c_1(\mathfrak{s})\cdot\Lambda$, Λ^2, and m.

General SO(3)-monopole cobordism formula VII

The left-hand side of the SO(3)-monopole cobordism formula (25) is obtained by computing the intersection number for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the link of the moduli subspace $\bar{M}_{\kappa}^{w}$ of anti-self-dual SO(3) connections.

One uses the fiber-bundle structure of the link over $\bar{M}_{\kappa}^{w}$ to compute the intersection number and show that this is equal to a multiple of the Donaldson invariant, $D_X^w(h^{\delta-2m}x^m)$.

The right-hand side of the SO(3)-monopole cobordism formula (25) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspaces $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$ of ideal Seiberg-Witten monopoles appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

General SO(3)-monopole cobordism formula VIII

One uses the fiber-bundle structure of the link over each Seiberg-Witten moduli space, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, to compute the intersection number and show that this is equal to a multiple of a Seiberg-Witten invariant, $SW'_X(K)$, for each $K \in H^2(X;\mathbb{Z})$ with $c_1(\mathfrak{s}) = K$.

SO(3)-monopole cobordism and algebraic geometry I

When X is a complex projective surface, T. Mochizuki [36] proved a formula (see Göttsche, Nakajima, and Yoshioka [23, Theorem 4.1]) expressing the Donaldson invariants in a form similar to our SO(3)-monopole cobordism formula (Theorem 3.7).

The coefficients in Mochizuki's formula are given as the residues of a generating function for integrals of $\mathbb{C}^*$-equivariant cohomology classes over the product of Hilbert schemes of points on X.

In [23, p. 309], Göttsche, Nakajima, and Yoshioka conjecture that the coefficients in Mochizuki's formula (which are meaningful for any standard four-manifold) and in our SO(3)-monopole cobordism formula are the same.

SO(3)-monopole cobordism and algebraic geometry II

Göttsche, Nakajima, and Yoshioka prove an explicit formula for complex projective surfaces relating Donaldson invariants and Seiberg-Witten invariants of standard four-manifolds of Seiberg-Witten simple type using

> *Nekrasov's deformed partition function for the $N = 2$ SUSY gauge theory with a single fundamental matter*

and verify Witten's Conjecture complex projective surfaces.

In [23, p. 323], Göttsche, Nakajima, and Yoshioka discuss the relationship between their approach, Mochizuki's formula, and our SO(3)-monopole cobordism formula.

SO(3)-monopole cobordism and algebraic geometry III

See also [22, pp. 344–347] for a related discussion concerning their wall-crossing formula for the Donaldson invariants of a four-manifold with $b^+ = 1$.

Witten's conjecture in special cases

Partial results towards Witten's Conjecture I

Because the general SO(3) monopole cobordism formula is complicated and the undetermined coefficients difficult to compute directly, it is natural ask whether any special cases of or partial results towards Witten's Conjecture can be extracted from the SO(3) monopole cobordism formula?

We shall describe a few special cases and ingredients in their proofs, as they help in understanding the proof of the general case.

One other approach to verifying Witten's Conjecture for certain classes of four-manifolds would be to separately compute the Donaldson and Seiberg-Witten invariants when they are already have both Kronheimer-Mrowka and Seiberg-Witten simple type.

Partial results towards Witten's Conjecture II

We shall return to this point, when we describe a useful family of four-manifolds described by Fintushel, Park, and Stern.

Because Seiberg-Witten basic classes and invariants are known for the Fintushel-Park-Stern four-manifolds, one can use them and the blow-up formula for Seiberg-Witten invariants to determine the unknown coefficients in the SO(3) monopole cobordism formula.

However, as we shall discuss, that procedure is quite involved and proceeds in two steps (first that Seiberg-Witten simple type $\Longrightarrow$ superconformal simple type and second that superconformal simple type $\Longrightarrow$ Witten's formula).

Therefore, we shall first discuss the following three special cases:

1. Witten's formula in low-degrees (no gluing)

Partial results towards Witten's Conjecture III

2. Witten's formula in low-degrees (gluing one-instantons)
3. Witten's formula for "many" four manifolds (general gluing of multi-instantons):
   - "Abundant" four-manifolds, or
   - Four-manifolds with $c_1^2(X) \geq \chi_h(X) - 3$.

Witten's formula in low degrees (no gluing) I

For $w \in H^2(X;\mathbb{Z})$, define the Seiberg-Witten series, for all $h \in H_2(X;\mathbb{R})$, by

$$\mathbf{SW}_X^w(h) := \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}),h\rangle}, \tag{27}$$

by analogy with the structure of the Donaldson series $\mathbf{D}_X^w(h)$ (see Kronheimer and Mrowka [27, Theorem 1.7]).

Let $B^\perp \subset H^2(X;\mathbb{Z})$ denote the orthogonal complement of the subset of Seiberg-Witten basic classes, B, with respect to the intersection form Q_X on $H^2(X;\mathbb{Z})$.

Witten's formula in low degrees (no gluing) II

Theorem 4.1 (Witten's formula in low degrees)

(See F and Leness [14, Theorem 1.1].) Let X be a standard four-manifold that is abundant and has Seiberg-Witten simple type. For any $\Lambda \in B^\perp$ and $w \in H^2(X;\mathbb{Z})$ for which $\Lambda^2 = 2-(\chi+\sigma)$ and $w - \Lambda \equiv w_2(X) \pmod 2$, and any $h \in H_2(X;\mathbb{R})$, one has

$$(28)\quad \begin{aligned} \mathbf{D}^w_X(h) &\equiv 0 \equiv \mathbf{SW}^w_X(h) \pmod{h^{c(X)-2}}, \\ \mathbf{D}^w_X(h) &\equiv 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}^w_X(h) \pmod{h^{c(X)}}. \end{aligned}$$

The order-of-vanishing assertion for the series $\mathbf{D}^w_X(h)$ and $\mathbf{SW}^w_X(h)$ in equation (28) was proved in joint work with Kronheimer and Mrowka [5].

Witten's formula in low degrees (no gluing) III

A four-manifold is abundant if the restriction of Q_X to $B^\perp$ contains a hyperbolic sublattice [13, Definition 1.2].

Thus, one can find $f_1, f_2 \in B^\perp$ such that $f_1 \cdot f_1 = f_2 \cdot f_2 = 0$ and $f_1 \cdot f_2 = 1$ and $\{f_1, f_2\} \cup B'(X)$ is linearly independent over $\mathbb{R}$.

The abundance condition ensures that there exist classes $\Lambda \in B^\perp$ with prescribed even square, such as $\Lambda^2 = 2 - (\chi + \sigma)$.

All compact, complex algebraic, simply connected surfaces with $b^+ \geq 3$ are abundant.

There exist simply connected four-manifolds with $b^+ \geq 3$ which are not abundant, but which nonetheless admit classes $\Lambda \in B^\perp$ with prescribed even squares [5, p. 175].

Witten's formula in low degrees (no gluing) IV

Equation (28) tells us that Witten's formula holds, modulo terms of degree greater than or equal to $c(X)$, at least for four-manifolds satisfying the hypotheses of Theorem 4.1.

Equation (28) is proved by considering Seiberg-Witten moduli spaces in the top level, $\ell = 0$, of the compactified SO(3) monopole moduli space, $\bar{\mathscr{M}}_{\mathfrak{t}}$.

In order to prove that Witten's Formula (2) holds modulo h^d for all $d \geq c(X)$, one needs to compute the contributions of Seiberg-Witten moduli spaces in arbitrary levels $\ell \geq 0$.

Equation (28) is a special case of a more general formula for Donaldson invariants which we proved as [14, Theorem 1.2] (F and Leness, 2001), that does not require X to have simple type.

Witten's formula in low degrees (no gluing) V

Hence [14, Theorem 1.2] provides insight into a potential proof of the Moore-Witten formula for the Donaldson series when X has finite type (rather than simple type).

The hypotheses of [14, Theorem 1.2] still include an important restriction which guarantees that the only Seiberg-Witten moduli spaces with non-trivial invariants lie in the top level ($\ell = 0$) of the SO(3)-monopole moduli space.

For $\Lambda \in H^2(X; \mathbb{Z})$, define

$$i(\Lambda) = \Lambda^2 + c(X) + \chi(X) + \sigma(X). \tag{29}$$

where $\chi(X)$ and $\sigma(X)$ are the Euler characteristic and signature.

Witten's formula in low degrees (no gluing) VI

If $S(X) \subset \mathrm{Spin}^c(X)$ is the subset yielding non-trivial Seiberg-Witten invariants of X, let

$$(30) \quad r(\Lambda, c_1(\mathfrak{s})) = -(c_1(\mathfrak{s}) - \Lambda)^2 - \tfrac{3}{4}(\chi + \sigma), \\ \text{and} \quad r(\Lambda) = \min_{\mathfrak{s} \in S(X)} r(\Lambda, c_1(\mathfrak{s})).$$

See [14, Remark 3.36] for a discussion of the significance of $r(\Lambda, c_1(\mathfrak{s}))$ and $r(\Lambda)$.

We then have, for $\Lambda \in B^\perp \subset H^2(X; \mathbb{Z})$ and X with Seiberg-Witten simple type, the following simplification of [14, Theorem 1.2]:

Witten's formula in low degrees (no gluing) VII

Theorem 4.2

(See F and Leness [14, Theorem 1.4].) Let X be a standard four-manifold with Seiberg-Witten simple type. Suppose that $\Lambda \in B^\perp$ and that $w \in H^2(X;\mathbb{Z})$ is a class with $w - \Lambda \equiv w_2(X) \pmod 2$. Let $\delta \geq 0$ and $0 \leq m \leq [\delta/2]$ be integers.

(a) *If $\delta < i(\Lambda)$ and $\delta < r(\Lambda)$, then for all $h \in H_2(X;\mathbb{R})$ we have*

(31)
$$D_X^w(h^{\delta-2m}x^m) = 0.$$

(b) *If $\delta < i(\Lambda)$ and $\delta = r(\Lambda)$ we have*

(32)
$$D_X^w(h^{\delta-2m}x^m) = 2^{1-\frac{1}{2}(c(X)+\delta)}(-1)^{m+1+\frac{1}{2}(\sigma-w^2)} \times \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}) - \Lambda, h\rangle^{\delta-2m}.$$

Witten's formula in low degrees (no gluing) VIII

Theorem 4.2

(c) *If* $r(\Lambda) < \delta \leq \frac{1}{2}(r(\Lambda) + c(X) - 2)$, *then equation* (32) *holds with* $D_X^w(h^{\delta-2m}x^m) = 0$.

Corollary 4.3 (Superconformal simple type for certain four-manifolds with Seiberg-Witten simple type)

(See F, Kronheimer, Leness, and Mrowka [5, Theorem 1.1].) Let X *be a standard four-manifold that is abundant* X *and has Seiberg-Witten simple type, with* $c(X) \geq 3$. *Then for any* $w \in H^2(X;\mathbb{Z})$ *with* $w \equiv w_2(X) \pmod 2$ *we have*

$$\mathbf{SW}_X^w(h) \equiv 0 \pmod{h^{c(X)-2}}.$$

Witten's formula in low-degrees (gluing one-instantons) I

Theorem 4.1 (assuming X is abundant) and Theorem 4.2 (*without* assuming X is abundant) apply to standard four-manifolds with Seiberg-Witten simple type.

Those results show that Witten's formula holds in low-degrees (as an equality with powers of h^d for small $d \geq 0$) and that certain special cases of the Mariño-Moore-Peradze superconformal type conjecture hold.

A key simplifying hypothesis in Theorems 4.1 and 4.2 was that Seiberg-Witten moduli spaces (with non-zero Seiberg-Witten invariants) only appear in the top level (namely $\ell = 0$) of the compactified moduli space of SO(3) monopoles.

Witten's formula in low-degrees (gluing one-instantons) II

By allowing Seiberg-Witten moduli spaces (with non-zero Seiberg-Witten invariants) to appear in lower levels (namely $\ell \geq 1$) of the compactified moduli space of SO(3) monopoles, one can extend the range of degrees for which Witten's formula holds and the range of cases for which the Mariño-Moore-Peradze superconformal type conjecture holds.

Allowing $\ell \geq 1$ requires the use of gluing to describe the topology of a neighborhood of a Seiberg-Witten moduli space, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

In the simplest case, $\ell = 1$, it suffices to construct gluing maps for SO(3) monopoles (see F and Leness [10]) corresponding to a

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Witten's formula in low-degrees (gluing one-instantons) III

"one-instanton bubble" (an anti-self-dual connection on an SU(2)-bundle E over S^4 with $c_2(E) = 1$).

Theorem 4.4

(See F and Leness [15, Theorem 1.1].) Let X be a standard four-manifold that is abundant and has Seiberg-Witten simple type. Then there exist $\Lambda \in B^\perp$ and $w \in H^2(X;\mathbb{Z})$ for which $\Lambda^2 = 4 - (\chi + \sigma)$ and $w - \Lambda \equiv w_2(X) \pmod 2$. For any such Λ and w, and any $h \in H_2(X;\mathbb{R})$, one has

(33)
$$\begin{aligned} \mathbf{D}_X^w(h) &\equiv 0 \equiv \mathbf{SW}_X^w(h) \pmod{h^{c(X)-2}}, \\ \mathbf{D}_X^w(h) &\equiv 2^{2-c(X)} e^{\frac{1}{2}h\cdot h}\mathbf{SW}_X^w(h) \pmod{h^{c(X)+2}}. \end{aligned}$$

Witten's Conjecture for "many" four manifolds I

Lastly, one may use the

- SO(3)-monopole cobordism formula in Theorem 3.7 (multi-instanton gluing),
- Blow-up formula for Donaldson invariants,
- Blow-up formula for Seiberg-Witten invariants, and
- A family of simple-type standard four-manifolds due to Fintushel, Park, and Stern [17]

Witten's Conjecture for "many" four manifolds II

to determine sufficiently many of the unknown coefficients in the SO(3)-monopole cobordism formula (Theorem 3.7) to prove

Theorem 4.5 (Witten's formula for "many" four manifolds)

(See F and Leness Main Theorem 1.2]FL6.) Let X be a standard four-manifold with Seiberg-Witten simple type which is abundant or has $c_1^2(X) \geq \chi_h(X) - 3$. Then X obeys Witten's Conjecture 1.1.

Construction of local and global gluing maps and obstruction sections for SO(3) monopoles

Local gluing maps for SO(3) monopoles I

When $\ell \geq 1$, the construction of links of Seiberg-Witten moduli subspaces,

$$M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \mathscr{M}_{\mathfrak{t}},$$

and the computation of intersection numbers for intersections of geometric representatives of cohomology classes on $\mathscr{M}_{\mathfrak{t}}^{*,0}$ with those links requires the construction of a (global) SO(3)-monopole gluing map (and obstruction section of an obstruction bundle, since gluing is always obstructed in the case of SO(3) monopoles).

We summarize the steps in the construction of the local SO(3)-monopole gluing map and obstruction section and proofs of their properties and hence completing the verification of

Local gluing maps for SO(3) monopoles II

Hypothesis 5.1 (Properties of local SO(3)-monopole gluing maps)

The local gluing map, constructed in [10], gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for each smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$.

These local gluing maps are the analogues for SO(3) monopoles of the local gluing maps for anti-self-dual SO(3) connections constructed by Taubes in [48, 49, 50], Donaldson [1], and Donaldson and Kronheimer in [3].

Local gluing maps for SO(3) monopoles III

Local splicing (or pregluing) map

This map is a smooth embedding from the local gluing data parameter space — a finite-dimensional, open, Riemannian manifold — into the configuration space of gauge-equivalence classes of SO(3) pairs.

The image of the map is given by gauge-equivalence classes of approximate SO(3) monopoles, $[A, \Phi]$, defined by a "cut-and-paste" construction.

We splice anti-self-dual SU(2) connections from S^4 onto background SO(3) monopoles on X (elements of $\mathscr{M}_{\mathfrak{t}(\ell)}$) at points in the support of

$$\mathbf{x} \in \Sigma \subset \mathrm{Sym}^{\ell}(X)$$

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Local gluing maps for SO(3) monopoles IV

to form gauge-equivalence classes of SO(3) pairs, $[A, \Phi]$, which are close to the stratum

$$\mathscr{M}_{\mathfrak{t}(\ell)} \times \Sigma \subset \bar{\mathscr{M}}_{\mathfrak{t}}.$$

See F and Leneess [6, 7, 10].

Local gluing maps for SO(3) monopoles V

Local gluing map

This is a smooth map from the gluing data parameter space defined by a single stratum,

$$\Sigma \subset \mathrm{Sym}^{\ell}(X)$$

into the configuration space of SO(3) pairs.

The image of the map is given by gauge-equivalence classes of *extended* SO(3)-*monopoles*, $[A+a, \Phi+\phi]$, obtained by solving the *extended* SO(3)-*monopole equations* for the perturbations, (a, ϕ),

$$\Pi^{\perp}_{A,\phi,\mu}\mathfrak{S}(A+a, \Phi+\phi) = 0,$$

Local gluing maps for SO(3) monopoles VI

rather than the SO(3)-monopole equations directly,

$$\mathfrak{S}(A + a, \Phi + \phi) = 0,$$

since $\operatorname{Coker} D\mathfrak{S}(A, \Phi) = \operatorname{Ran} \Pi_{A,\Phi,\mu}$ is non-zero, where $\mu > 0$ is a "small-eigenvalue" cut-off parameter.

With respect to local coordinates and bundle trivializations, these equations comprise an elliptic, quasi-linear, partial integro-differential system.

The gauge-equivalence classes of true SO(3) monopoles are given by the zero-locus of a local, smooth section of a finite-rank *local Kuranishi obstruction bundle* over the gluing data parameter space.

Local gluing maps for SO(3) monopoles VII

defined by L^2-orthogonal projection onto finite-dimensional, "small-eigenvalue" vector spaces (see [6, 10]).

Smooth embedding property of the local gluing map

One must compute the differential of the gluing map and prove that the differential is injective.

Surjectivity of the local gluing map

Every extended SO(3) monopole close enough to the Uhlenbeck boundary of $\mathscr{M}_{\mathfrak{t}}$ must lie in the image of the local gluing map.

Local gluing maps for SO(3) monopoles VIII

Continuity of the local gluing map and obstruction section

The gluing map and obstruction section must extend continuously to the compactification of the local gluing data space, which includes the Uhlenbeck compactification of moduli spaces of anti-self-dual connections on S^4.

Local gluing maps for SO(3) monopoles IX

Analytical difficulties in the construction of the SO(3)-monopole gluing maps and obstruction sections

1. *Small-eigenvalue obstructions to gluing.*

   The Laplacian, $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$, constructed from the differential, $d^1_{A,\Phi} = D\mathfrak{S}(A,\Phi)$, of the SO(3)-monopole map $\mathfrak{S}$, at an approximate SO(3) monopole, (A,Φ), has small eigenvalues which tend to zero when (A,Φ) bubbles and $\mathfrak{S}(A,\Phi)$ tends to zero.

   This phenomenon occurs for SO(3) monopoles because the

   - Dirac operators, when coupled with an anti-self-dual connections over S^4, always have non-trivial cokernels and

Local gluing maps for SO(3) monopoles X

- Seiberg-Witten monopoles need not be smooth points of their ambient moduli space of background SO(3) monopoles.

2. *Bubbling curvature component in Bochner-Weitzenböck formulae.*

   A key ingredient employed by Taubes in his solution to the anti-self-dual equation in [48, 49] is his use of the Bochner-Weitzenböck formula for the Laplacian, $d_A^+ d_A^{+,*}$, constructed from the differential, d_A^+, of the map, $A \mapsto F_A^+$, at an approximate anti-self-dual connection.

   While Taubes' Bochner-Weitzenböck formula only involves the *small*, self-dual curvature component, F_A^+, our

Local gluing maps for SO(3) monopoles XI

Bochner-Weitzenböck formula for $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$ also involves the *large* anti-self-dual curvature component, F_A^-.

3. *Seiberg-Witten moduli spaces of positive dimension and spectral flow.*

   When $\dim M_\mathfrak{s} > 0$, one cannot fix a single, uniform positive upper bound for the small eigenvalues of $d^1_{A,\Phi} d^{1,*}_{A,\Phi}$, due to spectral flow as the point $[A, \Phi]$ varies in an open neighborhood of $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ in the local gluing data parameter space.

Those issues are addressed in our article [10] and monograph [7].

Global gluing maps for SO(3) monopoles I

Building a global gluing map and obstruction section from the local gluing maps and obstruction sections

Hypothesis 5.1 describes a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ a smooth stratum while the proof of Theorem 3.7 (general SO(3)-monopole cobordism formula) requires a description of a neighborhood of the union of these strata, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$.

In [7], we proved how the local gluing data parameter spaces, splicing maps, obstruction bundles, and obstruction sections given by Hypothesis 5.1 for different $\Sigma \subset \mathrm{Sym}^{\ell}(X)$ fit together and extend over the Uhlenbeck compactification, $\bar{\mathscr{M}}_{\mathfrak{t}}$.

Global gluing maps for SO(3) monopoles II

The splicing maps are suitably deformed so that they obey a type of "cocycle condition" — to form *global* splicing maps and obstruction sections, thus solving the "overlap problem" identified by Kotschick and Morgan for gluing anti-self-connections in [24].

Using this construction, we computed the expressions for the intersection number yielding the SO(3)-monopole cobordism formula (25) and completing the proof of Theorem 3.7.

The authors are currently developing a proof of the required properties for the local gluing maps and obstruction sections for SO(3) monopoles (Hypothesis 5.1) in a book in progress [9].

Global gluing maps for SO(3) monopoles III

Remaining lectures

1. SO(3)-*monopole cobordism formula and superconformal simple type.*

   Verification using the SO(3)-monopole cobordism formula that all Seiberg-Witten simple type standard four-manifolds have superconformal simple type.

2. *Superconformal simple type and Witten's conjecture.*

   Verification using the SO(3)-monopole cobordism formula that all superconformal simple type standard four-manifolds satisfy Witten's formula.

Thank you for your attention!

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