Lecture 2: SO(3)-monopole cobordism formula and superconformal simple type

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Outline

Introduction and main results

Introduction I

Using supersymmetric quantum Yang-Mills field theory, Marinõ, Moore, and Peradze (1999) also showed that a certain low-degree polynomial part of the Donaldson-Seiberg-Witten series always vanishes [20, 21], a consequence of their notion of superconformal simple type.

Marinõ, Moore, and Peradze noted that this vanishing would confirm a conjecture (attributed to Fintushel and Stern) for a lower bound on the number of (Seiberg-Witten) basic classes of a four-dimensional manifold.

The purpose of our second lecture in this series is to describe a proof using SO(3) monopoles that all four-manifolds with

Introduction II

Seiberg-Witten simple type are necessarily superconformal simple type.

It is not known whether all four-manifolds also have Seiberg-Witten simple type.

Our lecture is primarily based no

- P. M. N. Feehan and T. G. Leness, *The* $\mathrm{SO}(3)$ *monopole cobordism and superconformal simple type*, arXiv:1408.5307 (submitted October 2014).

That article is in turn based on methods and results described earlier in

Introduction III

1. P. M. N. Feehan and T. G. Leness, *A general* $\mathrm{SO}(3)$*-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, Memoirs of the American Mathematical Society, in press, arXiv:math/0203047,
2. P. M. N. Feehan and T. G. Leness, *Witten's conjecture for many four-manifolds of simple type*, Journal of the European Mathematical Society **17** (2015), 899–923.

with supporting results and background material with earlier published articles with Leness cited therein.

As we shall see in our next lecture, the concept of superconformal simple type is important since it leads to a proof of Witten's formula relating Donaldson and Seiberg-Witten invariants:

Introduction IV

- P. M. N. Feehan and T. G. Leness, *Superconformal simple type and Witten's conjecture*, arXiv:1408.5085 (submitted October 2014).

Our proofs of these conjectures rely on an assumption of certain analytical properties gluing maps for SO(3) monopoles (see Hypothesis 2.5), analogous to properties proved by Donaldson and Taubes in simpler contexts of gluing maps for SO(3) anti-self-dual connections.

Verification of those analytical gluing map properties is work in progress [6] and appears well within reach.

Statements of main results I

A closed, oriented four-manifold X has an *intersection form*,

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

One lets $b^{\pm}(X)$ denote the dimensions of the maximal positive or negative subspaces of the form Q_X on $H_2(X;\mathbb{Z})$ and

$$e(X) := \sum_{i=0}^{4} (-1)^i b_i(X) \quad \text{and} \quad \sigma(X) := b^+(X) - b^-(X)$$

denote the *Euler characteristic* and *signature* of X, respectively.

Statements of main results II

We define the characteristic numbers,

$$
\begin{aligned}
c_1^2(X) &:= 2e(X) + 3\sigma(X), \\
\chi_h(X) &:= (e(X) + \sigma(X))/4, \qquad (1) \\
c(X) &:= \chi_h(X) - c_1^2(X).
\end{aligned}
$$

We call a four-manifold standard if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

For a standard four-manifold, the Seiberg-Witten invariants comprise a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

Statements of main results III

on the set of spin^c structures on X.

The set of Seiberg-Witten basic classes, $B(X)$, is the image under $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X, that is

$$B(X) = \{K \in H^2(X;\mathbb{Z}) : SW_X(K) \neq 0\}.$$

A manifold X has Seiberg-Witten simple type if

$$K^2 = c_1^2(X), \quad \forall\, K \in B(X).$$

Statements of main results IV

As defined by Mariño, Moore, and Peradze, [21, 20], a manifold X has superconformal simple type if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic,

$$\boxed{SW_X^{w,i}(h) = 0 \quad \text{for } 0 \leq i \leq c(X) - 4} \tag{2}$$

and all $h \in H_2(X;\mathbb{R})$, where

$$\boxed{SW_X^{w,i}(h) := \sum_{\mathfrak{s} \in \operatorname{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) \langle c_1(\mathfrak{s}), h\rangle^i.}$$

From [5], we have the

Statements of main results V

Theorem 1.1 (All standard four-manifolds with Seiberg-Witten simple type have superconformal simple type)

(See F and Leness [5, Theorem 1.1].) Assume Hypothesis 2.5. If X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

Hypothesis 2.5 asserts certain analytical properties of local gluing maps for SO(3) monopoles constructed by the authors in [7].

Proofs of these analytical properties, analogous to known properties of local gluing maps for anti-self-dual SO(3) connections and Seiberg-Witten monopoles, are being developed by us [6].

Statements of main results VI

Global gluing maps are used to describe the topology of neighborhoods of Seiberg-Witten monopoles appearing at all levels of the compactified moduli space of SO(3) monopoles and hence construct links of those singularities.

Marinõ, Moore, and Peradze had previously shown [21, Theorem 8.1.1] that if the set of Seiberg-Witten basic classes, $B(X)$, is non-empty and X has superconformal simple type, then

$$\boxed{|B(X)/\{\pm 1\}| \geq [c(X)/2].} \tag{3}$$

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For example, suppose X is the K3 surface. It is known that $B(X) = \{0\}$, so $|B(X)/\{\pm 1\}| = 1$, while

$$b_1(X) = 0, \quad b^+(X) = 3, \quad b^-(X) = 19,$$

which gives $e(X) = 24$, $\sigma(X) = -16$, $c_1^2(X) = 0$, $\chi_h(X) = 2$, and $c(X) = 2$, so $[c(X)/2] = 1$ and equality holds in (3).

Because $c(X) \leq 3$, the K3 surface is superconformal simple type by our definition.

Theorem 1.1 and [21, Theorem 8.1.1] yield a proof of the following result, first conjectured by Fintushel and Stern [15].

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Corollary 1.2 (Lower bound for the number of basic classes)

(See F and Leness [5, Corollary 1.2]) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume Hypothesis 2.5. If $B(X)$ is non-empty and $c(X) \geq 3$, then the number of basic classes obeys the lower bound (3).

On the other hand, we recall the

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Conjecture 1.3 (Witten's Conjecture)

Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then for any $w \in H^2(X;\mathbb{Z})$ the Donaldson invariants satisfy

$$(4)\quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}.$$

From [8] (and as we discuss in the next lecture), we have the

Statements of main results X

Theorem 1.4 (Superconformal simple type $\implies$ Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [8, Theorem 1.2].) Assume Hypothesis 2.5. If a standard four-manifold has superconformal simple type, then it satisfies Witten's Conjecture 1.3.

Combining Theorems 1.1 and 1.4 thus yields the following

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Corollary 1.5 (Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [8, Corollary 1.3] or [5, Corollary 1.4].) Assume Hypothesis 2.5. If X is a standard four-manifold of Seiberg-Witten simple type then X satisfies Witten's Conjecture 1.3.

SO(3)-monopole cobordism formula for link pairings

Seiberg-Witten invariants I

The Seiberg-Witten invariants comprise a map with finite support,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

from the set of spin^c structures on X.

A spin^c structure $\mathfrak{s} = (W^\pm, \rho_W)$ on X consists of a pair of complex rank-two vector bundles, $W^\pm \to X$, and a Clifford multiplication map, $\rho_W : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$.

If $\mathfrak{s} \in \mathrm{Spin}^c(X)$, then $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ is an integral lift of $w_2(TX) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$.

Seiberg-Witten invariants II

One calls $c_1(\mathfrak{s})$ a Seiberg-Witten basic class if $SW_X(\mathfrak{s}) \neq 0$. Define

$$B(X) := \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}. \tag{5}$$

If $H^2(X;\mathbb{Z})$ has 2-torsion, then $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ is not injective.

Because we will work with functions involving real homology and cohomology, we define

$$SW_X' : H^2(X;\mathbb{Z}) \to \mathbb{Z}, \quad K \mapsto \sum_{\mathfrak{s} \in c_1^{-1}(K)} SW_X(\mathfrak{s}). \tag{6}$$

Seiberg-Witten invariants III

Thus, we can rewrite the expression for $SW_X^{w,i}(h)$ in (2) as

$$SW_X^{w,i}(h) = \sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + w \cdot K)} SW_X'(K) \langle K, h \rangle^i. \tag{7}$$

A four-manifold, X, has Seiberg-Witten simple type if $SW_X(\mathfrak{s}) \neq 0$ implies that $c_1(\mathfrak{s})^2 = c_1^2(X)$.

Spin^{u}structures and SO(3) monopoles I

A spin^u structure $\mathfrak{t} = (V^\pm, \rho)$ on a four-manifold, X, is a pair of complex rank-four vector bundles, $V^\pm \to X$, with a Clifford module structure, $\rho : T^*X \to \mathrm{Hom}_{\mathbb{C}}(V^+, V^-)$.

In more familiar terms, for a spin^c structure $\mathfrak{s} = (W^\pm, \rho_W)$ on X, a spin^u structure is given by $V^\pm = W^\pm \otimes E$, where $E \to X$ is a complex rank-two vector bundle and the Clifford multiplication map is given by $\rho = \rho_W \otimes \mathrm{id}_E$.

Spin^u structures and SO(3) monopoles II

We define characteristic classes of a spin^u structure $\mathfrak{t} = (W^\pm \otimes E, \rho)$ by

$$p_1(\mathfrak{t}) := p_1(\mathfrak{su}(E)), \quad c_1(\mathfrak{t}) := c_1(W^+) + c_1(E),$$
$$w_2(\mathfrak{t}) := c_1(E) \pmod 2.$$

Spin^{u}structures and SO(3) monopoles III

Lemma 2.1 (Spin^u structures with prescribed characteristic classes)

(See F and Leness [5, Lemma 2.1].) Let X be a standard four-manifold. Given $\wp \in H^4(X;\mathbb{Z})$, $\Lambda \in H^2(X;\mathbb{Z})$, and $\mathfrak{w} \in H^2(X;\mathbb{Z}/2\mathbb{Z})$, there is a spin^u structure $\mathfrak{t}$ on X with $p_1(\mathfrak{t}) = \wp$, $c_1(\mathfrak{t}) = \Lambda$, and $w_2(\mathfrak{t}) = \mathfrak{w}$ if and only if:

1. *There is a class $w \in H^2(X;\mathbb{Z})$ with $\mathfrak{w} = w \pmod 2$,*
2. $\Lambda \equiv \mathfrak{w} + w_2(X) \pmod 2$,
3. $\wp \equiv \mathfrak{w}^2 \pmod 4$.

Space of SO(3) monopoles and S^1 fixed points I

For a spin^u structure $\mathfrak{t} = (W^\pm \otimes E, \rho)$ on X, the moduli space $\mathscr{M}_\mathfrak{t}$ of SO(3) monopoles on $\mathfrak{t}$ is the space of solutions, modulo gauge equivalence, to the SO(3)-monopole equations (see F and Leness [9, Equation (1.1)] or [10, Equation (2.32)]) for a pair (A, Φ) where A is a unitary connection on E and $\Phi \in \Omega^0(V^+)$.

Complex scalar multiplication on the section, Φ, defines an S^1 action on $\mathscr{M}_\mathfrak{t}$ with stabilizer $\{\pm 1\}$ away from two families of fixed point sets:

- zero-section points, $[A, 0]$, where $\Phi \equiv 0$ and
- reducible points, $[A, \Phi]$, where A is a reducible connection.

Space of SO(3) monopoles and S^1 fixed points II

The *subspace of zero-section points* is a manifold with a natural smooth structure diffeomorphic to the moduli space of anti-self-dual connections on $\mathfrak{su}(E)$ which we denote, following the notation of [19], by M_κ^w where $\kappa = -p_1(\mathfrak{t})/4$ and $w = c_1(E)$ (see F and Leness [10, Section 3.2]).

By [10, Lemma 2.17], the *subspace of reducible points* where A is reducible with respect to a splitting $V = (W \otimes L_1) \oplus (W \otimes L_2)$ is a manifold, $M_{\mathfrak{s}}$, which is compactly cobordant to the moduli space of Seiberg-Witten monopoles associated with the spin^c structure $\mathfrak{s}$ where $c_1(\mathfrak{s}) = c_1(W^+ \otimes L_1)$.

Space of SO(3) monopoles and S^1 fixed points III

By [11, Lemma 3.32], the possible splittings of $\mathfrak{t}$ are given by

$$\operatorname{Red}(\mathfrak{t}) = \{\mathfrak{s} \in \operatorname{Spin}^c : (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(\mathfrak{t})\}.$$

Hence, the subspace of reducible points is

$$M_{\mathfrak{t}}^{\text{red}} = \bigcup_{\mathfrak{s}\in\operatorname{Red}(\mathfrak{t})} M_{\mathfrak{s}}.$$

We define

$$\mathscr{M}_{\mathfrak{t}}^0 := \mathscr{M}_{\mathfrak{t}} \setminus M_{\kappa}^w, \quad \mathscr{M}_{\mathfrak{t}}^* := \mathscr{M}_{\mathfrak{t}} \setminus M_{\mathfrak{t}}^{\text{red}}, \quad \text{and} \quad \mathscr{M}_{\mathfrak{t}}^{*,0} := \mathscr{M}_{\mathfrak{t}}^0 \cap \mathscr{M}_{\mathfrak{t}}^*.$$

We recall the

Space of SO(3) monopoles and S^1 fixed points IV

Theorem 2.2

(See F [3], F and Leness [9], or Teleman [23].) Let $\mathfrak{t}$ be a spin^u structure on a standard four-manifold, X. For generic perturbations of the SO(3) *monopole equations, the moduli space, $\mathscr{M}_{\mathfrak{t}}^{*,0}$,is a smooth, orientable manifold of dimension*

$$\dim \mathscr{M}_{\mathfrak{t}} = 2d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}),$$

where, for $\chi_h(X)$ and $c_1^2(X)$ as in (1),

$$d_a(\mathfrak{t}) := \frac{1}{2}\dim M_\kappa^w = -p_1(\mathfrak{t}) - 3\chi_h(X), \tag{8a}$$

$$n_a(\mathfrak{t}) := \frac{1}{4}\left(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - c_1^2(X) + 8\chi_h(X)\right). \tag{8b}$$

Compactification of moduli space of SO(3) monopoles I

The moduli space $\mathscr{M}_\mathfrak{t}$ is not compact but admits an Uhlenbeck compactification as follows (see F and Leness [10, Section 2.2] or [9] for details).

For $\ell \geq 0$, let $\mathfrak{t}(\ell)$ be the spin^u structure on X with

$$p_1(\mathfrak{t}(\ell)) = p_1(\mathfrak{t}) + 4\ell, \quad c_1(\mathfrak{t}(\ell)) = c_1(\mathfrak{t}), \quad w_2(\mathfrak{t}(\ell)) = w_2(\mathfrak{t}).$$

Let $\mathrm{Sym}^\ell(X)$ be the ℓ-th symmetric product of X (that is, X^ℓ modulo the symmetric group on ℓ elements).

For $\ell = 0$, we define $\mathrm{Sym}^\ell(X)$ to be a point.

Compactification of moduli space of SO(3) monopoles II

The space of ideal SO(3) monopoles is defined by

$$I\mathscr{M}_{\mathfrak{t}} := \bigcup_{\ell=0}^{N} \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X).$$

We give $I\mathscr{M}_{\mathfrak{t}}$ the topology defined by Uhlenbeck convergence (see F and Leness [9, Definition 4.9]).

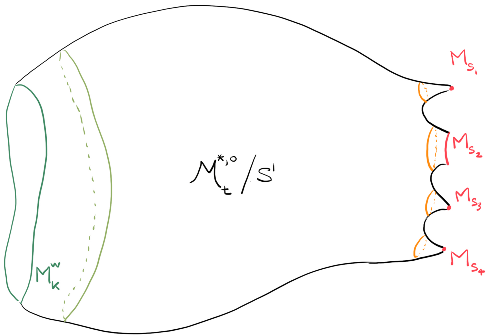
Theorem 2.3

[9] (See F and Leness [9].) Let X be a standard four-manifold with Riemannian metric, g. Let $\bar{\mathscr{M}}_{\mathfrak{t}} \subset I\mathscr{M}_{\mathfrak{t}}$ be the closure of $\mathscr{M}_{\mathfrak{t}}$ with respect to the Uhlenbeck topology. Then there is a non-negative integer, N, depending only on (X, g), $p_1(\mathfrak{t})$, and $c_1(\mathfrak{t})$ such that $\bar{\mathscr{M}}_{\mathfrak{t}}$ is compact.

Compactification of moduli space of SO(3) monopoles III

The following figure illustrates the SO(3)-monopole cobordism between codimension-one links in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ of $\bar{M}_\kappa^w$ and $M_{\mathfrak{s}_i} \times \mathrm{Sym}^\ell(X)$.

Compactification of moduli space of SO(3) monopoles IV



Compactification of moduli space of SO(3) monopoles V

The S^1 action on $\mathscr{M}_{\mathfrak{t}}$ extends continuously over $I\mathscr{M}_{\mathfrak{t}}$ and $\bar{\mathscr{M}}_{\mathfrak{t}}$, in particular, but $\bar{\mathscr{M}}_{\mathfrak{t}}$ contains fixed points of this S^1 action which are not contained in $\mathscr{M}_{\mathfrak{t}}$.

The closure of M^w_κ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ is $\bar{M}^w_\kappa$, the Uhlenbeck compactification of the moduli space of anti-self-dual connections as defined in [2].

There are additional reducible points in the lower levels of $I\mathscr{M}_{\mathfrak{t}}$ (where $\ell \geq 1$). Define

$$\overline{\mathrm{Red}}(\mathfrak{t}) := \{\mathfrak{s} \in \mathrm{Spin}^c(X) : (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 \geq p_1(\mathfrak{t})\}.$$

If we define the level, $\ell(\mathfrak{t},\mathfrak{s})$, in $\bar{\mathscr{M}}_{\mathfrak{t}}$ of the spin^c structure $\mathfrak{s}$ by

$$\ell(\mathfrak{t},\mathfrak{s}) := \frac{1}{4}\left((c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 - p_1(\mathfrak{t})\right), \tag{9}$$

Compactification of moduli space of SO(3) monopoles VI

then the set of strata of reducible points in $\bar{\mathscr{M}}_{\mathfrak{t}}$ are given by

$$(10) \qquad \bar{M}_{\mathfrak{t}}^{\mathrm{red}} := \bigcup_{\mathfrak{s}\in\overline{\mathrm{Red}}(\mathfrak{t})} M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell(\mathfrak{t},\mathfrak{s})}(X).$$

Note that for $\mathfrak{s} \in \overline{\mathrm{Red}}(\mathfrak{t})$, we have $\ell(\mathfrak{t},\mathfrak{s}) \geq 0$ by the definitions of $\overline{\mathrm{Red}}(\mathfrak{t})$ and $\ell(\mathfrak{t},\mathfrak{s})$.

By analogy with the corresponding definitions for $\mathscr{M}_{\mathfrak{t}}$, we write

$$\bar{\mathscr{M}}_{\mathfrak{t}}^{0} := \bar{\mathscr{M}}_{\mathfrak{t}} \setminus \bar{M}_{\kappa}^{w}, \quad \bar{\mathscr{M}}_{\mathfrak{t}}^{*} := \bar{\mathscr{M}}_{\mathfrak{t}} \setminus \bar{M}_{\mathfrak{t}}^{\mathrm{red}}, \quad \bar{\mathscr{M}}_{\mathfrak{t}}^{*,0} := \bar{\mathscr{M}}_{\mathfrak{t}}^{0} \cap \bar{\mathscr{M}}_{\mathfrak{t}}^{*},$$

and observe that the stabilizer of the S^1 action on $\bar{\mathscr{M}}_{\mathfrak{t}}^{*,0}$ is $\{\pm 1\}$.

Cohomology classes and geometric representatives I

The cohomology classes used to define Donaldson invariants extend to $\mathscr{M}_{\mathfrak{t}}^*/S^1$.

For $\beta \in H_\bullet(X;\mathbb{R})$, there is a cohomology class,

$$\mu_p(\beta) \in H^{4-\bullet}(\mathscr{M}_{\mathfrak{t}}^*/S^1;\mathbb{R}),$$

with geometric representative (in the sense of Donaldson [1], Kronheimer and Mrowka [19, p. 588] or F and Leness [11, Definition 3.4]),

$$\mathscr{V}(\beta) \subset \mathscr{M}_{\mathfrak{t}}^*/S^1.$$

Cohomology classes and geometric representatives II

For $h_i \in H_2(X;\mathbb{R})$ and a generator $x \in H_0(X;\mathbb{Z})$, we define

$$\begin{aligned}
&\mu_p(h_1 \dots h_{\delta-2m}x^m)\\
&\quad := \mu_p(h_1) \smile \dots \smile \mu_p(h_{\delta-2m}) \smile \bar{\mu}_p(x)^m \in H^{2\delta}(\mathscr{M}_{\mathfrak{t}}^*/S^1;\mathbb{R}),\\
&\mathscr{V}(h_1 \dots h_{\delta-2m}x^m)\\
&\quad := \mathscr{V}(h_1) \cap \dots \cap \mathscr{V}(h_{\delta-2m}) \cap \underbrace{\mathscr{V}(x) \cap \dots \cap \mathscr{V}(x)}_{m \text{ copies}},
\end{aligned}$$

and let

$$\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m}x^m)$$

be the closure of $\mathscr{V}(h_1 \dots h_{\delta-2m}x^m)$ in $\bar{\mathscr{M}}_{\mathfrak{t}}^*/S^1$.

Cohomology classes and geometric representatives III

Denote the first Chern class of the S^1 action on $\bar{\mathscr{M}}_{\mathfrak{t}}^{*,0}$ with multiplicity two by

$$\bar{\mu}_c \in H^2(\bar{\mathscr{M}}_{\mathfrak{t}}^{*,0}/S^1;\mathbb{Z}).$$

This cohomology class has a geometric representative $\bar{\mathscr{W}}$.

Link of the moduli space of anti-self-dual connections I

Proposition 2.4 (Vanishing of pairing with link of the moduli space of anti-self-dual connections)

(See F and Leness [11, Proposition 3.29] or [5, Proposition 2.4].) Let $\mathfrak{t}$ be a spin^u structure on a standard four-manifold X. For $\delta, \eta_c, m \in \mathbb{N}$, if

$$\delta - 2m \geq 0, \tag{11a}$$

$$\delta + \eta_c = \frac{1}{2}\dim \mathbf{L}^{\mathrm{asd}}_{\mathfrak{t}} = d_a(\mathfrak{t}) + n_a(\mathfrak{t}) - 1, \tag{11b}$$

$$\delta > \frac{1}{2}\dim M^w_\kappa = d_a(\mathfrak{t}) \geq 0, \tag{11c}$$

then

$$\#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m) \cap \bar{\mathscr{W}}^{\eta_c} \cap \mathbf{L}^{\mathrm{asd}}_{\mathfrak{t}}\right) = 0,$$

where $\#$ denotes the signed count of the points in the intersection.

Link of a moduli space of Seiberg-Witten monopoles I

Hypothesis 2.5 (Properties of local SO(3)-monopole gluing maps)

The local gluing map, constructed in [7], gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for each smooth stratum $\Sigma \subset \operatorname{Sym}^\ell(X)$.

Hypothesis 2.5 is discussed in greater detail in [4, Section 6.7].

The question of how to assemble the local gluing maps for neighborhoods of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$, as Σ ranges over all smooth strata of $\operatorname{Sym}^\ell(X)$, into a global gluing map for a neighborhood of $M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X)$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ is itself difficult — involving the "overlap problem" described in [12] — but one which we solve in [4].

Link of a moduli space of Seiberg-Witten monopoles II

Theorem 2.6 (Pairing with link of moduli space of Seiberg-Witten monopoles)

(See F and Leneß [4, Theorem 9.0.5] or [5, Theorem 2.6].) Let $\mathfrak{t}$ be a spin^u structure on a standard four-manifold X of Seiberg-Witten simple type and assume Hypothesis 2.5. Denote $\Lambda = c_1(\mathfrak{t})$ and $K = c_1(\mathfrak{s})$ for $\mathfrak{s} \in \overline{\mathrm{Red}}(\mathfrak{t})$. Let $\delta, \eta_c, m \in \mathbb{Z}_{\geq 0}$ satisfy $\delta - 2m \geq 0$ and

$$\delta + \eta_c = \frac{1}{2}\dim \mathbf{L}_{\mathfrak{t},\mathfrak{s}} = d_a(\mathfrak{t}) + n_a(\mathfrak{t}) - 1.$$

Let $\ell = \ell(\mathfrak{t},\mathfrak{s})$ be as defined in (9).

Link of a moduli space of Seiberg-Witten monopoles III

Theorem 2.6 (Pairing with link of moduli space of Seiberg-Witten monopoles)

Then, for any integral lift $w \in H^2(X;\mathbb{Z})$ of $w_2(\mathfrak{t})$, and any $h \in H_2(X;\mathbb{R})$ and generator $x \in H_0(X;\mathbb{Z})$,

$$\begin{aligned}(12)\qquad & \#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m) \cap \bar{\mathscr{W}}^{\eta_c} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) \\ & = SW_X(\mathfrak{s}) \sum_{\substack{i+j+2k \\ =\delta-2m}} a_{i,j,k}(\chi_h, c_1^2, K\cdot\Lambda, \Lambda^2, m, \ell)\langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k,\end{aligned}$$

where $\#$ denotes the signed count of points in the intersection and where for each triple of non-negative integers, $i, j, k \in \mathbb{N}$, the coefficients,

$$a_{i,j,k} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R},$$

are universal functions of χ_h, c_1^2, $c_1(\mathfrak{s})\cdot\Lambda$, Λ^2, m, ℓ and vanish if $k > \ell(\mathfrak{t},\mathfrak{s})$.

Link of a moduli space of Seiberg-Witten monopoles IV

The right-hand side of (12) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspace $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$ of ideal Seiberg-Witten monopole appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

One uses the fiber-bundle structure of the link over the Seiberg-Witten moduli subspace, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, to compute the intersection number and show that this is equal to a multiple of the Seiberg-Witten invariant, $SW_X(\mathfrak{s})$.

SO(3)-monopole cobordism formula for link-pairings I

The compactification $\bar{\mathscr{M}}_{\mathfrak{t}}^{*,0}/S^1$ defines a compact cobordism, stratified by smooth oriented manifolds, between

$$\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}} \quad \text{and} \quad \bigcup_{\mathfrak{s}\in\overline{\mathrm{Red}}(\mathfrak{t})} \mathbf{L}_{\mathfrak{t},\mathfrak{s}}.$$

For $\delta+\eta_c=\frac{1}{2}\dim \mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}$, this cobordism gives the following equality (see F and Leness [4, Equation (1.6.1)]),

$$\boxed{\begin{aligned}&\#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m)\cap\bar{\mathscr{W}}^{\eta_c}\cap\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}\right)\\ &\quad=-\sum_{\mathfrak{s}\in\overline{\mathrm{Red}}(\mathfrak{t})}(-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-c_1(\mathfrak{t}))\cdot c_1(\mathfrak{s}))}\\ &\qquad\qquad\times\#\left(\bar{\mathscr{V}}(h^{\delta-2m}x^m)\cap\bar{\mathscr{W}}^{\eta_c}\cap\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right).\end{aligned}} \tag{13}$$

SO(3)-monopole cobordism formula for link-pairings II

The left-hand side of (13) is obtained by computing the intersection number for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the link of the moduli subspace $\bar{M}_\kappa^w$ of anti-self-dual SO(3) connections.

The right-hand side of (13) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspaces $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ of ideal Seiberg-Witten monopoles appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ I

We will derive a formula (see the forthcoming (18)) relating the Seiberg-Witten polynomials $SW_X^{w,i}$ defined in (2) and the intersection form of X.

We do so by applying the cobordism formula (13) in a case where Proposition 2.4 (*vanishing of intersection number with link of ASD moduli space*) implies the left-hand-side of (13) vanishes.

To extract a formula from the resulting vanishing sum that includes the Seiberg-Witten polynomials, $SW_X^{w,i}$, we apply Theorem 2.6 (*intersection number with link of Seiberg-Witten moduli space*) to the terms on the right-hand-side of (13).

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ II

In the resulting sum over $\overline{\mathrm{Red}}(\mathfrak{t})$, the coefficients, $a_{i,j,k}$, appearing in equation (12) in Theorem 2.6 depend on $c_1(\mathfrak{t}) \cdot c_1(\mathfrak{s})$.

This dependence prevents the desired extraction of $SW_X^{w,i}$ from the cobordism sum (this is a technical point arising in the calculation — see F and Leness [5, Remark 3.4]).

To ensure that $c_1(\mathfrak{t}) \cdot c_1(\mathfrak{s})$ is constant as $c_1(\mathfrak{s})$ varies in $B(X)$ without further assumptions on $B(X)$, we shall assume $c_1(\mathfrak{t}) = 0$.

We begin by establishing the existence of a family of spin^u structures with $c_1(\mathfrak{t}) = 0$.

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SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ III

Lemma 3.1 (Existence of a family of spin^u structures)

(See F and Leness [5, Proposition 3.1].) Let X be a standard four-manifold. For every $n \in \mathbb{N}$ there is a spin^u structure $\mathfrak{t}_n$ on X satisfying

$$(14) \quad c_1(\mathfrak{t}_n) = 0, \quad p_1(\mathfrak{t}_n) = 4n + c_1^2(X) - 8\chi_h(X), \quad w_2(\mathfrak{t}_n) = w_2(X),$$

and such that $n_a(\mathfrak{t}_n) = n$, where $n_a(\mathfrak{t})$ is the index defined in (8b).

To apply Theorem 2.6 to the cobordism formula (13) for a spin^u structure $\mathfrak{t}_n$ satisfying (14), we compute the level in $\bar{\mathscr{M}}_{\mathfrak{t}_n}$ of a spin^c structure $\mathfrak{s}$.

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ IV

Lemma 3.2 (Computation of the level of Seiberg-Witten moduli space in a compactified moduli space of SO(3) monopoles)

(See F and Leness [5, Lemma 3.2].) Let X be a standard four-manifold of Seiberg-Witten simple type. For a non-negative integer, n, let $\mathfrak{t}_n$ be a spin^u structure on X satisfying (14). For $c_1(\mathfrak{s}) \in B(X)$, the level $\ell = \ell(\mathfrak{t}_n, \mathfrak{s})$ in $\bar{\mathscr{M}}_{\mathfrak{t}_n}$ of $\mathfrak{s}$ is

$$\ell(\mathfrak{t}_n, \mathfrak{s}) = 2\chi_h(X) - n. \tag{15}$$

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ V

Combining

- Equation (13) (SO(3)-*monopole cobordism formula*),
- Proposition 2.4 (*vanishing of intersection number with link of ASD moduli space*),
- Theorem 2.6 (*intersection number with link of Seiberg-Witten moduli space*),

then gives the following

SO(3)-monopole cobordism with $c_1(\mathfrak{t}) = 0$ VI

Theorem 3.3 (SO(3)-monopole cobordism formula vanishing)

(See F and Leness [5, Theorem 3.3].) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume that $m, n \in \mathbb{N}$ satisfy

$$n \leq 2\chi_h(X), \tag{16a}$$

$$1 < n, \tag{16b}$$

$$0 \leq c(X) - n - 2m - 1. \tag{16c}$$

We abbreviate the coefficients in equation (12) *in Theorem 2.6 by*

$$a_{i,0,k} := a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n). \tag{17}$$

Then, for $A = c(X) - n - 2m - 1$ and $w \in H^2(X; \mathbb{Z})$ characteristic,

$$0 = \sum_{k=0}^{2\chi_h(X)-n} a_{A+2k,0,2\chi_h(X)-n-k} SW_X^{w,A+2k}(h) Q_X(h)^{2\chi_h(X)-n+k}. \tag{18}$$

Computation of the leading-order term in the SO(3)-monopole cobordism formula for link pairings

Leading term in cobordism formula for link pairings I

To show that equation (18) is non-trivial, we now demonstrate, in a computation similar to that due to Kotschick and Morgan [18, Theorem 6.1.1], that the coefficient of the term in (18) including the highest power of Q_X is non-zero.

Leading term in cobordism formula for link pairings II

Proposition 4.1 (Leading-order term in the SO(3)-monopole cobordism formula (18) for link pairings)

(See F and Leness [5, Proposition 4.1].) Continue the notation and assumptions of Theorem 3.3. In addition, assume that there is $K \in B(X)$ with $K \neq 0$. Let m and n be non-negative integers satisfying the conditions (16). Define $A := c(X) - n - 2m - 1$, and $\delta := c(X) + 4\chi_h(X) - 3n - 1$, and $\ell = 2\chi_h(X) - n$. Then

$$a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) = (-1)^{m+\ell} 2^{\ell-\delta} \frac{(\delta - 2m)!}{\ell! A!}. \tag{20}$$

Although the computation of the precise value of the coefficient in (20) is delicate, we need only show that $a_{A,0,\ell}$ is non-zero.

Leading term in cobordism formula for link pairings III

The methods discussed below allow one to compute the coefficients $a_{i,j,\ell}$ in greater generality (for example, without the assumption that $c_1(\mathfrak{t}) = 0$), but not complete generality (mainly due to the complicated stratified-space structure of the symmetric products, $\mathrm{Sym}^\ell(X)$).

Because Theorem 1.1 does not require greater generality and indeed requires the assumption that $c_1(\mathfrak{t}) = 0$ (as one can show), we omit the proof of a more general result in the interest of clarity.

Topology of a neighborhood of $M_\mathfrak{s} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_\mathfrak{t}$ I

The proof of Proposition 4.1 requires knowledge of the topology of an open neighborhood of

$$M_\mathfrak{s} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_\mathfrak{t}/S^1,$$

where $\mathbf{x} \in \mathrm{Sym}^\ell(X)$.

While we need to consider arbitrary *lower levels*, $\ell \geq 0$, in the Uhlenbeck compactification,

$$\bar{\mathscr{M}}_\mathfrak{t} \subset \bigcup_{\ell=0}^{N} \mathscr{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X),$$

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ II

the proof of Proposition 4.1 only requires us to consider points,

$$\mathbf{x} \in \mathrm{Sym}^{\ell}(X),$$

in the *top stratum* of $\mathrm{Sym}^{\ell}(X)$ (all points distinct).

Although the definition of the intersection numbers appearing in the SO(3)-monopole cobordism formula (13) (and thus defining the coefficient (20)) requires a smooth structure on the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, the forthcoming equality (35) turns these intersection numbers into a cohomological pairing.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ III

Lower-level Seiberg-Witten moduli spaces and invariants

Because $\dim M_{\mathfrak{s}} = 0$ (by our simple type hypothesis), the virtual normal bundle construction of [10, Theorem 3.21] (F and Leness) gives a homeomorphism between a neighborhood in $\mathscr{M}_{\mathfrak{t}(\ell)}/S^1$ of a point in $M_{\mathfrak{s}}$ and $\chi^{-1}_{\mathfrak{t}(\ell),\mathfrak{s}}(0)/S^1$, where

(21) $$\chi_{\mathfrak{t}(\ell),\mathfrak{s}} : \mathbb{C}^{r_N} \to \mathbb{C}^{r_\Xi}$$

is a continuous, S^1-equivariant map with $0 \in g^{-1}_{\mathfrak{t}(\ell),\mathfrak{s}}(0)$ and which is smooth away from the origin and vanishes transversely away from the origin.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ IV

The dimensions satisfy

$$r_N - r_\Xi = \frac{1}{2}\mathscr{M}_{\mathfrak{t}(\ell)} = \frac{1}{2}\mathscr{M}_{\mathfrak{t}} - 3\ell. \tag{22}$$

We further note that because $\dim M_{\mathfrak{s}} = 0$ and $M_{\mathfrak{s}}$ is compact and oriented, $M_{\mathfrak{s}}$ is a finite set of points.

If $1 \in H^0(M_{\mathfrak{s}}; \mathbb{Z})$ is a generator given by an orientation of $M_{\mathfrak{s}}$, then

$$\langle 1, [M_{\mathfrak{s}}] \rangle = SW_X(\mathfrak{s}), \tag{23}$$

as this pairing is just the count with sign of the points in the oriented moduli space $M_{\mathfrak{s}}$.

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Topology of a neighborhood of $M_\mathfrak{s} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_\mathfrak{t}$ V

Open neighborhood of $M_\mathfrak{s} \times \{\mathbf{x}\}$ in $\bar{\mathscr{M}}_\mathfrak{t}$

In [4, Chapter 5] (F and Leness), we constructed a virtual neighborhood $\bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$ of $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathscr{M}}_\mathfrak{t}$ admitting a continuous, surjective map (see F and Leness [4, Lemma 5.8.2]),

$$\pi_X : \bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}} \to \mathrm{Sym}^\ell(X). \tag{24}$$

The space, $\bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$, is stratified by smooth manifolds and contains a subspace, $\bar{\mathscr{O}}_{\mathfrak{t},\mathfrak{s}}$, which is homeomorphic to a neighborhood $\bar{\mathscr{U}}_{\mathfrak{t},\mathfrak{s}}$ of $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathscr{M}}_\mathfrak{t}$.

Let $\mathscr{O}_{\mathfrak{t},\mathfrak{s}}$ be the intersection of $\bar{\mathscr{O}}_{\mathfrak{t},\mathfrak{s}}$ with the top stratum of $\bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ VI

Then $\mathscr{O}_{\mathfrak{t},\mathfrak{s}}$ is the zero locus of a transversely vanishing section of a vector bundle of rank $2r_{\Xi} + 2\ell$ over the top stratum of $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ and $\mathscr{O}_{\mathfrak{t},\mathfrak{s}}$ is diffeomorphic to the top stratum of $\bar{\mathscr{U}}_{\mathfrak{t},\mathfrak{s}}$.

The top stratum of $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ has dimension determined by

$$\frac{1}{2}\dim \mathscr{M}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} = \frac{1}{2}\dim \mathscr{M}_{\mathfrak{t}} + r_{\Xi} + \ell.$$

There is an S^1 action on $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ which restricts to the S^1 action on $\bar{\mathscr{M}}_{\mathfrak{t}}$ discussed earlier.

This S^1 action is free on the complement of its fixed point set, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ VII

Let $\Delta \subset \mathrm{Sym}^\ell(X)$ be the *big diagonal*, given by points $\{x_1, \dots, x_\ell\}$ where $x_i = x_j$ for some $i \neq j$.

For $\mathbf{x} \in \mathrm{Sym}^\ell(X) \setminus \Delta$, let U be an open set,

$$\mathbf{x} \in U \Subset \mathrm{Sym}^\ell(X) \setminus \Delta,$$

and let $\tilde{U} \subset X^\ell$ be the pre-image of U under the branched cover $X^\ell \to \mathrm{Sym}^\ell(X)$.

Let $\mathrm{CSO}(3)$ be the open cone on $\mathrm{SO}(3)$.

For U sufficiently small, we define

$$N(\mathfrak{t}, \mathfrak{s}, U) := M_{\mathfrak{s}} \times \mathbb{C}^{r_N} \times \mathrm{CSO}(3)^\ell \times_{\mathfrak{S}_\ell} \tilde{U}, \tag{25}$$

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ VIII

where $\mathfrak{S}_\ell$ is the symmetric group on ℓ elements, acting diagonally by permutation on the ℓ factors in $\mathrm{CSO}(3)^\ell$ and $\tilde{U}$.

Because U is contained in the top stratum of $\mathrm{Sym}^\ell(X)$, the construction of $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ in [4, Section 5.1.5] (F and Leness) and the map π_X in [4, Lemma 5.8.2] (F and Leness) imply that there is a commutative diagram,

$$\begin{tikzcd} N(\mathfrak{t},\mathfrak{s},U) \arrow[r, "\gamma"] \arrow[d] & \bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}} \arrow[d, "\pi_X"'] \\ U \arrow[r] & \mathrm{Sym}^\ell(X) \end{tikzcd} \tag{26}$$

where

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ IX

1. The horizontal maps are open embeddings,
2. The vertical map on the left is projection onto the factor U,
3. The image of γ is a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\}$ in $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$,
4. The embedding γ is equivariant with respect to the diagonal S^1 action on the factors of $\mathbb{C}$ and SO(3) in (25) and the S^1 action on $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ X

Observe that because U is in the top stratum of $\mathrm{Sym}^{\ell}(X)$, the group $\mathfrak{S}_\ell$ acts freely on $\tilde{U}$.

Hence, for $\mathbf{x} \in U$ the pre-image of $\mathbf{x}$ under the left vertical arrow in the diagram (26) is

$$M_{\mathfrak{s}} \times \mathbb{C}^{r_N} \times \mathrm{CSO}(3)^{\ell} \times \{\mathbf{x}\}.$$

The commutativity of the diagram (26) implies that for $\mathbf{x} \in U$, the embedding γ defines a homeomorphism,

$$M_{\mathfrak{s}} \times \mathbb{C}^{r_N} \times \mathrm{CSO}(3)^{\ell} \times \{\mathbf{x}\} \cong \pi_X^{-1}(\mathbf{x}). \tag{27}$$

Note that for $\mathbf{x} \in U$ represented by $\{x_1, \dots, x_\ell\}$, each x_l has multiplicity one, by definition of Δ, for $1 \leq l \leq \ell$.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ XI

For the cone point $c \in \mathrm{CSO}(3)$, define $\mathbf{c}_\ell \in \mathrm{CSO}(3)^\ell$ by

$$\mathbf{c}_\ell = \underbrace{\{c\} \times \{c\} \times \cdots \times \{c\}}_{\ell \text{ copies}} \in \mathrm{CSO}(3)^\ell.$$

Because $\mathbf{c}_\ell$ is a fixed point of the $\mathfrak{S}_\ell$ action on $\mathrm{CSO}(3)^\ell$,

$$\begin{aligned} (28) \quad \gamma^{-1}\left(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)\right) &= M_{\mathfrak{s}} \times \{0\} \times \{\mathbf{c}_\ell\} \times U \\ &\subset M_{\mathfrak{s}} \times \mathbb{C}^{r_N} \times \mathrm{CSO}(3)^\ell \times_{\mathfrak{S}_\ell} \tilde{U}, \end{aligned}$$

where γ is the embedding in (26).

Topology of a neighborhood of $M_\mathfrak{s} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_\mathfrak{t}$ XII

The link and its branched cover

In [4, Proposition 8.0.4] (F and Leness), we constructed a link,

$$\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}} \subset \bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}/S^1$$

of

$$M_\mathfrak{s} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}/S^1.$$

We will need the following description of

$$\pi_X^{-1}(\mathbf{x}) \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$$

and a branched cover of this space.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ XIII

Lemma 4.2

(See F and Leness [5, Lemma 4.5].) For $\mathbf{x} \in \operatorname{Sym}^{\ell}(X) \setminus \Delta$*, the space* $\pi_X^{-1}(\mathbf{x}) \cap \bar{\mathbf{L}}^{\text{vir}}_{\mathfrak{t},\mathfrak{s}}$ *is homeomorphic to the link of*

$$M_{\mathfrak{s}} \times \{0\} \times \{\mathbf{c}_{\ell}\} \times \{\mathbf{x}\} \quad \text{in} \quad M_{\mathfrak{s}} \times \mathbb{C}^{r_N} \times_{S^1} \mathrm{CSO}(3)^{\ell} \times \{\mathbf{x}\}.$$

The computations in our proof of Proposition 4.1 require the following branched cover of this link.

Topology of a neighborhood of $M_{\mathfrak{s}} \times \{\mathbf{x}\} \subset \bar{\mathscr{M}}_{\mathfrak{t}}$ XIV

Lemma 4.3

(See F and Leness [5, Lemma 4.6].) There is a degree $(-1)^{\ell}2^{r_N+\ell-1}$ *branched cover*

$$\tilde{f} : M_{\mathfrak{s}} \times \mathbb{CP}^{r_N+2\ell-1} \to \pi_X^{-1}(\mathbf{x}) \cap \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}.$$

If ν *is the first Chern class of the* S^1 *action on* $\bar{\mathscr{M}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$*, then*

$$\tilde{f}^*\nu = 1 \times 2\tilde{\nu}, \tag{29}$$

where $1 \in H^0(M_{\mathfrak{s}}; \mathbb{Z})$ *satisfies* (23) *and* $\tilde{\nu} \in H^2(\mathbb{CP}^{r_N+2\ell-1}; \mathbb{Z})$ *satisfies*

$$\langle \tilde{\nu}^{r_N+2\ell-1}, [\mathbb{CP}^{r_N+2\ell-1}] \rangle = (-1)^{r_N+2\ell-1}. \tag{30}$$

Multilinear algebra I

The proof of Proposition 4.1 requires us to consider the intersection number with $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ in (12) as a symmetric multilinear map on $H_2(X;\mathbb{R})$ rather than a polynomial.

We thus introduce some notation to translate between the two concepts.

For a finite-dimensional, real vector space V, let $S_d(V)$ be the vector space of d-linear, symmetric maps, $M : V^{\otimes d} \to \mathbb{R}$, and let $P_d(V)$ be the vector space of degree-d homogeneous polynomials on V.

Multilinear algebra II

The map $\Phi : S_d(V) \to P_d(V)$ defined by $\Phi(M)(v) = M(v, \dots, v)$ is an isomorphism of vector spaces (see Friedman and Morgan [16, Section 6.1.1]).

For $M_i \in S^{d_i}(V)$, we define a product on $S_\bullet(V) = \oplus_{d \geq 0} S_d(V)$ by

$$
\begin{aligned}
&(M_1 M_2)(h_1, \dots, h_{d_1+d_2}) \\
(31) \qquad &:= \frac{1}{(d_1 + d_2)!} \sum_{\sigma \in \mathfrak{S}_{d_1+d_2}} M_1(h_{\sigma(1)}, \dots, h_{\sigma(d_1)}) \\
&\qquad \times M_2(h_{\sigma(d_1+1)}, \dots, h_{\sigma(d_1+d_2)}),
\end{aligned}
$$

where $\mathfrak{S}_d$ is the symmetric group on d elements.

Multilinear algebra III

When $S_\bullet(V)$ has this product and $P_\bullet(V)$ has its usual product, Φ is an algebra isomorphism.

Multilinear algebra IV

Lemma 4.4

(See F and Leness [5, Lemma 4.7].) Continue the assumptions and notation of Proposition 4.1. For $n \in \mathbb{N}$ as in Proposition 4.1 let $\mathfrak{t}_n$ be a spin^u structure satisfying (14)*. Then,*

(32)
$$\begin{aligned}
&\#\left(\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m}x^m) \cap \mathbf{L}_{\mathfrak{t}_n,\mathfrak{s}}\right) \\
&= \frac{SW_X(\mathfrak{s})}{(\delta-2m)!} \sum_{\substack{i+2k \\ =\delta-2m}} \sum_{\sigma \in \mathfrak{S}_{\delta-2m}} a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) \\
&\quad \times \prod_{u=1}^{i} \langle K, h_{\sigma(u)} \rangle \prod_{u=1}^{(\delta-2m-i)/2} Q_X\left(h_{\sigma(i+2u-1)}, h_{\sigma(i+2u)}\right)
\end{aligned}$$

where $\mathfrak{S}_{\delta-2m}$ is the symmetric group on $(\delta-2m)$ elements and $K = c_1(\mathfrak{s})$.

Multilinear algebra V

The following corollary will yield the coefficient appearing in (20) in Proposition 4.1.

Multilinear algebra VI

Corollary 4.5 (Intersection pairing)

(See F and Leness [5, Corollary 4.8].) Continue the notation and hypotheses of Lemma 4.4 and abbreviate $A = \delta - 2m - 2\ell$. *There is a class* $h \in \operatorname{Ker} K \subset H_2(X;\mathbb{R})$ *with* $Q_X(h) = 1$ *and if*

$$h_u = h \in \operatorname{Ker} K \subset H_2(X;\mathbb{R}) \quad \text{for } A+1 \le u \le \delta - 2m, \tag{33}$$

then

$$\begin{aligned} &\#\left(\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m} x^m) \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) \\ &= \frac{SW_X(\mathfrak{s}) A! (2\ell)!}{(\delta - 2m)!} a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) \prod_{u=1}^{A} \langle K, h_u \rangle. \end{aligned} \tag{34}$$

Cohomology classes and duality I

By [4, Proposition 8.0.4] (F and Leness), there are a topological space, $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}} \subset \bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}/S^1$, with fundamental class $[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}]$ and cohomology classes

$$\bar{\mu}_p(h_i), \quad \bar{\mu}_c, \quad \bar{e}_I, \quad \bar{e}_s \in H^\bullet\left(\bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}/S^1 \setminus \left(M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)\right); \mathbb{R}\right),$$

such that

$$\begin{aligned} &\#\left(\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m} x^m) \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) \\ &\quad = \langle \bar{\mu}_p(h_1) \smile \cdots \smile \bar{\mu}_p(h_{\delta-2m}) \smile \bar{\mu}_p(x)^m \smile \bar{e}_I \smile \bar{e}_s, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}]\rangle. \end{aligned} \tag{35}$$

We note that the cohomology classes $\bar{\mu}_p(h_i)$ and $\bar{\mu}_p(x)$ are extensions of the classes $\mu_p(h_i)$ and $\mu_p(x)$ defined earlier.

Cohomology classes and duality II

For $\beta \in H_\bullet(X;\mathbb{R})$, the cohomology class

$$S^\ell(\beta) \in H^{4-\bullet}(\mathrm{Sym}^\ell(X);\mathbb{R})$$

is defined by the property that, for $\tilde{\pi} : X^\ell \to \mathrm{Sym}^\ell(X)$ denoting the degree-$\ell!$ branched covering map,

$$\tilde{\pi}^* S^\ell(\beta) = \sum_{i=1}^{n} \pi_i^* \,\mathrm{PD}[\beta],$$

where $\pi_i : X^\ell \to X$ is projection onto the i-th factor.

Cohomology classes and duality III

Thus (compare Kotschick and Morgan [18, p. 454]),

(36)
$$
\begin{aligned}
&\langle S^\ell(h_1) \smile \cdots \smile S^\ell(h_{2\ell+k}), [\mathrm{Sym}^\ell(X)]\rangle \\
&\quad = \begin{cases} \dfrac{(2\ell)!}{\ell! 2^\ell} Q_X^\ell(h_1,\dots,h_{2\ell})\,\mathrm{PD}[\mathbf{x}] & \text{if } k=0, \\ 0 & \text{if } k>0, \end{cases}
\end{aligned}
$$

where $\mathbf{x} \in \mathrm{Sym}^\ell(X) \setminus \Delta$ is a point in the top stratum.

Note that if $h_u = h$ for $u = 1,\dots,2\ell$, then by the definition of the product in (31),

(37)
$$
Q_X^\ell(h_1,\dots,h_{2\ell}) = Q_X(h)^\ell.
$$

Cohomology classes and duality IV

From [4, Equations (8.3.21), (8.3.24), and (8.3.25)] and [4, Lemma 8.4.1] (F and Leness) we have, denoting

$$K = c_1(\mathfrak{s}), \quad \Lambda = c_1(\mathfrak{t}), \quad h \in H_2(X;\mathbb{R}),$$

and a generator $x \in H_0(X;\mathbb{Z})$,

(38)
$$\begin{aligned} \bar{\mu}_p(h) &= \frac{1}{2}\langle \Lambda - K, h\rangle \nu + \pi_X^* S^\ell(h), \\ \bar{\mu}_p(x) &= -\frac{1}{4}\nu^2 + \pi_X^* S^\ell(x), \\ \bar{e}_{\mathfrak{s}} &= (-\nu)^{r_\Xi}, \end{aligned}$$

where ν is the first Chern class of the S^1 action on $\bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}}$ and $pi_X : \bar{\mathscr{M}}_{\mathfrak{t},\mathfrak{s}}^{\mathrm{vir}} \to \mathrm{Sym}^\ell(X)$ is defined in (24).

Outline of remainder of leading-order term computation I

We now outline the proof of Proposition 4.1.

For $n \in \mathbb{N}$ as in Proposition 4.1 (*leading-order term in the* SO(3)*-monopole cobordism formula* (18) *for link pairings*), we choose $\mathfrak{t}_n$ be a spin^u structure satisfying the condition (14).

We apply Corollary 4.5 (*intersection pairing*) to verify the expression (20) for the coefficient $a_{A,0,\ell}$ to give

(39)
$$\begin{aligned}&\#\left(\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m}x^m) \cap \mathbf{L}_{\mathfrak{t}_n,\mathfrak{s}}\right)\\ &\quad = (-1)^A \frac{SW_X(\mathfrak{s})A!(2\ell)!}{(\delta-2m)!} a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell).\end{aligned}$$

Outline of remainder of leading-order term computation II

We now compute the *left-hand-side* of (39), namely

$$\#\left(\bar{\mathscr{V}}(h_1 \ldots h_{\delta-2m}x^m) \cap \mathbf{L}_{\mathfrak{t}_n,\mathfrak{s}}\right).$$

From the proof of Corollary 4.5, there are homology classes $h_0, h_0' \in H_2(X;\mathbb{R})$ which satisfy

$$\text{(40)} \qquad \langle K, h_0\rangle = 0, \quad Q_X(h_0) = 1, \quad \langle K, h_0'\rangle = -1.$$

We define

$$\text{(41)} \qquad h_u := \begin{cases} h_0' & \text{for } 1 \le u \le A, \\ h_0 & \text{for } A+1 \le u \le \delta - 2m. \end{cases}$$

Outline of remainder of leading-order term computation III

Using the topology of the link we compute that, under the assumptions (40) on h_u,

(42)
$$\begin{aligned}&\#\left(\bar{\mathscr{V}}(h_1 \dots h_{\delta-2m}x^m) \cap \mathbf{L}_{\mathfrak{t}_n,\mathfrak{s}}\right)\\ &\quad= \frac{(2\ell)!}{\ell!2^\ell}(-1)^{m+r_\Xi+r_N+1}2^{-A-2m}SW_X(\mathfrak{s}).\end{aligned}$$

Comparing (39) and (42) gives

$$\begin{aligned}(-1)^A &\frac{SW_X(\mathfrak{s})A!(2\ell)!}{(\delta-2m)!}a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell)\\ &= \frac{(2\ell)!}{\ell!2^\ell}(-1)^{m+r_\Xi+r_N+1}2^{-A-2m}SW_X(\mathfrak{s}),\end{aligned}$$

Outline of remainder of leading-order term computation IV

which we solve to get

$$(43)\quad a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) = \frac{(\delta - 2m)!}{\ell! A!}(-1)^{A+m+r_\Xi+r_N+1} 2^{-A-2m-\ell}.$$

We can arrange that

$$-A - 2m - \ell = \ell - \delta$$

while dimension counting implies that

$$A + m + r_\Xi + r_N + 1 \equiv \ell + m \pmod 2.$$

Hence, (43) yields the desired equality (20), completing the proof of Proposition 4.1.

Vanishing coefficients in the SO(3)-monopole cobordism formula for link pairings

Vanishing coefficients in formula for link pairings I

We now determine the coefficients $a_{i,0,k}$ with $i \geq c(X) - 3$ appearing in Equation (18) in Theorem 3.3, namely

$$\boxed{0 = \sum_{k=0}^{2\chi_h(X)-n} a_{A+2k,0,2\chi_h(X)-n-k} SW_X^{w,A+2k}(h) Q_X(h)^{2\chi_h(X)-n+k}}$$

The techniques used in the proof of [13, Proposition 4.8] (F and Leness) also determine the coefficients $a_{i,0,k}$ with $i \geq c(X) - 3$ appearing in (18).

Vanishing coefficients in formula for link pairings II

Proposition 5.1 (Vanishing coefficients in SO(3)-monopole cobordism formula for link pairings)

(See F and Leness [5, Proposition 5.1].) Continue the hypothesis and notation of Theorem 3.3. In addition, assume $c(X) \geq 3$ and

$$n \equiv 1 \pmod 2. \tag{44}$$

Then for $p \geq c(X) - 3$ and $k \geq 0$ an integer such that $p + 2k = c(X) + 4\chi_h(X) - 3n - 1 - 2m$,

$$a_{p,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h - n) = 0.$$

Vanishing coefficients in formula for link pairings III

We prove Proposition 5.1 by showing that on certain standard four-manifolds, the following vanishing result arising in the proof of Theorem 3.3 (SO(3)-*monopole cobordism formula vanishing*),

$$
\begin{aligned}
(45)\quad 0 = &\sum_{K \in B(X)} (-1)^{\frac{1}{2}(w^2 + w\cdot K)} SW_X'(K) \\
&\times \sum_{\substack{i+2k \\ =\delta-2m}} a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h(X) - n) \langle K, h\rangle^i Q_X(h)^k.
\end{aligned}
$$

forces each of the coefficients in the sum to be zero by using the following generalization of [16, Lemma VI.2.4] (Friedman and Morgan).

Vanishing coefficients in formula for link pairings IV

Lemma 5.2 (Algebraic independence)

(See F and Leness [13, Lemma 4.1] or [5, Lemma 5.2].) Let V be a finite-dimensional real vector space. Let $T_1, \dots, T_n$ be linearly independent elements of the dual space V^*. Let Q be a quadratic form on V which is non-zero on $\cap_{i=1}^n \operatorname{Ker} T_i$. Then $T_1, \dots, T_n, Q$ are algebraically independent in the sense that if $F(z_0, \dots, z_n) \in \mathbb{R}[z_0, \dots, z_n]$ and $F(Q, T_1, \dots, T_n) : V \to \mathbb{R}$ is the zero map, then $F(z_0, \dots, z_n)$ is the zero element of $\mathbb{R}[z_0, \dots, z_n]$.

In [13, Section 4.2], we used the manifolds constructed by Fintushel, Park and Stern in [14] to give the following useful family of standard four-manifolds.

Vanishing coefficients in formula for link pairings V

Lemma 5.3 (Fintushel-Park-Stern family of standard four-manifolds with Seiberg-Witten simple type)

(See F and Leness [5, Lemma 5.3].) For every integer $q \geq 2$, there is a standard four-manifold X_q of Seiberg-Witten simple type satisfying

(46a) $\chi(X_q) = q$ *and* $c(X_q) = 3$,

(46b) $B(X_q) = \{\pm K\}$ *and* $K \neq 0$,

(46c) *The restriction of* Q_{X_q} *to* $\operatorname{Ker} K \subset H_2(X_q;\mathbb{R})$ *is non-zero.*

Vanishing coefficients in formula for link pairings VI

We write the blow-up of X_q at r points as $X_q(r)$, so

$$\begin{aligned}\chi_h(X_q(r)) &= \chi_h(X_q) = q,\\ c_1^2(X_q(r)) &= c_1^2(X_q) - r,\\ c(X_q(r)) &= c(X_q) + r = r + 3,\end{aligned}$$

where we recall from (1) that $c(X) := \chi_h(X) - c_1^2(X)$.

We consider both the homology and cohomology of X_q as subspaces of the homology and cohomology of $X_q(r)$, respectively.

Let $e_u^* \in H^2(X_q(r); \mathbb{Z})$ be the Poincaré dual of the u-th exceptional class.

Vanishing coefficients in formula for link pairings VII

Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^r \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the u-th factor.

For $\varphi \in (\mathbb{Z}/2\mathbb{Z})^r$ and $K \in B(X_q)$, we define

$$(47) \qquad K_\varphi := K + \sum_{u=1}^{r} (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^{r} e_u^*.$$

Then, by the blow-up formula for Seiberg-Witten invariants (see Frøyshov [17, Theorem 14.1.1]),

$$(48) \qquad \begin{aligned} B'(X_q(r)) &= \{K_\varphi : \varphi \in (\mathbb{Z}/2\mathbb{Z})^r\}, \\ SW_{X_q(r)}(K_\varphi) &= SW_{X_q}(K). \end{aligned}$$

In preparation for our application of Lemma 5.2, we have the

Vanishing coefficients in formula for link pairings VIII

Lemma 5.4 (Algebraically independent functionals and intersection form on $H_2(X_q(r);\mathbb{R})$)

(See F and Leness [5, Lemma 5.4].) Let $q \geq 2$ and $r \geq 0$ be integers. Let $X_q(r)$ be the blow-up of the four-manifold X_q given in Lemma 5.3 at r points. Then the set

$$\{K, e_1^*, \ldots, e_r^*, Q_{X_q(r)}\}$$

is algebraically independent in the sense of Lemma 5.2 for the vector space $H_2(X_q(r);\mathbb{R})$.

Vanishing coefficients in formula for link pairings IX

Outline of the Proof of Proposition 5.1.

Because $c(X) \geq 3$, if $q = \chi_h(X)$ and $r = c(X) - 3 \geq 0$, then

$$\chi_h(X) = \chi_h(X_q(r)) \quad \text{and} \quad c_1^2(X) = c_1^2(X_q(r)) \tag{49}$$

by Lemma 5.3 and so

$$\begin{aligned} &a_{i,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, \ell) \\ &\qquad = a_{i,0,k}(\chi_h(X_q(r)), c_1^2(X_q(r)), 0, 0, m, \ell). \end{aligned} \tag{50}$$

As in the proof of Theorem 3.3, the assumptions on m and n allow us to apply the SO(3)-monopole cobordism formula (13) with a

Vanishing coefficients in formula for link pairings X

spin^u structure $\mathfrak{t}_n$ on $X_q(r)$ satisfying (14), $\tilde{w} \in H^2(X_q(r);\mathbb{Z})$ characteristic,

$$\begin{aligned}\delta &:= c(X) + 4\chi_h(X) - 3n - 1\\ &= c(X_q(r)) + 4\chi_h(X_q(r)) - 3n - 1 \quad \text{(by (49))},\end{aligned}$$

and $\ell(\mathfrak{t}_n, \mathfrak{s}) = 2\chi_h(X_q(r)) - n$ from (15) to get (see (45))

$$\begin{aligned}0 = &\sum_{K\in B(X_q(r))} (-1)^{\frac{1}{2}(\tilde{w}^2+\tilde{w}\cdot K)} SW'_{X_q(r)}(K)\\ &\times \sum_{\substack{i+2k\\=\delta-2m}} a_{i,0,k}(\chi_h(X_q(r)), c_1^2(X_q(r)), 0, 0, m, \ell)\langle K, h\rangle^i Q_{X_q(r)}(h)^k.\end{aligned} \tag{51}$$

Vanishing coefficients in formula for link pairings XI

Because $SW_X(-K) = (-1)^{\chi_h(X)} SW_X(K)$ by [22, Corollary 6.8.4], the set $B(X_q(r))$ is closed under the action of $\{\pm 1\}$.

Let $B'(X_q(r))$ be a fundamental domain for the action of $\{\pm 1\}$ on $B(X_q(r))$.

We rewrite (51) as a sum over $B'(X_q(r))$ by combining the terms given by K and $-K$.

To apply Lemma 5.2 to (51) and get information about the coefficients $a_{i,0,k}$, we will replace $B'(X_q(r))$ with the set $\{K, e_1^*, \ldots, e_r^*\}$ appearing in Lemma 5.4.

By Lemma 5.4, the set $\{K, e_1^*, \ldots, e_r^*, Q_{X_q(r)}\}$ is algebraically independent and, after some calculation, we eventually see that

Vanishing coefficients in formula for link pairings XII

(51) implies that the coefficient $a_{p,0,k}$ must vanish, for $p \geq c(X) - 3$ as in the statement of Proposition 5.1. $\square$

Simple Type $\implies$ Superconformal Simple Type

Simple Type $\implies$ Superconformal Simple Type I

We shall prove Theorem 1.1 (simple type $\implies$ superconformal simple type) by applying the computations of the coefficients in

- Proposition 4.1 (formula for $a_{A,0,\ell}(\chi_h(X), c_1^2(X), 0, 0, m, \ell)$),
- Proposition 5.1 ($a_{p,0,k}(\chi_h(X), c_1^2(X), 0, 0, m, 2\chi_h - n) = 0$),

to the vanishing sum formula (18), namely

$$\boxed{0 = \sum_{k=0}^{2\chi_h(X)-n} a_{A+2k,0,2\chi_h(X)-n-k} SW_X^{w,A+2k}(h) Q_X(h)^{2\chi_h(X)-n+k}.}$$

Simple Type $\implies$ Superconformal Simple Type II

To apply Proposition 4.1, we need to assume that there is a class $K \in B(X)$ with $K \neq 0$.

This assumption is valid if we can replace X with its blow-up $\widetilde{X}$.

In the following, we show that the superconformal simple type property is invariant under blow-up, allowing the desired replacement of X with $\widetilde{X}$ in the proof of Theorem 1.1.

The “only if” direction in Lemma below was proved by Mariño, Moore, and Peradze as [21, Theorem 7.3.1].

Simple Type $\implies$ Superconformal Simple Type III

Lemma 6.1 (Invariance of the superconformal simple type property under blow-up)

(See F and Leness [5, Lemma 6.1] and Mariño, Moore, and Peradze [21, Theorem 7.3.1].) Let X be a standard four-manifold of Seiberg-Witten simple type with $c(X) \geq 3$. Then X has superconformal simple type if and only if its blow-up $\widetilde{X}$ does.

Half of the polynomials $SW_X^{w,i}$ vanish for the parity reasons.

(This observation appears in the remarks following [21, Proposition 6.1.3] due to Mariño, Moore, and Peradze.)

Simple Type $\implies$ Superconformal Simple Type IV

Lemma 6.2 (Vanishing of half of the polynomials $SW_X^{w,i}$ by parity)

(See F and Leness [5, Lemma 6.2] and Mariño, Moore, and Peradze [21].) If a standard four-manifold X has Seiberg-Witten simple type, $i \geq 0$ is any integer obeying $c(X) + i \equiv 1 \pmod 2$, and $w \in H^2(X;\mathbb{Z})$ is characteristic, then $SW_X^{w,i}$ vanishes.

The vanishing sum formula (18) will give information about the Seiberg-Witten polynomial $SW_X^{w,A}$ of degree

$$A = c(X) - n - 2m - 1$$

which appears in this sum with a non-zero coefficient.

Simple Type $\implies$ Superconformal Simple Type V

We write $A = c(X) - 2v$ where v is a non-negative integer such that $2v = n + 2m + 1$ as in the statement of Theorem 3.3 and note some of the values for this degree to which we can apply Theorem 3.3 and Proposition 5.1.

If $n = 3$, then the equality $2v = n + 2m + 1$ implies $m = v - 2$.

Lemma 6.3

(See F and Leness [5, Lemma 6.3] Let X be a standard four-manifold with $c(X) \geq 3$. For any $v \in \mathbb{N}$ with $4 \leq 2v \leq c(X)$, the natural numbers $n = 3$ and $m = v - 2$ satisfy the conditions (16) in Theorem 3.3 and the parity condition (44) in Proposition 5.1.

Simple Type $\implies$ Superconformal Simple Type VI

Outline of Remainder of Proof of Theorem 1.1 (simple type $\implies$ superconformal simple type)

By Lemma 6.1, it suffices to prove that the blow-up of X has superconformal simple type.

Because $c_1^2(\widetilde{X}) = c_1^2(X) - 1$, we can assume $c_1^2(X) \neq 0$ by replacing X with its blow up if necessary.

If we assume $c_1^2(X) \neq 0$ and $K \in B(X)$, then $K^2 = c_1^2(X) \neq 0$ by our assumption that X has Seiberg-Witten simple type, so $K \neq 0$.

Thus, we can assume $0 \notin B(X)$ by replacing X with its blow-up.

We now abbreviate $c = c(X)$ and $\chi_h = \chi_h(X)$.

Simple Type $\implies$ Superconformal Simple Type VII

If $w \in H^2(X;\mathbb{Z})$ is characteristic, then $SW_X^{w,i}$ vanishes unless $i \equiv c \pmod 2$ by Lemma 6.2 (*vanishing of half of the polynomials* $SW_X^{w,i}$ *by parity*).

Thus, it suffices to prove that $SW_X^{w,c-2v} = 0$ for $4 \leq 2v \leq c$, which we will do by induction on v.

By Lemma 6.3, the values $n = 3$ and $m = v - 2$ satisfy the conditions (16) in Theorem 3.3 (SO(3)-*monopole cobordism formula vanishing*).

Simple Type $\implies$ Superconformal Simple Type VIII

Substituting these values into (18) (noting that $A = c - n - 2m - 1 = c - 2v$), yields

$$
\begin{aligned}
(52)\quad & 0 \\
& = \sum_{k=0}^{2\chi_h - 3} a_{c-2v+2k,0,2\chi_h-3-k} SW_X^{w,c-2v+2k}(h) Q_X(h)^{2\chi_h-3+k},
\end{aligned}
$$

where the coefficients $a_{i,0,k}$ are defined in (17).

Because $n = 3$ satisfies the assumption (44), Proposition 5.1 implies that

$$
(53)\qquad a_{c-2v+2k,0,2\chi_h-3-k} = 0 \quad \text{for } 2k - 2v \geq -3.
$$

Simple Type $\implies$ Superconformal Simple Type IX

Because of our assumption that $0 \notin B(X)$, an application of Proposition 4.1 (*leading-order term in the* SO(3)*-monopole cobordism formula* (18) *for link pairings*) with $n = 3$ gives

$$a_{c-2v,0,2\chi_h-3} \neq 0. \tag{54}$$

We now begin the induction on v.

If $2v = 4$, the identity (52) becomes

$$\begin{aligned} 0 &= \sum_{k=0}^{2\chi_h-3} a_{c-4+2k,0,2\chi_h-3-k} SW_X^{w,c-4+2k}(h) Q_X(h)^{2\chi_h-3-k} \\ &= a_{c-4,0,2\chi_h-3} SW_X^{w,c-4}(h) Q_X(h)^{2\chi_h-3} \quad \text{(by (53))}, \end{aligned}$$

Simple Type $\implies$ Superconformal Simple Type X

that is,

$$0 = a_{c-4,0,2\chi_h-3} SW_X^{w,c-4}(h) Q_X(h)^{2\chi_h-3}. \tag{55}$$

Because $2v = 4$, equations (54) and (55) imply that

$$SW_X^{w,c-4}(h) Q_X(h)^{2\chi_h-3} = 0 \quad \text{for all } h \in H_2(X;\mathbb{R}).$$

If $Z \subset H_2(X;\mathbb{R})$ is the (codimension-one) zero locus of Q_X, the preceding equality implies that the polynomial $SW_X^{w,c-4}$ vanishes on the open, dense subset $H_2(X;\mathbb{R}) \setminus Z$ of $H_2(X;\mathbb{R})$.

Hence $SW_X^{w,c-4}$ vanishes on $H_2(X;\mathbb{R})$, completing the proof of the initial case of the induction on v.

Simple Type $\implies$ Superconformal Simple Type XI

For our induction hypothesis, we assume that $SW_X^{w,c-2v'} = 0$ for all v' with $4 \leq 2v' < 2v \leq c$.

Calculation then reveals that

$$0 = SW_X^{w,c-2v}(h) Q_X(h)^{2\chi_h - 3}. \tag{56}$$

If Z is the zero locus of Q_X, then (56) implies that $SW_X^{w,c-2v}$ vanishes on the open dense subset $H_2(X;\mathbb{R}) \setminus Z$ of $H_2(X;\mathbb{R})$.

Hence $SW_X^{w,c-2v}$ vanishes identically on $H_2(X;\mathbb{R})$, completing the induction and the proof of Theorem 1.1.

Thank you for your attention!

Bibliography

S. K. Donaldson, *Connections, cohomology and the intersection forms of 4-manifolds*, J. Differential Geom. **24** (1986), no. 3, 275–341. MR 868974

S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990. MR 1079726 (92a:57036)

P. M. N. Feehan, *Generic metrics, irreducible rank-one* $\mathrm{PU}(2)$ *monopoles, and transversality*, Comm. Anal. Geom. **8** (2000), 905–967, arXiv:math/9809001.

P. M. N. Feehan and T. G. Leness, *A general* $\mathrm{SO}(3)$*-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, Memoirs of the American Mathematical Society, in press, arXiv:math/0203047.

______, *The* $\mathrm{SO}(3)$ *monopole cobordism and superconformal simple type*, arXiv:1408.5307.

______, *Nonlinear elliptic partial differential equations and applications to gluing* $\mathrm{SO}(3)$ *monopoles and anti-self-dual connections*, book in preparation, based in part on arXiv:math/9812060 and arXiv:math/9907107.

______, $\mathrm{PU}(2)$ *monopoles. III: Existence of gluing and obstruction maps*, arXiv:math/9907107.

______, *Superconformal simple type and Witten's conjecture*, arXiv:1408.5085.

______, $\mathrm{PU}(2)$ *monopoles. I. Regularity, Uhlenbeck compactness, and transversality*, J. Differential Geom. **49** (1998), 265–410. MR 1664908 (2000e:57052)

______, $\mathrm{PU}(2)$ *monopoles and links of top-level Seiberg-Witten moduli spaces*, J. Reine Angew. Math. **538** (2001), 57–133, arXiv:math/0007190.

______, $\mathrm{PU}(2)$ *monopoles. II. Top-level Seiberg-Witten moduli spaces and Witten's conjecture in low degrees*, J. Reine Angew. Math. **538** (2001), 135–212, arXiv:dg-ga/9712005.

______, $\mathrm{SO}(3)$-*monopoles: the overlap problem*, Geometry and topology of manifolds, Fields Inst. Commun., vol. 47, Amer. Math. Soc., Providence, RI, 2005, arXiv:1211.0480, pp. 97–118.

______, *Witten's conjecture for many four-manifolds of simple type*, Journal of the European Mathematical Society **17** (2015), no. 4, 899923, arXiv:math/0609530.

R. Fintushel, J. Park, and R. J. Stern, *Rational surfaces and symplectic 4-manifolds with one basic class*, Algebr. Geom. Topol. **2** (2002), 391–402 (electronic), arXiv:math/0202105.

R. Fintushel and R. Stern, *The canonical class of a symplectic 4-manifold*, Turkish J. Math. **25** (2001), 137–145, math.uci.edu/ rstern.

R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 27, Springer-Verlag, Berlin, 1994. MR 1288304

K. A. Frøyshov, *Compactness and gluing theory for monopoles*, Geometry & Topology Monographs, vol. 15, Geometry & Topology Publications, Coventry, 2008, available at `msp.warwick.ac.uk/gtm/2008/15/`. MR 2465077 (2010a:57050)

D. Kotschick and J. W. Morgan, *SO(3) invariants for four-manifolds with $b^+ = 1$, II*, J. Differential Geom. **39** (1994), 433–456.

P. B. Kronheimer and T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), 573–734. MR 1338483 (96e:57019)

M. Mariño, G. Moore, and G. Peradze, *Four-manifold geography and superconformal symmetry*, Math. Res. Lett. **6** (1999), 429–437, arXiv:math/9812042.

- ______, *Superconformal invariance and the geography of four-manifolds*, Comm. Math. Phys. **205** (1999), 691–735, arXiv:hep-th/9812055.
- J. W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Princeton Univ. Press, Princeton, NJ, 1996.
- A. Teleman, *Moduli spaces of* $\mathrm{PU}(2)$*-monopoles*, Asian J. Math. **4** (2000), 391–435, arXiv:math/9906163.