Lecture 3: Superconformal simple type and Witten's conjecture

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Outline

Introduction and main results

Introduction I

In his article [25], Witten (1994)

- Gave a formula expressing the Donaldson series in terms of Seiberg-Witten invariants for standard four-manifolds,
- Outlined an argument based on supersymmetric quantum field theory, his previous work [24] on topological quantum field theories (TQFT), and his work with Seiberg [22, 23] explaining how to derive this formula.

In a later article [19], Moore and Witten

- extended the scope of Witten's previous formula by allowing four-dimensional manifolds with $b^1 \neq 0$ and $b^+ = 1$, and
- provided the details underlying the derivation of these formulae using supersymmetric quantum field theory.

Introduction II

The purpose of our third lecture in this series is to describe a proof using SO(3) monopoles that for all standard four-manifolds with *superconformal simple type*, the Donaldson series is given by *Witten's formula*.

Without knowing whether or not all four-manifolds have superconformal simple type, one may use SO(3) monopoles to prove that for "many" four-manifolds (we shall quantify "how many" later), the Donaldson series is given by Witten's formula.

In our previous lecture, we explained how to use SO(3) monopoles to prove that all standard four-manifolds with Seiberg-Witten simple type necessarily have *superconformal simple type*.

Hence, by combining the main result of our previous lecture,

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Seiberg-Witten simple type $\Longrightarrow$ *Superconformal simple type,*

and the main result of this lecture,

Superconformal simple type $\Longrightarrow$ *Witten's formula,*

we obtain the desired

Seiberg-Witten simple type $\Longrightarrow$ *Witten's formula.*

It is not known whether all four-manifolds also have Seiberg-Witten simple type.

Our lecture is primarily based on

- P. M. N. Feehan and T. G. Leness, *Superconformal simple type and Witten's conjecture*, arXiv:1408.5085 (in review since October 2014).

Introduction IV

That article is in turn based on methods and results described earlier in

1. P. M. N. Feehan and T. G. Leness, *A general* $\mathrm{SO}(3)$*-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, Memoirs of the American Mathematical Society, in press, arXiv:math/0203047,
2. P. M. N. Feehan and T. G. Leness, *Witten's conjecture for many four-manifolds of simple type*, Journal of the European Mathematical Society **17** (2015), 899–923.

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with supporting results and background material with earlier published articles with Leness cited therein.

Our proofs of these conjectures rely on an assumption of certain analytical properties gluing maps for SO(3) monopoles (see Hypothesis 2.6), analogous to properties proved by Donaldson and Taubes in simpler contexts of gluing maps for SO(3) anti-self-dual connections.

Verification of those analytical gluing map properties is work in progress [4] and appears well within reach.

Statements of main results I

A closed, oriented four-manifold X has an *intersection form*,

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}.$$

One lets $b^{\pm}(X)$ denote the dimensions of the maximal positive or negative subspaces of the form Q_X on $H_2(X;\mathbb{Z})$ and

$$e(X) := \sum_{i=0}^{4}(-1)^i b_i(X) \quad \text{and} \quad \sigma(X) := b^+(X) - b^-(X)$$

denote the *Euler characteristic* and *signature* of X, respectively.

Statements of main results II

We define the characteristic numbers,

$$
\begin{aligned}
c_1^2(X) &:= 2e(X) + 3\sigma(X), \\
\chi_h(X) &:= (e(X) + \sigma(X))/4, \qquad (1) \\
c(X) &:= \chi_h(X) - c_1^2(X).
\end{aligned}
$$

We call a four-manifold standard if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

For a standard four-manifold, the Seiberg-Witten invariants comprise a function,

$$SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

on the set of spin^c structures on X.

Statements of main results III

The set of Seiberg-Witten basic classes, $B(X)$, is the image under $c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z})$ of the support of SW_X, that is

$$B(X) := \{K \in H^2(X;\mathbb{Z}) : SW_X(K) \neq 0\}.$$

A manifold X has Seiberg-Witten simple type if $K^2 = c_1^2(X)$ for all $K \in B(X)$.

Statements of main results IV

Conjecture 1.1 (Witten's Conjecture)

Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then for any $w \in H^2(X;\mathbb{Z})$ the Donaldson invariants satisfy

$$\text{(2)} \quad \mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \times \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}),h\rangle}.$$

As defined by Mariño, Moore, and Peradze, [18, 17], a manifold X has superconformal simple type if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic,

$$\text{(3)} \quad \boxed{SW_X^{w,i}(h) = 0 \quad \text{for } i \leq c(X) - 4}$$

Statements of main results V

and all $h \in H_2(X;\mathbb{R})$, where

$$SW_X^{w,i}(h) := \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})\langle c_1(\mathfrak{s}), h\rangle^i$$

Our main goal in this lecture is to describe the proof of the

Theorem 1.2 (Superconformal simple type $\Longrightarrow$ Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [6, Theorem 1.2].) Assume Hypothesis 2.6. If a standard four-manifold has superconformal simple type, then it satisfies Witten's Conjecture 1.1.

Statements of main results VI

Hypothesis 2.6 asserts certain analytical properties of local gluing maps for SO(3) monopoles constructed by the authors in [5].

Proofs of these analytical properties, analogous to known properties of local gluing maps for anti-self-dual SO(3) connections and Seiberg-Witten monopoles, are being developed by us [4].

Global gluing maps are used to describe the topology of neighborhoods of Seiberg-Witten monopoles appearing at all levels of the compactified moduli space of SO(3) monopoles and hence construct links of those singularities.

On the other hand, from [3] (F and Leness) and our previous lecture, we have

Statements of main results VII

Theorem 1.3 (All standard four-manifolds with Seiberg-Witten simple type have superconformal simple type)

(See F and Leness [6, Theorem 1.1].) Assume Hypothesis 2.6. If X is a standard four-manifold of Seiberg-Witten simple type, then X has superconformal simple type.

Combining Theorems 1.2 and 1.3 yields the following

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Corollary 1.4 (Witten's Conjecture holds for all standard four-manifolds)

(See F and Leness [6, Corollary 1.3] or [6, Corollary 1.4].) Assume Hypothesis 2.6. If X is a standard four-manifold of Seiberg-Witten simple type then X satisfies Witten's Conjecture 1.1.

Refinements of Witten's formula, superconformal simple type, and the SO(3)-monopole cobordism formula

Refinements of Witten's formula and SCST I

The following result allows us to use a convenient choice of w:

Proposition 2.1 (Independence of Witten's Conjecture 1.1 from choice of class w)

(See F and Leness [7, Proposition 2.5] or [6, Proposition 2.2].) Let X be a standard four-manifold of Seiberg-Witten simple type. If Witten's Conjecture 1.1 holds for one $w \in H^2(X;\mathbb{Z})$, then it holds for all $w \in H^2(X;\mathbb{Z})$.

One can view Proposition 2.1 as a partial analogue of the following result due to Kronheimer and Mrowka [15]:

> *If a standard four-manifold is KM-simple type for one $w \in H^2(X;\mathbb{Z})$, then it is KM-simple type for all w.*

Refinements of Witten's formula and SCST II

The following result allows us to replace a manifold by its blow-up without loss of generality.

Theorem 2.2 (Invariance of Witten's Conjecture 1.1 under blow-up)

(See Fintushel and Stern [10, Theorem 8.9].) Let X be a standard four-manifold. Then Witten's Conjecture 1.1 holds for X if and only if it holds for the blow-up, $\widetilde{X}$.

One can view Theorem 2.2 as a partial analogue of the following result due to Kronheimer and Mrowka [15] (in the "only if" direction and [7, Proposition 2.6] in the "if" direction):

> *A standard four-manifold is KM-simple type if and only if its blow-up is KM-simple type.*

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Refinements of Witten's formula and SCST III

A standard four-manifold X has superconformal simple type if $c(X) \leq 3$ or $c(X) \geq 4$ and for $w \in H^2(X;\mathbb{Z})$ characteristic and all $h \in H_2(X;\mathbb{R})$,

$$SW_X^{w,i}(h) = 0 \quad \text{for } 0 \leq i \leq c(X) - 4 \tag{4}$$

where

$$SW_X^{w,i}(h) := \sum_{K \in B(X)} (-1)^{\varepsilon(w,K)} SW_X'(K) \langle K, h \rangle^i$$

Observe that we have rewritten (3) as a sum over $B(X)$ using the expression

$$SW_X' : H^2(X;\mathbb{Z}) \to \mathbb{Z}, \quad SW_X'(K) = \sum_{\mathfrak{s} \in c_1^{-1}(K)} SW_X(\mathfrak{s}). \tag{5}$$

Refinements of Witten's formula and SCST IV

We further note that the property (4) is invariant under blow-up.

Lemma 2.3 (Invariance of the superconformal simple type property under blow-up)

(See Mariño, Moore, and Peradze [18, Theorem 7.3.1] or F and Leness [3, Lemma 6.1].) A standard manifold, X, has superconformal simple type if and only if its blow-up, $\widetilde{X}$, has superconformal simple type.

This is a convenient point at which to recall versions of the blow-up formulae for Donaldson and Seiberg-Witten invariants, since these formulae are used to verify invariance of Witten's Formula (2) and the Superconformal Simple Type property under blow-ups.

Refinements of Witten's formula and SCST V

Let $\widetilde{X} \to X$ be the blow-up of X at one point, let $e \in H_2(\widetilde{X};\mathbb{Z})$ be the fundamental class of the exceptional curve, and let $e^* \in H^2(\widetilde{X};\mathbb{Z})$ be the Poincaré dual of e.

Using the direct sum decomposition of the homology and cohomology of $\widetilde{X} = X\#\overline{\mathbb{CP}}^2$, we can consider both the homology and cohomology of X as subspaces of those of $\widetilde{X}$,

$$H_\bullet(X) \subset H_\bullet(\tilde{X}) \quad \text{and} \quad H^\bullet(X) \subset H^\bullet(\tilde{X}).$$

Denote $\tilde{w} := w + e^*$. The simplest blow-up formula for Donaldson invariants (see Kotschick [14] or Leness [16] for SO(3) invariants and Ozsváth [21] for SU(2) invariants) gives

$$\text{(6)} \qquad \boxed{D_X^w(h^{\delta-2m}x^m) = D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m}ex^m).}$$

Refinements of Witten's formula and SCST VI

Versions of the blow-up formula for Seiberg-Witten invariants have been established by Fintushel and Stern [9], Nicolaescu [20, Theorem 4.6.7], and Frøyshov [12, Theorem 14.1.1] (in increasing generality).

The following is a special case of their results.

Refinements of Witten's formula and SCST VII

Theorem 2.4 (Blow-up formula for Seiberg-Witten invariants)

Let X be a standard four-manifold and let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ be its blow-up. Then $\widetilde{X}$ has Seiberg-Witten simple type if and only if that is true for X. If X has Seiberg-Witten simple type, then

$$B(\widetilde{X}) = \{K \pm e^* : K \in B(X)\}, \tag{7}$$

where $e^* \in H^2(\widetilde{X}; \mathbb{Z})$ is the Poincaré dual of the exceptional curve, and if $K \in B(X)$, then

$$SW'_{\widetilde{X}}(K \pm e^*) = SW'_X(K).$$

The significance of Theorem 2.4 lies in its universality; more general versions, with more complicated statements, hold without the assumption of simple type.

Refinements of SO(3)-monopole cobordism formula I

It will be more convenient to have Witten's Formula (2) expressed at the level of the Donaldson polynomial invariants rather than the Donaldson power series which they form.

Let $B'(X)$ be a fundamental domain for the action of $\{\pm 1\}$ on the set of Seiberg-Witten basic classes, $B(X)$.

Refinements of SO(3)-monopole cobordism formula II

Lemma 2.5 (Witten's Formula (2) expressed at the level of the Donaldson polynomial invariants)

(See F and Leness [7, Lemma 4.2] or [6, Lemma 2.4].) Let X be a standard four-manifold. Then X satisfies equation (2) and has Kronheimer-Mrowka simple type if and only if the Donaldson invariants of X satisfy $D^w_X(h^{\delta-2m}x^m) = 0$ for $\delta \not\equiv -w^2 - 3\chi_h \pmod 4$ and for $\delta \equiv -w^2 - 3\chi_h \pmod 4$ satisfy

$$(8)\quad D^w_X(h^{\delta-2m}x^m) = \sum_{\substack{i+2k \\ =\delta-2m}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)}\nu(K) \times \frac{SW'_X(K)(\delta-2m)!}{2^{k+c(X)-3-m}k!i!}\langle K,h\rangle^i Q_X(h)^k,$$

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Refinements of SO(3)-monopole cobordism formula III

Lemma 2.5 (Witten's Formula (2) expressed at the level of the Donaldson polynomial invariants)

where

$$\varepsilon(w,K) := \frac{1}{2}(w^2 + w\cdot K), \tag{9}$$

and

$$\nu(K) = \begin{cases} \frac{1}{2} & \text{if } K = 0, \\ 1 & \text{if } K \neq 0. \end{cases} \tag{10}$$

We recall the

Refinements of SO(3)-monopole cobordism formula IV

Hypothesis 2.6 (Properties of local SO(3)-monopole gluing maps)

(See F and Leneess [6, Hypothesis 3.1].) The local gluing map, constructed in [5], gives a continuous parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathscr{M}}_{\mathfrak{t}}$ for each smooth stratum $\Sigma \subset \mathrm{Sym}^{\ell}(X)$.

The SO(3)-monopole cobordism formula given below provides an expression for a Donaldson invariant in terms of the Seiberg-Witten invariants.

Refinements of SO(3)-monopole cobordism formula V

Theorem 2.7 (SO(3)-monopole cobordism formula)

(See F and Leness [2, Main Theorem] or [6, Theorem 3.2].) Let X be a standard four-manifold of Seiberg-Witten simple type. Assume Hypothesis 2.6. Assume further that $w, \Lambda \in H^2(X;\mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfy

$$w - \Lambda \equiv w_2(X) \pmod 2, \tag{11a}$$

$$I(\Lambda) = \Lambda^2 + c(X) + 4\chi_h(X) > \delta, \tag{11b}$$

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4, \tag{11c}$$

$$\delta - 2m \geq 0. \tag{11d}$$

Then, for any $h \in H_2(X;\mathbb{R})$ and positive generator $x \in H_0(X;\mathbb{Z})$,

Refinements of SO(3)-monopole cobordism formula VI

Theorem 2.7 (SO(3)-monopole cobordism formula)

$$D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B(X)} (-1)^{\frac{1}{2}(w^2-\sigma)+\frac{1}{2}(w^2+(w-\Lambda)\cdot K)} SW_X'(K) \times f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h), \quad (12)$$

where the map,

$$f_{\delta,m}(h) : \mathbb{Z}\times\mathbb{Z}\times H^2(X;\mathbb{Z})\times H^2(X;\mathbb{Z}) \to \mathbb{R}[h],$$

takes values in the ring of polynomials in the variable h with real coefficients, is universal (independent of X) and is given by

Refinements of SO(3)-monopole cobordism formula VII

Theorem 2.7 (SO(3)-monopole cobordism formula)

$$
\begin{aligned}
(13)\quad & f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h) \\
&\quad := \sum_{\substack{i+j+2k \\ =\delta-2m}} a_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m) \\
&\qquad\qquad \times \langle K, h\rangle^i \langle \Lambda, h\rangle^j Q_X(h)^k.
\end{aligned}
$$

For each triple, $i, j, k \in \mathbb{N}$, the coefficients,

$$a_{i,j,k} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{N} \to \mathbb{R},$$

are universal (independent of X) real analytic functions of the variables $\chi_h(X)$, $c_1^2(X)$, $c_1(\mathfrak{s})\cdot\Lambda$, Λ^2, and m.

Refinements of SO(3)-monopole cobordism formula VIII

The left-hand side of the SO(3)-monopole cobordism formula (12) is obtained by computing the intersection number for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the link of the moduli subspace $\bar{M}_\kappa^w$ of anti-self-dual SO(3) connections.

One uses the fiber-bundle structure of the link over $\bar{M}_\kappa^w$ to compute the intersection number and show that this is equal to a multiple of the Donaldson invariant, $D_X^w(h^{\delta-2m}x^m)$.

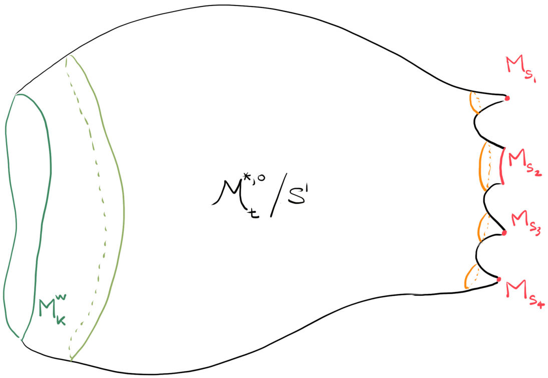
The right-hand side of the SO(3)-monopole cobordism formula (12) is obtained by computing the intersection numbers for geometric representatives on $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ with the links of the moduli subspaces $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ of ideal Seiberg-Witten monopoles appearing in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$.

Refinements of SO(3)-monopole cobordism formula IX

One uses the fiber-bundle structure of the link over each Seiberg-Witten moduli space, $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$, to compute the intersection number and show that this is equal to a multiple of a Seiberg-Witten invariant, $SW'_X(K)$, for each $K \in H^2(X;\mathbb{Z})$ with $c_1(\mathfrak{s}) = K$.

The following figure illustrates the SO(3)-monopole cobordism between codimension-one links in $\bar{\mathscr{M}}_{\mathfrak{t}}/S^1$ of $\bar{M}^w_\kappa$ and $M_{\mathfrak{s}_i} \times \mathrm{Sym}^{\ell}(X)$.

Refinements of SO(3)-monopole cobordism formula X



Refinements of SO(3)-monopole cobordism formula XI

We rewrite the SO(3)-monopole cobordism formula (12) for $D_X^w(h^{\delta-2m}x^m))$ as a sum over $B'(X) \subset B(X)$, a fundamental domain for the action of $\{\pm 1\}$.

To this end, we define (compare [7, Equation (4.4)])

$$
\begin{aligned}
b_{i,j,k}&(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m) \\
&:= (-1)^{c(X)+i} a_{i,j,k}(\chi_h(X), c_1^2(X), -K \cdot \Lambda, \Lambda^2, m) \\
&\qquad + a_{i,j,k}(\chi_h(X), c_1^2(X), K \cdot \Lambda, \Lambda^2, m),
\end{aligned}
$$

where $a_{i,j,k}$ are the coefficients appearing in the expression (13) for

$$
f_{\delta,m}(\chi_h(X), c_1^2(X), K, \Lambda)(h)
$$

in the SO(3)-monopole cobordism formula (12) for $D_X^w(h^{\delta-2m}x^m)$.

Refinements of SO(3)-monopole cobordism formula XII

To simplify the orientation factor in (12), we define

$$\begin{aligned}(14)\quad & \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m) \\ & \quad := (-1)^{\frac{1}{2}(\Lambda^2+\Lambda\cdot K)} b_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).\end{aligned}$$

Observe that

$$\begin{aligned}(15)\quad & \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), -K\cdot\Lambda, \Lambda^2, m) \\ & \quad = (-1)^{c(X)+i+\Lambda\cdot K} \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).\end{aligned}$$

We now rewrite the SO(3)-monopole cobordism formula (12) as a sum over $B'(X)$.

Refinements of SO(3)-monopole cobordism formula XIII

Lemma 2.8 (SO(3)-monopole cobordism formula on fundamental domain)

(See F and Leness [6, Lemma 3.4].) Assume the hypotheses of Theorem 2.7 (the SO(3)-monopole cobordism formula). Denote the coefficients in (15) *more concisely by*

$$\tilde{b}_{i,j,k}(K\cdot\Lambda) := \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X), K\cdot\Lambda, \Lambda^2, m).$$

Then, for $\varepsilon(w,K) = \frac{1}{2}(w^2 + w\cdot K)$ *as in* (9) *and* $\nu(K)$ *as in* (10),

$$\begin{aligned} D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B'(X)} \sum_{\substack{i+j+2k\\=\delta-2m}} \nu(K)(-1)^{\varepsilon(w,K)} SW_X'(K) \\ \times \tilde{b}_{i,j,k}(K\cdot\Lambda)\langle K,h\rangle^i\langle\Lambda,h\rangle^j Q_X(h)^k. \end{aligned} \tag{16}$$

Refinements of SO(3)-monopole cobordism formula XIV

The proof of Lemma 2.8 is relatively simple and does not involve anything deep.

The following lemma allows us to ignore the coefficients $\tilde{b}_{0,j,k}$ in the formula (16) for $D_X^w(h^{\delta-2m}x^m)$ for the purpose of proving Theorem 1.2 and Corollary 1.4.

Refinements of SO(3)-monopole cobordism formula XV

Lemma 2.9 (Eliminating the coefficients $\tilde{b}_{0,j,k}$ in the formula (16) for $D_X^w(h^{\delta-2m}x^m)$)

(See F and Leness [6, Lemma 3.5].) Continue the notation and hypotheses of Lemma 2.8. Then,

$$(17)\quad D_X^w(h^{\delta-2m}x^m) = \sum_{K\in B'(X)} \sum_{\substack{i+j+2k\\=\delta-2m}} (-1)^{\varepsilon(w,K)} SW_X'(K) \frac{2(i+1)}{(\delta-2m+1)} \times \tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X)-1, K\cdot\Lambda, \Lambda^2, m) \times \langle K,h\rangle^i \langle\Lambda,h\rangle^j Q_X(h)^k.$$

Refinements of SO(3)-monopole cobordism formula XVI

Idea of Proof of Lemma 2.9. The argument is a little more involved than that of the proof of Lemma 2.8, relying on more than elementary algebra.

The key ingredients include the *blow-up formulae* for Donaldson and Seiberg-Witten invariants.

Let $\widetilde{X} \to X$ be the blow-up of X at one point, let $e \in H_2(\widetilde{X};\mathbb{Z})$ be the fundamental class of the exceptional curve, and let $e^* \in H^2(\widetilde{X};\mathbb{Z})$ be the Poincaré dual of e.

Using the direct sum decomposition of the homology and cohomology of $\widetilde{X}$, we will consider both the homology and cohomology of X as subspaces of those of $\widetilde{X}$.

Refinements of SO(3)-monopole cobordism formula XVII

Denote $\tilde{w} := w + e^*$. The blow-up formula for Donaldson invariants (see Kotschick [14] or Leness [16]) gives

$$(18) \qquad D_X^w(h^{\delta-2m}x^m) = D_{\tilde{X}}^{\tilde{w}}(h^{\delta-2m}ex^m).$$

The blow-up formula for Seiberg-Witten invariants (see Frøyshov [12, Theorem 14.1.1]) gives,

$$(19) \quad B'(\widetilde{X}) = \{K_\varphi = K + (-1)^\varphi e^* : K \in B'(X),\ \varphi \in \mathbb{Z}/2\mathbb{Z}\}.$$

and if $K \in B(X)$, then

$$SW'_{\widetilde{X}}(K \pm e^*) = SW'_X(K).$$

Combining these blow-up formulae with Lemma 2.8 eventually yields the desired expression (17) for $D_X^w(h^{\delta-2m}x^m)$.

Constraining the coefficients

Our next goal is to show that the coefficients $\tilde{b}_{i,j,k}$ appearing in (16) which are not determined by [7, Proposition 4.8] satisfy a difference equation in the parameter $K \cdot \Lambda$ and thus can be written as a *polynomial* in $K \cdot \Lambda$.

We recall from [7] that [7, Proposition 4.8] allowed us to determined the unknown coefficients $\tilde{b}_{i,j,k}$ in the SO(3)-monopole cobordism formula (16) for $D_X^w(h^{\delta-2m}x^m)$ with

$$i \geq c(X) - 3 > 0$$

but not those coefficients $\tilde{b}_{i,j,k}$ with

$$0 \leq i < c(X) - 3,$$

when $c(X) - 3 > 0$.

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Algebraic preliminaries and difference equations

Algebraic preliminaries and difference equations I

To determine the unknown coefficients $\tilde{b}_{i,j,k}$ appearing in the SO(3)-monopole cobordism formula (16) for $D_X^w(h^{\delta-2m}x^m)$, we compare

- Witten's formula (8) for $D_X^w(h^{\delta-2m}x^m)$, and
- SO(3)-monopole cobordism formula (16) for $D_X^w(h^{\delta-2m}x^m)$,

on manifolds where Witten's Conjecture 1.1 is known to hold.

To help us determine the coefficients $\tilde{b}_{i,j,k}$, we appeal to the following generalization of Friedman and Morgan [11, Lemma VI.2.4].

Algebraic preliminaries and difference equations II

Lemma 3.1 (Algebraic independence)

(See F and Leness [7, Lemma 4.1] or [6, Lemma 4.1].) Let V be a finite-dimensional real vector space. Let $T_1, \dots, T_n$ be linearly independent elements of the dual space V^*. Let Q be a quadratic form on V which is non-zero on $\cap_{i=1}^n \operatorname{Ker} T_i$. Then $T_1, \dots, T_n, Q$ are algebraically independent in the sense that if $F(z_0, \dots, z_n) \in \mathbb{R}[z_0, \dots, z_n]$ and $F(Q, T_1, \dots, T_n) : V \to \mathbb{R}$ is the zero map, then $F(z_0, \dots, z_n)$ is the zero element of $\mathbb{R}[z_0, \dots, z_n]$.

We review some notation and results for difference operators (see F and Leness [7, Section 4.3] and [6, Section 4.2] and Elaydi [1] for related results on difference operators and difference equations).

Algebraic preliminaries and difference equations III

For $f : \mathbb{Z} \to \mathbb{R}$ in the context of difference equations, it is customary to denote the difference and shift operators by [1, Section 2.1]

$$\nabla f(x) := f(x+1) - f(x), \quad Ef(x) := f(x+1), \quad \forall x \in \mathbb{Z}.$$

(Or often $\Delta f(x) := f(x+1) - f(x)$.) For $p, q \in \mathbb{Z}$, by analogy we define

$$(\nabla_p^q f)(x) := f(x) + (-1)^q f(x+p), \quad \forall x \in \mathbb{Z}.$$

For $a \in \mathbb{Z}/2\mathbb{Z}$ and $p \in \mathbb{Z}$, define $pa, ap \in \mathbb{Z}$ by

$$(20) \quad pa = ap = -\frac{1}{2}\left(-1 + (-1)^a\right)p = \begin{cases} 0 & \text{if } a \equiv 0 \pmod 2, \\ p & \text{if } a \equiv 1 \pmod 2. \end{cases}$$

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Algebraic preliminaries and difference equations IV

We recall the

Lemma 3.2

(See F and Leness [7, Lemma 4.6] or [6, Lemma 4.2].) For all $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n) \in \mathbb{Z}^n$, there holds

$$\sum_{\varphi \in (\mathbb{Z}/2\mathbb{Z})^n} (-1)^{\sum_{u=1}^n q_u \pi_u(\varphi)} f\left(x + \sum_{u=1}^n p_u \pi_u(\varphi)\right) = (\nabla_{p_1}^{q_1} \nabla_{p_2}^{q_2} \ldots \nabla_{p_n}^{q_n} f)(x),$$

where $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$ is projection onto the u-th factor and, for a constant function, C, there holds

(21)

$$(\nabla_{p_n}^{q_n} \nabla_{p_{n-1}}^{q_{n-1}} \ldots \nabla_{p_1}^{q_1} C) = \begin{cases} 0, & \text{if } \exists u \text{ with } 1 \leq u \leq n \text{ and } q_u \equiv 1 \pmod 2, \\ 2^n C, & \text{if } q_u \equiv 0 \pmod 2 \ \forall\, u \text{ with } 1 \leq u \leq n. \end{cases}$$

Algebraic preliminaries and difference equations V

We will also use the following similar result (compare Elaydi [1, Lemma 2.22]).

Lemma 3.3

(See F and Leness [6, Lemma 4.3].) For $f : \mathbb{Z} \to \mathbb{Z}$ and $\lambda \in \mathbb{Z}$, there holds

$$\left(\left(\nabla^1_\lambda\right)^n f\right)(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + i\lambda).$$

We add the following

Algebraic preliminaries and difference equations VI

Lemma 3.4

(See F and Leness [6, Lemma 4.4].) Let $\lambda \in \mathbb{Z}$ and $p : \mathbb{Z} \to \mathbb{R}$ be a function.

1. *If $\nabla^1_\lambda p(x)$ is a polynomial of degree n in x, then $p(\lambda x)$ is a polynomial of degree $n+1$;*
2. *If $\nabla^1_\lambda p(x) = 0$, then $p(\lambda x)$ is constant.*

Algebraic preliminaries and difference equations VII

Corollary 3.5

(See Corollary [6, Corollary 4.5].) For $\lambda \neq 0$, let $c : \mathbb{Z} \to \mathbb{R}$ be a function satisfying,

$$\underbrace{(\nabla^1_\lambda \nabla^1_\lambda \cdots \nabla^1_\lambda c)}_{n \text{ copies}}(\lambda x) = 0,$$

for all $x \in \mathbb{Z}$. Then $c_\lambda(x) = c(\lambda x)$ is a polynomial in x of degree $n-1$.

The proofs of Lemmas 3.2, 3.3, and 3.4, and Corollary 3.5 are relatively straightforward and proceed by analogy with standard methods used for difference equations (compare Elaydi [1]).

The example manifolds

The example manifolds I

We shall use the manifolds constructed by Fintushel, Park and Stern in [8] to give a family of standard four-manifolds, X_q, for $q = 2, 3, \dots,$ obeying the following conditions (see F and Leness [7, Section 4.2] and [6, Section 4.3]):

1. X_q satisfies Witten's Conjecture 1.1;
2. For $q = 2, 3, \dots,$ one has $\chi_h(X_q) = q$ and $c(X_q) = 3$;
3. $B'(X_q) = \{K\}$ with $K \neq 0$;
4. For each q, there are classes $f_1, f_2 \in H^2(X_q; \mathbb{Z})$ satisfying

(22a) $f_1 \cdot f_2 = 1, \quad f_i^2 = 0,$ and $f_i \cdot K = 0$ for $i = 1, 2,$

(22b) $\{f_1, f_2, K\}$ linearly independent subset of $H^2(X_q; \mathbb{R}),$

(22c) Restriction of Q_{X_q} to $\operatorname{Ker} f_1 \cap \operatorname{Ker} f_2 \cap \operatorname{Ker} K$ is non-zero.

The example manifolds II

Let $X_q(n)$ be the blow-up of X_q at n points,

$$X_q(n) := X_q \# \underbrace{\overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2}_{n \text{ copies}}. \tag{23}$$

Then $X_q(n)$ is a standard four-manifold of Seiberg-Witten simple type and satisfies Witten's Conjecture 1.1 (see F and Leness [7, Theorem 2.7] or [6, Theorem 2.3]), with

$$\chi_h(X_q(n)) = q, \quad c_1^2(X_q(n)) = q - n - 3, \quad \text{and} \quad c(X_q(n)) = n + 3. \tag{24}$$

We will consider both the homology and cohomology of X_q as subspaces of those of $X_q(n)$.

The example manifolds III

Let $e_u^* \in H^2(X_q(n);\mathbb{Z})$ be the Poincaré dual of the u-th exceptional class.

Let $\pi_u : (\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$ be projection onto the u-th factor.

For $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$, we define

$$(25) \qquad K_\varphi := K + \sum_{u=1}^{n} (-1)^{\pi_u(\varphi)} e_u^* \quad \text{and} \quad K_0 := K + \sum_{u=1}^{n} e_u^*.$$

By the blow-up formula for Seiberg-Witten invariants (see Frøyshov [12, Theorem 14.1.1])

$$(26) \qquad B'(X_q(n)) = \{K_\varphi : \varphi \in (\mathbb{Z}/2\mathbb{Z})^n\},$$

The example manifolds IV

and, for all $\varphi \in (\mathbb{Z}/2\mathbb{Z})^n$,

(27) $$SW'_{X_q(n)}(K_\varphi) = SW'_{X_q}(K).$$

Because $X_q(n)$ has Seiberg-Witten simple type, we have

(28) $$K_\varphi^2 = c_1^2(X_q(n)) \quad \text{for all } \varphi \in (\mathbb{Z}/2\mathbb{Z})^n.$$

In addition, because $K \neq 0$, we see that

(29) $$0 \notin B'(X_q(n)).$$

Because the manifolds $X_q(n)$ satisfy Witten's Conjecture 1.1, then

- Lemma 3.1 (the algebraic determination lemma),
- Witten's formula (8) for $D_X^w(h^{\delta-2m}x^m)$, and

The example manifolds V

- SO(3)-monopole cobordism formula (16) for $D_X^w(h^{\delta-2m}x^m)$,

when applied to the manifolds $X_q(n)$, will show that the coefficients $\tilde{b}_{i,j,k}$ satisfy certain *difference equations*.

Those difference equations will allow us to prove Theorem 1.2 (that Superconformal Simple Type $\implies$ Witten's Formula).

For $n \geq 2$, the set $B'(X_q(n))$ is not linearly independent in $H^2(X_q(n);\mathbb{R})$.

To apply the algebraic Lemma 3.1 (and determine unknown coefficients), we need to replace $B'(X_q(n))$ with a linearly independent set, namely $\{K \pm e_1^*, e_2^*, \ldots, e_n^*\}$.

The example manifolds VI

To this end, we give the following formula for the Donaldson invariants of $X_q(n)$ computed by the SO(3)-monopole cobordism formula.

The example manifolds VII

Lemma 3.6 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

(See F and Leness [6, Lemma 4.6].) For $n, q \in \mathbb{Z}$ with $n \geq 1$ and $q \geq 2$, let $X_q(n)$ be the manifold defined in (23). For $\Lambda, w \in H^2(X_q; \mathbb{Z})$ and $\delta, m \in \mathbb{N}$ satisfying $\Lambda - w \equiv w_2(X_q) \pmod 2$ and $\delta - 2m \geq 0$, define $\tilde{w}, \tilde{\Lambda} \in H^2(X_q(n); \mathbb{Z})$ by

$$\tilde{w} := w + \sum_{u=1}^{n} w_u e_u^* \quad \text{and} \quad \tilde{\Lambda} := \Lambda + \sum_{u=1}^{n} \lambda_u e_u^*, \tag{30}$$

where $w_u, \lambda_u \in \mathbb{Z}$ and $w_u + \lambda_u \equiv 1 \pmod 2$ for $u = 1, \ldots, n$. We assume that

$$\Lambda^2 > \delta - (n+3) - 4q + \sum_{u=1}^{n} \lambda_u^2, \tag{31a}$$

$$\delta \equiv -w^2 + \sum_{u=1}^{n} w_u^2 - 3q \pmod 4. \tag{31b}$$

The example manifolds VIII

Lemma 3.6 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

Denote $x := \tilde{K}_\varphi \cdot \tilde{\Lambda}$ *and, for* $i, j, k \in \mathbb{N}$ *satisfying* $i + j + 2k + 2m = \delta$, *write*

$$\tilde{b}_{i,j,k}(x) = \tilde{b}_{i,j,k}(\chi_h(X_q(n)), c_1^2(X_q(n)), x, \tilde{\Lambda}^2, m).$$

Then, for $x_0 = K_0 \cdot \tilde{\Lambda}$ *where* K_0 *is defined in* (25),

The example manifolds IX

Lemma 3.6 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

$$
\begin{aligned}
&\sum_{\substack{i_1+\cdots+i_n+2k\\=\delta-2m}} \frac{(\delta-2m)!}{2^{k+n-m}k!i_1!\cdots i_n!} p^{\tilde{w}}(i_2,\ldots,i_n)\left(\prod_{u=2}^{n}\langle e_u^*,h\rangle^{i_u}\right) Q_{X_q(n)}(h)^k \\
&\quad\times\left(\langle K+e_1^*,h\rangle^{i_1}+(-1)^{w_1}\langle K-e_1^*,h\rangle^{i_1}\right) \\
&= \sum_{\substack{i_1+\cdots+i_n+j+2k\\=\delta-2m}} \binom{i_1+\cdots+i_n}{i_1,\ldots,i_n}\langle\tilde{\Lambda},h\rangle^j\left(\prod_{u=2}^{n}\langle e_u^*,h\rangle^{i_u}\right) Q_{X_q(n)}(h)^k \\
&\quad\times\left(\nabla_{2\lambda_2}^{i_2+w_2}\cdots\nabla_{2\lambda_n}^{i_n+w_n}\tilde{b}_{i,j,k}(x_0)\langle K+e_1^*,h\rangle^{i_1}\right. \\
&\qquad\left.+(-1)^{w_1}\nabla_{2\lambda_2}^{i_2+w_2}\cdots\nabla_{2\lambda_n}^{i_n+w_n}\tilde{b}_{i,j,k}(x_0+2\lambda_1)\langle K-e_1^*,h\rangle^{i_1}\right),
\end{aligned}
\tag{32}
$$

The example manifolds X

Lemma 3.6 (Donaldson invariants of $X_q(n)$ via SO(3)-monopole cobordism)

are both equal to the following multiple of the Donaldson invariant,

$$\frac{(-1)^{\varepsilon(\tilde{w},\varphi_0)}}{SW'_{X_q}(K)} D^{\tilde{w}}_{X_q(n)}(h^{\delta-2m}x^m),$$

where $\tilde{\Lambda}$ *is as defined in* (30) *and*

$$(33) \qquad p^{\tilde{w}}(i_2,\dots,i_n) = \begin{cases} 0 & \text{if } \exists u \text{ with } 2 \le u \le n \text{ and } w_u + i_u \equiv 1 \pmod 2, \\ 2^{n-1} & \text{if } w_u + i_u \equiv 0 \pmod 2 \ \forall u \text{ with } 2 \le u \le n. \end{cases}$$

We recall a result giving the coefficients $\tilde{b}_{i,j,k}$ for $i \geq c(X) - 3$.

The example manifolds XI

Proposition 3.7 (Computation of coefficients $\tilde{b}_{i,j,k}$ for $i \geq c(X) - 3$)

(See F and Leness [7, Proposition 4.8] or [6, Proposition 4.7].) Let $n > 0$ and $q \geq 2$ be integers. If x, y are integers and i, j, k, m are non-negative integers satisfying, for $A := i + j + 2k + 2m$,

$$i \geq n, \tag{34a}$$

$$y > A - 4q - 3 - n, \tag{34b}$$

$$A \geq 2m, \tag{34c}$$

$$x \equiv y \equiv 0 \pmod 2, \tag{34d}$$

then the coefficients $\tilde{b}_{i,j,k}(\chi_h, c_1^2, \Lambda \cdot K, \Lambda^2, m)$ defined in (14) are given by

$$\tilde{b}_{i,j,k}(q, q-3-n, x, y, m) = \begin{cases} \dfrac{(A-2m)!}{k!i!} 2^{m-k-n} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

The example manifolds XII

Because of the condition (34a), Proposition 3.7 only determines the coefficients $\tilde{b}_{i,j,k}$ with $i \geq c(X) - 3$.

Lemma 2.9 allows us to *ignore* the coefficients $\tilde{b}_{i,j,k}$ with $i = 0$, that is, $\tilde{b}_{0,j,k}$ when proving Theorem 1.2 and Corollary 1.4.

We next derive a difference equation satisfied by the coefficients $\tilde{b}_{i,j,k}$ with $1 \leq i < c(X) - 3$.

The example manifolds XIII

Proposition 3.8 (Difference equation for $\tilde{b}_{i,j,k}$ with $1 \leq i < c(X) - 3$)

(See F and Leness [6, Proposition 4.7].) Let $n > 1$ and $q \geq 2$ be integers. If x, y are integers and p, j, k, m are non-negative integers satisfying, for $A := p + j + 2k + 2m$,

(35a) $$1 \leq p \leq n - 1,$$

(35b) $$y > A - 4q - n - 3,$$

(35c) $$y \equiv A - (n + 3) \pmod 4,$$

(35d) $$x - y \equiv 0 \pmod 2,$$

and we abbreviate $\tilde{b}_{p,j,k}(x) = \tilde{b}_{p,j,k}(q, q - n - 3, x, y, m)$, then

(36) $$\left(\nabla^1_4\right)^{n-p} \tilde{b}_{p,j,k}(x) = 0.$$

The example manifolds XIV

Proposition 3.8 and the difference equation given by Corollary 3.5 allow us to write the coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$.

We will combine this fact with Lemma 4.1 (forthcoming) to show that, for manifolds of superconformal simple type, the coefficients $\tilde{b}_{i,j,k}$ with $i \leq c(X) - 4$ do not contribute to the SO(3)-monopole cobordism expression (16) for the Donaldson invariant $D_X^w(h^{\delta-2m}x^m)$.

The example manifolds XV

Corollary 3.9 (Coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$)

(See F and Leness [6, Corollary 4.11].) Continue the assumptions of Proposition 3.8. In addition, assume

1. *There is a class $K_0 \in B(X)$ such that $\Lambda \cdot K_0 = 0$;*
2. *For all $K \in B(X)$, we have $\Lambda \cdot K \equiv 0 \pmod 4$.*

Then for $1 \leq i \leq n-1$, the function $\tilde{b}_{i,j,k}$ is a polynomial of degree $n-1-i$ in $\Lambda \cdot K$ and thus

$$\tilde{b}_{i,j,k}(q, q-n-3, K \cdot \Lambda, \Lambda^2, m) = \sum_{u=0}^{n-1-i} \tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) \langle K, h_\Lambda \rangle^u, \tag{37}$$

where $h_\Lambda = \mathrm{PD}[\Lambda]$ is the Poincaré dual of Λ and if $u \equiv n+i \pmod 2$, then

$$\tilde{b}_{u,i,j,k}(q, q-n-3, \Lambda^2, m) = 0. \tag{38}$$

Proof of Witten's Conjecture

Proof of Witten's Conjecture I

We begin by establishing the following algebraic consequence of superconformal simple type which will allow us to show that Witten's Formula (2) holds even without determining the coefficients $\tilde{b}_{i,j,k}$ with $i < c(X) - 3$ in the SO(3)-monopole cobordism formula (16) for $D_X^w(h^{\delta-2m}x^m)$.

Proof of Witten's Conjecture II

Lemma 4.1 (An algebraic consequence of superconformal simple type)

(See F and Leness [6, Lemma 5.1].) Let X be a standard four-manifold of superconformal simple type. Assume $0 \notin B(X)$. If $w \in H^2(X, \mathbb{Z})$ is characteristic and $j, u \in \mathbb{N}$ satisfy $j + u < c(X) - 3$ and $j + u \equiv c(X) \pmod 2$, then

$$\sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h_1 \rangle^j \langle K, h_2 \rangle^u = 0, \tag{39}$$

for any $h_1, h_2 \in H_2(X; \mathbb{R})$.

The following lemma allows us to apply Corollary 3.9.

Proof of Witten's Conjecture III

Lemma 4.2 (Existence of positive Λ classes orthogonal to basic classes)

(See F and Leness [6, Lemma 5.2].) Let X be a standard four-manifold with odd intersection form. Then for any $K \in B(X)$, there is a class $\Lambda \in H^2(X;\mathbb{Z})$ with $\Lambda^2 > 0$ and $\Lambda \cdot K = 0$.

Corollary 3.9 (coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$) and Lemma 4.1 provide the basis of the proof of our main result.

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Proof of Witten's Conjecture IV

Outline of Remainder of Proof of Theorem 1.2 (Superconformal Simple Type $\implies$ Witten's Conjecture).

We have discussed all of the key ingredients, so it remains to assemble them and hence deduce Witten's Formula (2), assuming the superconformal simple type property (whose proof we outlined earlier in these lectures).

Replace X by its blow-up $X\#\overline{\mathbb{CP}}^2$ when convenient

By Theorem 2.2 (Witten's Conjecture 1.1 preserved under blow-up), we may blow up X without loss of generality.

According to Lemma 2.3, the superconformal simple type condition is preserved under blow-up.

Proof of Witten's Conjecture V

If $\widetilde{X}$ is the blow-up of X, then the characterization of $B(\widetilde{X})$ in (7) implies that $0 \notin B(\widetilde{X})$.

Thus, by replacing X with its blow-up if necessary, we may assume without loss of generality that $c_1^2(X) \neq 0$, Q_X is odd, $c(X) \geq 5$, $0 \notin B(X)$ and $\nu(K) = 1$, where $\nu(K)$ is defined in (10) for each $K \in B(X)$.

By Proposition 2.1, it suffices to prove that equation (8) in Lemma 2.5 (Witten's Formula (2) for a Donaldson invariant) holds when $w \in H^2(X;\mathbb{Z})$ is characteristic.

Proof of Witten's Conjecture VI

Preliminary reductions and simplifications, possibly after replacing X by its blow-up

Because w is characteristic (by the preceding reduction), we have

$$\begin{aligned} w^2 &\equiv \sigma(X) \pmod 8 \quad \text{(by [13, Lemma 1.2.20])} \\ &= c_1^2(X) - 8\chi_h(X) \quad \text{(by (1))} \\ &\equiv c_1^2(X) \pmod 8. \end{aligned}$$

Thus,

$$D_X^w(h^{\delta-2m}x^m) = 0$$

unless

$$\delta \equiv -w^2 - 3\chi_h(X) \equiv \chi_h(X) - c_1^2(X) - 4\chi_h(X) \equiv c(X) \pmod 4.$$

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Proof of Witten's Conjecture VII

Hence, we only need find Donaldson invariants $D_X^w(h^{\delta-2m}x^m)$ with

$$(40) \qquad \delta \geq 2m \quad \text{and} \quad \delta \equiv -w^2 - 3\chi_h(X) \equiv c(X) \pmod{4}.$$

To apply Lemma 2.9 (a refined version of the SO(3)-monopole cobordism formula allowing us to ignore coefficients $\tilde{b}_{0,j,k}$) to compute $D_X^w(h^{\delta-2m}x^m)$, we abbreviate

$$(41) \qquad \tilde{b}_{i,j,k}(\Lambda \cdot K) = \tilde{b}_{i,j,k}(\chi_h(X), c_1^2(X) - 1, \Lambda \cdot K, \Lambda^2, m),$$

and verify we can find $\Lambda \in H^2(X;\mathbb{Z})$ obeying the conditions of

- Theorem 2.7 (SO(3)-*monopole cobordism formula*) and hence those of Lemma 2.9, and
- Corollary 3.9 (*coefficients* $\tilde{b}_{i,j,k}$ *as polynomials in* $\Lambda \cdot K$).

Proof of Witten's Conjecture VIII

Verifying hypotheses of SO(3)-monopole cobordism formula

By Lemma 4.2 and our observation that by replacing X with its blow-up if necessary we can assume that Q_X is odd and there are classes $K_0 \in B(X)$ and $\Lambda_0 \in H^2(X;\mathbb{Z})$ with

$$\Lambda_0^2 > 0 \quad \text{and} \quad \Lambda_0 \cdot K_0 = 0.$$

Because any $K \in B(X)$ can be written as $K = K_0 + 2L_K$ for $L_K \in H^2(X;\mathbb{Z})$, if $\Lambda = 2b\Lambda_0$ where $b \in \mathbb{N}$, then

$$(42) \quad K_0 \cdot \Lambda = 0 \quad \text{and} \quad K \cdot \Lambda \equiv 0 \pmod 4 \text{ for all } K \in B(X),$$

so Λ satisfies two of the assumptions of Corollary 3.9 (*coefficients* $\tilde{b}_{i,j,k}$ *as polynomials in* $\Lambda \cdot K$).

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Proof of Witten's Conjecture IX

If $w \in H^2(X;\mathbb{Z})$ is characteristic and $\Lambda = 2b\Lambda_0$, where $b \in \mathbb{N}$ and $\Lambda_0^2 > 0$, then $\Lambda - w \equiv w_2(X) \pmod 2$ and so condition (11a) holds in Theorem 2.7 (SO(3)-*monopole cobordism formula*).

Given δ, by choosing b sufficiently large, we can ensure

$$\Lambda^2 + c(X) + 4\chi_h(X) > \delta, \tag{43}$$

so condition (11b) holds in Theorem 2.7 (SO(3)-*monopole cobordism formula*).

Conditions (11c) and (11d) in Theorem 2.7, that

$$\delta \equiv -w^2 - 3\chi_h(X) \pmod 4 \quad \text{and} \quad \delta - 2m \geq 0,$$

respectively, follow from (40).

Proof of Witten's Conjecture X

Thus, Lemma 2.9 (*the refined version of the* SO(3)*-monopole cobordism formula allowing us to ignore coefficients* $\tilde{b}_{0,j,k}$) gives

(44)
$$D_X^w(h^{\delta-2m}x^m) = \sum_{\substack{i+j+2k\\=\delta-2m}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW_X'(K)}{\delta-2m+1} \tilde{b}_{i+1,j,k}(K\cdot\Lambda) \times \langle K,h\rangle^i \langle \Lambda,h\rangle^j Q_X(h)^k.$$

Computation of the coefficients $\tilde{b}_{i+1,j,k}$ in (44)

We now verify that we can apply

- Proposition 3.7 (*coefficients* $\tilde{b}_{i,j,k}$ *for* $i \geq c(X)-3$),
- Proposition 3.8 (*difference equation for* $\tilde{b}_{i,j,k}$ *with* $1 \leq i < c(X)-3$), and

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Proof of Witten's Conjecture XI

- Corollary 3.9 (*coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$*),

to compute the coefficients $\tilde{b}_{i+1,j,k}$ in (44).

The indices i, j, k, m appearing in (44) satisfy

(45) $$i + 1 + j + 2k + 2m = \delta + 1.$$

To match the notation of Propositions 3.7 and 3.8, we will write the first two arguments of the coefficients in (41) as

(46) $$q := \chi_h(X),$$

and $c_1^2(X) - 1 = q - 3 - n$, where

(47) $$n := \chi_h(X) - c_1^2(X) - 2 = c(X) - 2.$$

Proof of Witten's Conjecture XII

The definitions (46) and (47), the property that $b^+(X) \geq 3$ for standard manifolds, and our earlier observation that we can assume $c(X) \geq 5$ imply that

$$q \geq 2 \quad \text{and} \quad n \geq 2, \tag{48}$$

as required in Propositions 3.7 and 3.8.

We now verify the hypotheses of Proposition 3.7 for the coefficients $\tilde{b}_{i+1,j,k}$ in (41) with $i \geq c(X) - 3$.

The condition (34a) in Proposition 3.7 holds because by (47),

$$i + 1 \geq c(X) - 2 = n$$

Proof of Witten's Conjecture XIII

In the notation of Proposition 3.7 for $\tilde{b}_{i+1,j,k}$, we have

$$A = i + 1 + j + k + 2m$$

and so $A = \delta + 1$ by (45).

The property (43) of Λ^2 and (46) imply that

$$(49) \quad \Lambda^2 > \delta - c(X) - 4q = \delta - n - 2 - 4q = A - n - 3 - 4q,$$

so condition (34b) in Proposition 3.7 holds.

The condition $A \geq 2m$ for (34c) in Proposition 3.7 holds by (40).

Our choice of $\Lambda = 2\Lambda_0$ implies that

$$\Lambda^2 \equiv \Lambda \cdot K \equiv 0 \pmod 2, \quad \forall K \in B(X),$$

Proof of Witten's Conjecture XIV

and thus condition (34d) holds as well, noting that $x = \Lambda^2$ and $y = \Lambda \cdot K$.

Hence, Proposition 3.7 (*coefficients* $\tilde{b}_{i,j,k}$ *for* $i \geq c(X) - 3$) and the equality $A = \delta + 1$ imply that, for all $i \geq c(X) - 3$, we have

$$\tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) = \begin{cases} \dfrac{(\delta + 1 - 2m)!}{k!(i+1)!} 2^{m-k-c(X)+2} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases} \tag{50}$$

Proof of Witten's Conjecture XV

Verifying the hypotheses of Proposition 3.8 and Corollary 3.9

We now verify the hypotheses of

- Proposition 3.8 ($\tilde{b}_{i,j,k}$ *difference equation*, $1 \le i < c(X) - 3$),
- Corollary 3.9 ($\tilde{b}_{i,j,k}$ *as polynomials in* $\Lambda \cdot K$),

Observe that $i + 1 \le c(X) - 3 = n - 1$ by (47), so condition (35a) in Proposition 3.8 holds.

The inequality in (49) implies that condition (35b) in Proposition 3.8 holds.

Because $A = \delta + 1 \equiv c(X) + 1 \equiv n + 3 \pmod 4$ by (40) and (47), the fact that $\Lambda^2 = (2\Lambda_0)^2 \equiv 0 \pmod 4$ implies

$$\Lambda^2 \equiv 0 \equiv A - (n + 3) \pmod 4,$$

Proof of Witten's Conjecture XVI

and thus condition (35c) in Proposition 3.8 holds.

We already showed that condition (34d) in Proposition 3.8 holds and that implies condition (35d) in Proposition 3.8 holds too.

Therefore, Proposition 3.8 (*difference equation*) applies to compute the coefficients $\tilde{b}_{i+1,j,k}$ with $i \leq c(X) - 3$.

The hypotheses of Corollary 3.9 are those of Proposition 3.8 and the conditions we have previously verified in (42).

Proof of Witten's Conjecture XVII

Thus, Corollary 3.9 (*coefficients $\tilde{b}_{i,j,k}$ as polynomials in $\Lambda \cdot K$*) implies that the coefficients $\tilde{b}_{i+1,j,k}$ with $i \leq c(X) - 3$ can be written as

$$\begin{aligned} &\tilde{b}_{i+1,j,k}(\chi_h(X), c_1^2(X) - 1, K \cdot \Lambda, \Lambda^2, m) \\ &= \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k}(q, q-n-3, \Lambda^2, m)\langle K, h_\Lambda \rangle^u, \end{aligned} \tag{51}$$

where $h_\Lambda = \mathrm{PD}[\Lambda] \in H_2(X; \mathbb{R})$.

Proof of Witten's Conjecture XVIII

Computation of the Donaldson invariant $D_X^w(h^{\delta-2m}x^m)$ by simplifying the right-hand-side of Equation (44)

We now abbreviate,

$$\tilde{b}_{u,i+1,j,k} := \tilde{b}_{u,i+1,j,k}(q, q-n-3, \Lambda^2, m),$$

and split the sum on the right-hand-side of (44) into two parts (one for $i \leq c(X) - 4$ and one for $i \geq c(X) - 3$):

Proof of Witten's Conjecture XIX

$$
\begin{aligned}
D_X^w(h^{\delta-2m}x^m) = &\sum_{\substack{i+j+2k\\=\delta-2m,\\ i\le c(X)-4}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW_X'(K)}{\delta-2m+1} \\
&\times \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k} \langle K,h\rangle^i \langle K,h_\Lambda\rangle^u \langle \Lambda,h\rangle^j Q_X(h)^k \\
&+ \sum_{\substack{i+j+2k\\=\delta-2m,\\ i\ge c(X)-3}} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)} \frac{2(i+1)SW_X'(K)}{\delta-2m+1} \\
&\times \tilde{b}_{i+1,j,k}(K\cdot\Lambda)\langle K,h\rangle^i \langle K,h\rangle^j Q_X(h)^k.
\end{aligned}
\tag{52}
$$

Proof of Witten's Conjecture XX

Verify that sum terms in Equation (52) with $i \leq c(X) - 4$ is zero

Because the coefficients $\tilde{b}_{u,i+1,j,k}$ do not depend on $\Lambda \cdot K$, we can rewrite the first sum on the right-hand-side of (52) as

(53)
$$\sum_{\substack{i+j+2k \\ =\delta-2m, \\ i\leq c(X)-4}} \frac{2(i+1)SW_X'(K)}{\delta-2m+1}\langle\Lambda,h\rangle^j Q_X(h)^k \times \sum_{u=0}^{c(X)-4-i} \tilde{b}_{u,i+1,j,k} \sum_{K\in B'(X)} (-1)^{\varepsilon(w,K)} SW_X'(K)\langle K,h\rangle^i \langle K,h_\Lambda\rangle^u.$$

By (38) and the equality $n \equiv c(X)$ from (47),

(54)
$$\tilde{b}_{u,i+1,j,k} = 0 \quad \text{if } u \equiv c(X)+i+1 \pmod 2.$$

Proof of Witten's Conjecture XXI

We consider the terms in the sum (53) with $u \equiv n + i \pmod 2$.

For u and i satisfying $0 \leq u + i \leq c(X) - 4$, and $u \equiv n + i \pmod 2$, and $w \in H^2(X; \mathbb{Z})$ characteristic, Lemma 4.1 (*algebraic consequence of superconformal simple type*) implies that

$$\sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW'_X(K) \langle K, h \rangle^i \langle K, h_\Lambda \rangle^u = 0.$$

Because $0 \leq u \leq c(X) - 4 - i$ and thus $0 \leq u + i \leq c(X) - 4$ for all terms in the sum (53), the preceding equality and (54) imply that the sum (53) vanishes.

Hence, the sum of terms in (52) with $i \leq c(X) - 4$ vanishes.

Proof of Witten's Conjecture XXII

Simplifying the sum of terms in (52) with $i \geq c(X) - 3$

By employing the vanishing of the sum of terms in (52) with $i \leq c(X) - 4$ and the formula (50) for the coefficients $\tilde{b}_{i+1,j,k}$, we can rewrite Equation (52) for the Donaldson invariant as

$$
\begin{aligned}
&D_X^w(h^{\delta-2m}x^m) \\
&= \sum_{\substack{i+2k \\ =\delta-2m, \\ i \geq c(X)-3}} \sum_{K \in B'(X)} (-1)^{\varepsilon(w,K)} SW_X'(K) \frac{(\delta - 2m)!}{k!i!2^{k+c(X)-3-m}} \langle K, h \rangle^i Q_X(h)^k.
\end{aligned}
$$

Comparing the preceding expression for $D_X^w(h^{\delta-2m}x^m)$ with Equation (8) in Lemma 2.5 (*Witten's Formula* (2) *expressed at the level of the Donaldson polynomial invariants*) and observing that

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Proof of Witten's Conjecture XXIII

the terms in (8) with $i \leq c(X) - 4$ also vanish by Lemma 4.1 (*algebraic consequence of superconformal simple type*), shows that Witten's Conjecture 1.1 holds.

This completes the proof of Theorem 1.2 (Superconformal Simple Type $\implies$ Witten's Conjecture). □

Thank you for your attention!

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