ON ONE RESULT OF ENRIQUES AND ITS APPLICATION TO INTEGRAL GEOMETRY

Simon Gindikin

Abstract. We discuss a classical result of F.Enriques on subvarieties in projective space of minimal degree and its important role in integral geometry
2000 Mathematics Subject Classification. 14M20, 44A12

My visit to Bologna was a pleasant occasion to recall a surprising result of F.Enriques which I believe must play an important role in mathematics. Some results of A.Goncharov in integral geometry deliver credence to such a possibility. This result of Enriques is in the line with some fundamental classical facts. Analytic geometry explained the fundamental role of lines and conics in geometry: they are two simplest classes of curves from algebraic point of view - curves of 1st and 2nd degree. A naive possibility that curves of 3rd degree will continue this series did not realize in spite of some remarkable examples and results about them starting with Newton's classification. May be lines and conics are representatives of another world of "good" curves?

Lines and conics share an interesting property: they are projective curves of minimal degree. Let us remember that the degree of a curve is the number of its intersection points with generic hyperplanes. We say that a curve in $\mathbb{C}P^n$ has the minimal degree if it has a minimal degree between curves which do not lie in hyperplanes. Apparently, it means that its degree is equal n. So lines and conics are curves of minimal degree corresponding to $n = 1, 2$. It turns out that all curves of minimal degree for each n preserve some important properties of lines and conics. They all are rational and mutually projectively equivalent. As canonical realizations of curves of minimal degree it is possible to take the rational curves of Hesse (1866) H_n which in homogeneous coordinates admit the representation

$$z_0 = \tau_0^n, z_1 = \tau_0^{n-1}\tau_1, ..., z_n = \tau_1^n, \qquad (\tau_0, \tau_1) \in \mathbb{C}P^1$$

Of course this curve admits also a representation as an intersection of quadrics.

F.Enriques [E] found in 1896 that this result on curves admits a complete generalization on varieties of minimal degree of any dimension. Earlier del Pezzo found the similar result for 2-dimensional projective surfaces. The degree of a subvariety X in $\mathbb{C}P^n$ of codimension k is the intersection number with generic k-planes; X has the minimal degree if it does not lie in a hyperplane and has the minimal degree between all such curves, that is its degree is equal to $k+1$. Enriques proved that in each dimension any subvariety of minimal degree is projectively equivalent to one of a finite number of canonical subvarieties which admit a very simple and explicit description.We now turn to these descriptions.

Let us start from the Veronese surface V in $\mathbb{C}P^5$. Its canonical realization is

$$x_0 = \tau_0^2, x_1 = \tau_0\tau_1, x_2 = \tau_0\tau_2, x_3 = \tau_1^2, x_4 = \tau_1\tau_2, x_5 = \tau_2^2,$$

where (τ_0, τ_1, τ_2) are homogeneous parameters on V. Its hyperplane sections are conics.We will call by Veronese surfaces all surfaces which in for appropriate coordinates and parameters admit such representation. We will consider also cones in spaces $\mathbb{C}^{5+r}$ over the Veronese surface.

The next class of subvarieties which participate in the Enriques'result is a generalization of 2-dimensional hyperboloids.We also describe them by a canonical representation in appropriate coordinates. Let $r_1 + \cdots + r_k = n + 1 - k$ and homogeneous coordinates be split on k groups $(x_l^{(j)}), l \leq r_j + 1, j \leq k$. Let us call by the canonical Enriques subvariety $E(r_1, \dots, r_k)$ the subvariety defining parametrically:

$$x_l^{(j)} = \lambda_l \tau_0^{r_j - l} \tau_1^l, \qquad \tau = (\tau_0, \tau_1) \in \mathbb{C}^2 \backslash \{0\}, \quad \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k.$$

Such a way we have j planes in $\mathbb{C}P^n$ of dimensions $r_1, \dots, r_k$ correspondingly and in in each of them we take the corresponding Hesse curve H_{r_j}. Points on these k curves corresponding to the same value of the projective parameter τ generate a $(k-1)$-plane $L(\tau)$. Their union for all τ is just $E(r)$. Let us remark that $E(1,1)$ is 2-dimensional hyperboloid in $\mathbb{C}P^3$ and we have its fibering on linear generators from one family. Subvarieties projectively equivalent to $S(r)$ we will call by Enriques subvarieties. Of course, the geometrical fibering on a rational curve of planes $L(\tau)$ exists for all them. Now we can state the basic result.

Enriques Theorem. *Any subvariety of minimal degree in projective space belongs to one of the next types: projective spaces, quadrics, cones over Veronese surfaces, or Enriques subvarieties.*

In the case of surfaces as we mentioned above this result was obtained by del Pezzo.

To see the connections of Enriques Theorem with integral geometry and other analytic problems let us replace the projective $(k-1)$-planes $L(\tau)$ by corresponding k-subspaces $\tilde{L}(\tau)$ and let $M(\tau)$ be dual $(n+1-k)$-subspaces. In such a way we consider the rational curve of dual subspaces. This family appears in another important construction.On Grothendick Theorem each vector bundle on the projective line $\mathbb{C}P^1$ is equivalent to a bundle $\mathcal{O}(r_1) \oplus \cdots \oplus \mathcal{O}(r_k)$. If Γ is the space of sections of this bundle , $\Gamma(\tau)$ is the subspace of sections equal zero at the point τ(τ are homogeneous coordinates on $\mathbb{C}P^1$), then the rational system of subspaces $\Gamma(\tau)$ coincides with the system of subspaces $M(\tau)$.

We will consider an associated curved construction. Let Ξ be a family of curves on a manifold X and let U_ξ be a curve.Let NE_ξ be its normal bundle and $\Gamma = \Gamma(NE_\xi)$ be the space of sections of Γ. Informally, elements of Γ are infinitesimal deformations of the curve U_ξ. Let $\tilde{\Gamma}$ be the subspace of deformations extendable up to local deformations in Ξ and $\tilde{\Gamma}(x), x \in U_\xi$, be subspaces of sections equal zero in a generic(relative to the family Ξ) point x of the curve U_ξ. We will call the family Ξ by *the full family of rational curves* if the system of subspaces $\tilde{\Gamma}(x)$ for generic x (relative to Ξ) can be include in a system of subspaces $M(\tau)$ (in appropriate coordinates). In such a way the family Ξ in the infinitesimal neighborhood of any curve looks as all sections of a vector bundle on a projective line. Let us emphasize that this definition is local and birationally invariant. On the other hand curves in a (local) full family of rational curves admit a canonical extension up to complete rational curves. Full systems of rational curves have a system of invariants $r_1, \ldots, r_k$. It is convenient for us suppose that $r_j \geq 0$, although only the case $r_j > 0$ corresponds to subvarieties of minimal degree.

Full families of rational curves are connected with many important analytic problems. Twistor construction of nonlinear graviton is based on the fact that such families for $r_1 = r_2 = 1, \dim X = 3, \dim \Xi = 4$ induce self-dual conformal metrics on Ξ. It gives a possibility to construct some solution of self-dual Einstein equation. J.Bernstein and myself proved that the class of full families of rational curves coincides with the class of admissible families in the sense of integral geometry. The admissibility means that for such families the problem of integral geometry (the integration with appropriate measures) admits local Radon type inversion formula.There are other problems connecting with these families.

It is a reason why it would be important to produce examples of full families of rational curves. Firstly, the family of all rational curves of some type on a variety is full.As examples we can take either all lines in $\mathbb{C}P^n$, or all conics in $\mathbb{C}P^n$, or all sections of a vector bundle on $\mathbb{C}P^1$. It is remarkable that also some subfamilies of full families of rational curves continue to be full. J.Bernstein and myself gave a complete description of such full subfamilies. We give here this description for a generic case. Let Ξ be a full family of rational curves on X and let $S_1, \ldots, S_l, T_1, \ldots, T_m, \operatorname{codim} S_i > 1, \operatorname{codim} T_j = 1$ be some submanifolds of X. Let$\nu_1, \ldots, \nu_m$ be some positive integers. Denote as $\tilde{\Xi}$ the subfamily of such curves which intersect all S_i and tangent to all T_j with multiplicities ν_j correspondingly. Then $\tilde{\Xi}$ be a full family of rational curves and each generic full subfamily in Ξ can be obtain by such a construction [Gi]. The general construction of full subvarieties is given on the language of towers of σ-processes.It is a way to include all possible complicate interactions of intersections and tangencies.

These results give a possibility produce a broad collection of examples, including examples of families of curves with explicit inversion formulas for corresponding problems of integral geometry and explicit solutions of the self-dual Einstein equation.However it would be important, at least under very strong restrictions , give a classification of such families. Goncharov [Go] showed that in the case of algebraic families of rational curves on algebraic surfaces it is possible give very explicit description. He proved that if X is a surface of minimal degree in $\mathbb{C}P^n$ and $T_1, \ldots, T_m$ are (irreducible) algebraic curves on X and $\nu_1, \ldots, \nu_m$ are positive integers, then the family Ξ of hyperplane sections of X, tangent to $T_1, \ldots, T_m$ with multiplicities $\nu_1, \ldots, \nu_m$ correspondingly. What is most surprising, that he proved the inverse theorem. Namely, any full algebraic family of rational curves on an algebraic surface is canonically birationally isomorphic to a described above family of rational curves on a surface of minimal degree. This result gives the complete description of algebraic admissible families of curves on surfaces. It gives also the complete and explicit description of algebraic Einstein-Weyl 3-folds. This result bases on Hitchin's reduction of the integration of the Einstein-Weyl equation to to the construction of full 3-parametric families of rational curves on 3-folds.

The restriction in these results, that we consider curves on surfaces, is very essential. Most strong results in integral geometry relate with families of curves. Only a few results are known about families of submanifolds of dimension more than 1. Between them there is Goncharov's theorem, that the family of hyperplane sections of a subvariety of minimal degree in $\mathbb{C}P^n$ which are tangent to some fixed hypersurfaces, is admissible. However, in this general situation there is no ideas as to how to obtain inverse results and to give general descriptions. Nevertheless, the connection of subvarieties of minimal degree with integral geometry looks to be a very deep phenomena.

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Departm. of Math., Hill Center, Rutgers University, 110 Frelinghysen Road, Piscataway, NJ 08854-8019

E-mail address: gindikin@math.rutgers.edu