STEIN EXTENSIONS OF RIEMANN SYMMETRIC SPACES AND DUALITIES OF ORBITS ON FLAG MANIFOLDS

SIMON GINDIKIN* AND TOSHIHIKO MATSUKI

Abstract. It is known that $K_{\mathbb{C}}$-orbits S and $G_{\mathbb{R}}$-orbits S' on a complex flag manifold are in one-to-one correspondence by the condition that $S \cap S'$ is non-empty and compact. We may replace $K_{\mathbb{C}}$ by some conjugate $xK_{\mathbb{C}}x^{-1}$ so that the correspondence is preserved. We will show that this replacement is related to the domain introduced by Akhiezer and Gindikin.

1. Introduction

Let $G_{\mathbb{C}}$ be a connected complex semisimple Lie group and $G_{\mathbb{R}}$ a connected real form of $G_{\mathbb{C}}$. Let $K_{\mathbb{C}}$ be the complexification in $G_{\mathbb{C}}$ of a maximal compact subgroup K of $G_{\mathbb{R}}$. Let $X = G_{\mathbb{C}}/P$ be a flag manifold of $G_{\mathbb{C}}$ where P is an arbitrary parabolic subgroup of $G_{\mathbb{C}}$. Then there exists a natural one-to-one correspondence between the set $K_{\mathbb{C}}\backslash X$ of $K_{\mathbb{C}}$-orbits S and the set $G_{\mathbb{R}}\backslash X$ of $G_{\mathbb{R}}$-orbits S' on X given by the condition:

$$S \leftrightarrow S' \iff S \cap S' \text{ is non-empty and compact} \tag{1.1}$$

([M4], c.f. [MUV] for an extension to sheaf theory). It is possible to interpret this correspondence as the duality between $K_{\mathbb{C}}$-orbits and $G_{\mathbb{R}}$-orbits on flag manifolds $X = G_{\mathbb{C}}/P$: dual orbits have non-empty compact intersections.

It turns out that for a fixed group $G_{\mathbb{R}}$, this duality between its orbits with $K_{\mathbb{C}}$-orbits is inherited by some conjugate subgroups $xK_{\mathbb{C}}x^{-1}$ in $G_{\mathbb{C}}$. In other words, let $M_{\mathbb{R}} = G_{\mathbb{R}}/K$ be the Riemann symmetric space, canonically embedded in its complexification $M_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. Then in (1.1), instead of $K_{\mathbb{C}}$, we can, of course, take $K_{\mathbb{C}}(z)$ the isotropy subgroup of any point $z \in M_{\mathbb{R}}$. Our aim is to find the set $\widetilde{M} \supset M_{\mathbb{R}}$ of such $z \in M_{\mathbb{C}}$ that (1.1) holds for $K_{\mathbb{C}}(z)$. This problem is one of several problems about Stein extensions $\widetilde{M}$ of Riemann symmetric spaces $M_{\mathbb{R}}$, on which can be extended different analytic or geometrical constructions on $M_{\mathbb{R}}$. There is an informal conjecture that is an universal Stein extension for many such problems which in [G] was suggested to call by complex crowns of Riemann symmetric spaces.

For every $K_{\mathbb{C}}$-orbit S on $X = G_{\mathbb{C}}/P$, define a subset

$$C(S) = \{x \in G_{\mathbb{C}} \mid xS \cap S' \text{ is non-empty and compact}\}$$

of $G_{\mathbb{C}}$ where S' is the $G_{\mathbb{R}}$-orbit on X given by (1.1).

Remark 1.1. (1) Let $h \in G_{\mathbb{R}}$ and $k \in K_{\mathbb{C}}$. Then we have

$$hxkS \cap S' = hxS \cap hS' = h(xS \cap S').$$

* The first author is partly supported by NSF grant DMS 00-70816.

Hence the sets $C(S)$ are left $G_{\mathbb{R}}$-invariant and right $K_{\mathbb{C}}$-invariant. So we can also define a $G_{\mathbb{R}}$-invariant subset $\widetilde{M}(S) = C(S)/K_{\mathbb{C}}$ of $M_{\mathbb{C}}$

(2) Let Z denote the center of $G_{\mathbb{C}}$. Then the sets $C(S)$ are Z-invariant because $Z \subset P$. Hence the sets $C(S)$ depend only on Lie algebras. Note that the $K_{\mathbb{C}}$-orbit structure and $G_{\mathbb{R}}$-orbit structure on X depend only on Lie algebras since we assume that $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ are connected.

Now we can propose the following problem.

Problem 1.2. *What is the set*

$$C = \bigcap C(S) \ ?$$

Here we take the intersection for all $K_{\mathbb{C}}$-orbits S on X and all flag manifolds $X = G_{\mathbb{C}}/P$ of $G_{\mathbb{C}}$. Clearly, $\widetilde{M} = \bigcap \widetilde{M}(S) = C/K_{\mathbb{C}}$.

Our conjecture is that the sets C and $\widetilde{M}$ correspond to the domain D_0 from [AG]. This domain includes $M_{\mathbb{R}}$ and is the maximal domain which unifies $G_{\mathbb{R}}$-orbits with compact isotropy subgroups. These orbits admit the next description.

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ with respect to K. Let $\mathfrak{t}$ be a maximal abelian subspace in $i\mathfrak{p}$. Put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma\}$$

where Σ is the restricted root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}$. Then we can define a left $G_{\mathbb{R}}$-invariant right $K_{\mathbb{C}}$-invariant open subset in $G_{\mathbb{C}}$ by

$$\widetilde{D_0} = G_{\mathbb{R}}(\exp \mathfrak{t}^+)K_{\mathbb{C}}$$

and a $G_{\mathbb{R}}$-invariant open subset $D_0 = \widetilde{D_0}/K_{\mathbb{C}}$ in $M_{\mathbb{C}}$.

Conjecture 1.3. $C = \widetilde{D_0}Z$. *(Equivalently, $\widetilde{M} = ZD_0$.)*

The aim of this paper is to prove:

Theorem 1.4. $C = \widetilde{D_0}Z$ *for all classical cases and exceptional Hermitian cases.*

We will prove this theorem by case-by-case checkings in Section 2-7 and by a general trick in Section 8. We will deal with the case where $G_{\mathbb{R}}$ is Hermitian type in Section 2, $SL(n,\mathbb{R})$-case in Section 3, $SO(p,q)$-case in Section 4, cases of complex classical Lie groups in Section 5, $GL(n,\mathbb{H})$-case in Section 6 and $Sp(p,q)$-case in Section 7. (The essential part in $SL(n,\mathbb{R})$-case was already proved in [AG], Example 1.)

Let us explain a structure of these results. Let B denote a Borel subgroup of $G_{\mathbb{C}}$ and let C_B denote the intersection of all $C(S)$ such that S are closed $K_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/B$.

In Section 2, we assume that $G_{\mathbb{R}}$ is Hermitian type. This case in many relations is a special one. Firstly, we consider the natural embedding

$$\iota : M_{\mathbb{C}} \to G_{\mathbb{C}}/P \times G_{\mathbb{C}}/\overline{P}$$

([WZ]) where P and $\overline{P}$ are the usual maximal parabolic subgroups of $G_{\mathbb{C}}$ containing $K_{\mathbb{C}}$. Then we will prove in Proposition 2.2 that

$$\iota(D_0) = G_{\mathbb{R}}P/P \times G_{\mathbb{R}}\overline{P}/\overline{P} \ (\cong M_{\mathbb{R}} \times \overline{M_{\mathbb{R}}}). \tag{1.2}$$

Let B be a Borel subgroup of $G_{\mathbb{C}}$ contained in P and let consider two special closed $K_{\mathbb{C}}$-orbits $S_1 = K_{\mathbb{C}}B/B = P/B$ and $S_2 = K_{\mathbb{C}}w_0B/B = \overline{P}w_0/B$ on $X = G_{\mathbb{C}}/B$ where w_0 is the longest element in the Weyl group. (The corresponding open $G_{\mathbb{R}}$-orbits S_1' and S_2' are used to realize holomorphic and anti-holomorphic discrete series representations of $G_{\mathbb{R}}$.) Then the above realization (1.2) of D_0 clearly implies that

$$\widetilde{D_0} = C(S_1) \cap C(S_2).$$

The second part of the section is dedicated to the proof of

$$C_B = C(S_1) \cap C(S_2)$$

(Proposition 2.4) which was already proved in [WZ] for classical cases by case-by-case checkings. But we will give, in this paper, a simple general proof.

Remark 1.5. Let Q be a parabolic subgroup of $G_{\mathbb{C}}$ containing B and $\pi : G_{\mathbb{C}}/B \to G_{\mathbb{C}}/Q$ the canonical projection. Then a $K_{\mathbb{C}}$-orbit S (resp. $G_{\mathbb{R}}$-orbit S') is called to be of holomorphic type if

$$S = \pi(S_1) \quad \text{or} \quad \pi(S_2) \qquad (\text{resp. } S' = \pi(S_1') \quad \text{or} \quad \pi(S_2')).$$

Clearly, this definition is equal to the one in [WZ] when S' is open in $G_{\mathbb{C}}/Q$. By this definition, S is automatically closed and S' is open in $G_{\mathbb{C}}/Q$. By the result in [WZ], we have

$$\widetilde{M}(S) = \begin{cases} G_{\mathbb{R}}P/K_{\mathbb{C}} & \text{if } S = \pi(S_1), \\ G_{\mathbb{R}}\overline{P}/K_{\mathbb{C}} & \text{if } S = \pi(S_2). \end{cases}$$

All five sections from 3 to 7 have the same structure. We first take a minimal flag manifold X_0 such that the $K_{\mathbb{C}}$-orbit structure on X_0 is nontrivial and take a closed $K_{\mathbb{C}}$-orbit S_0 (S_+ in Section 7). Then we will show that $C(S_0)$ is essentially equal to $\widetilde{D_0}Z$. (In Section 5, 6 and 7, they are equal. In Section 3 and 4, they are not equal in general but

$$C(S_0)_0 = \widetilde{D_0}. \tag{1.3}$$

where $C(S)_0$ denote the connected component of $C(S)$ containing the identity.) The most essential idea in these sections is that the problem is reduced to some nonstandard problems of linear algebra in a classical style as in [AG], Example 1 which we will solve by elementary calculus.

Here are samples of these problems. For $SL(n,\mathbb{R})$, we need to describe such (complex) nondegenerate symmetric forms m that corresponding quadrics (${}^t xmx = 0$) in the projective space $P^{n-1}(\mathbb{C})$ have no real points. Of course this set contains the symmetric space $M_{\mathbb{R}}$ of real positive matrixes.

For $SL(n,\mathbb{C})$, we need to describe such $m \in SL(n,\mathbb{C})$ that (${}^t\overline{x}mx = 0$) has no real points in the complex projective space. Again hermitian positive m (Riemann symmetric space $M_{\mathbb{R}}$) satisfy these condition but we want to describe all of them.

One more example: for $SO(p,q)$ we take the quadric corresponding to the canonical (p,q)-form in the complex projective space and describe $(p-1)$-sections on which the corresponding hermitian form will be positive. Evidently this set includes the real Grassmannian $Gr_p(\mathbb{R})$. In all cases these sets are $G_{\mathbb{R}}$-invariant and we describe them by a parameterization of orbits, which has a similar structure in all these and some other cases.

Next we consider full flag manifolds $X = G_{\mathbb{C}}/B$ and show that

$$C(S) = \widetilde{D_0}Z \tag{1.4}$$

for all closed $K_{\mathbb{C}}$-orbits S on X.

In Section 8, we will show in general (Proposition 8.1) that

$$C = C_B.$$

So we have proved Theorem 1.4 for all classical groups and exceptional groups of Hermitian type.

Moreover we will show in Proposition 8.3 that

$$C(S)_0 = C_0 \tag{1.5}$$

for the open $K_{\mathbb{C}}$-orbit S on $X = G_{\mathbb{C}}/B$ for any $G_{\mathbb{R}}$. This result has an interesting application (see below).

In the appendix, we will discuss some geometric consideration for $SO(p,q)$-case: We may embed D_0 into the Grassmann manifold $X_1 = Gr_{p,p+q}$ of $SL(p+q,C)$ in this case. It turns out that the image is bigger than the expected Hermitian symmetric domain which is an open $SU(p,q)$-orbit on X_1. Though this consideration is independent of our conjecture, it seems to be important for geometric studies on D_0.

We have seen in (1.3), (1.4) and (1.5) that for typical orbits S we have

$$C(S)_0 = \widetilde{D_0}. \tag{1.6}$$

We can also prove it for all closed $K_{\mathbb{C}}$-orbits S on $X = G_{\mathbb{C}}/P$ (P is arbitrary) of nonholomorphic type (c.f. [WZ]) for all classical cases. It will be published in a subsequent paper. Of course, we assume that X is not $K_{\mathbb{C}}$-homogeneous. For non-Hermitian cases, every closed $K_{\mathbb{C}}$-orbit (open $G_{\mathbb{R}}$-orbit) is nonholomorphic type.

By (1.6), we expect that

Conjecture 1.6. *$C(S)_0 = \widetilde{D_0}$ for all $K_{\mathbb{C}}$-orbits S on $X = G_{\mathbb{C}}/P$ of nonholomorphic type. (Every non-closed $K_{\mathbb{C}}$-orbit on X and each orbit for groups of non Hermitian type is defined to be of nonholomorphic type by Remark 1.5.)*

Apparently, the new element which we need for proving of this conjecture is a proof of the inclusions $C(S)_0 \subset \widetilde{D_0}$. (The opposite inclusion is already proved in Theorem 1.4 for all classical cases and exceptional Hermitian cases.)

This problem (Conjecture 1.6) for different types of orbits corresponds to different geometrical problems, some of which were studied extensively. Open $G_{\mathbb{R}}$-orbits S' are called by flag domains and they are connected with realizations of discrete series and some other unitary representations. Then xS are compact complex cycles and we have the problem on the parameterization of cycles in flag domains. Wolf proved that for these cases $C(S)$ are Stein manifolds ([W]). Wolf and Zierau proved that $\widetilde{M}(S)_0 \supset M_{\mathbb{R}} \times \overline{M_{\mathbb{R}}}$ for Hermitian classical groups and orbits S of nonholomorphic type ([WZ]). By our computation, it seems that their proof of the opposite inclusion contains a gap.

Another interesting special case is when S is open. There is only one such orbit and it is Zariski open. Correspondingly, there is a unique closed $G_{\mathbb{R}}$-orbit S' which is contained in S. The supplement $\widetilde{S}$ to S can be defined by an polynomial equation $D(g) = 0$, where the function D satisfies the condition:

$$D(kgp) = D(g)\chi(p)$$

for a character χ of P. (This function D is a $K_{\mathbb{C}}$-invariant element of a class-one finite dimensional representation of $G_{\mathbb{C}}$ which is realized, by the Borel-Weil theorem, on the space of holomorphic cross sections of the holomorphic line bundle over $G_{\mathbb{C}}/P$ with some strictly dominant class-one integral character of P.) In the case $P = B$, this condition is the obstruction to the existence of the Iwasawa decomposition on $G_{\mathbb{C}}$. Then we can define $C(S)$ as

$$C(S) = \{x \mid D(x^{-1}g) \neq 0,\ g \in S'\}$$

which is clearly a Stein manifold because $C(S)$ is obtained by deleting from $G_{\mathbb{C}}$ an infinite family of analytic hypersurfaces. If $P = B$, then we proved in Proposition 8.3 that $C(S)_0 = C_0$ and we proved $C_0 = \widetilde{D_0}$ for all classical groups. Hence $D_0 = \widetilde{D_0}/K_{\mathbb{C}}$ is a Stein manifold for all these groups. In such a way, we prove the conjecture from [AG] for all these groups. The functions D (let us call them determinant functions) must play essential role in the analysis on flag manifolds and symmetric spaces and deserve a special investigation. In simplest cases they reduce to the norm-functions on symmetric cones (Jordan algebras); for $SU(p,q)$ they constructed in [BGW].

Finally, we want to explain, by the following simplest example, the relation to the structure of the double coset decomposition $G_{\mathbb{R}}\backslash G_{\mathbb{C}}/K_{\mathbb{C}}$ which was studied in [M5].

Example 1.7. Let $G_{\mathbb{C}} = SL(2,\mathbb{C}), G_{\mathbb{R}} = SL(2,\mathbb{R})$ and $K_{\mathbb{C}} = SO(2,\mathbb{C})$. Then the flag manifold $X = G_{\mathbb{C}}/P \cong P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is decomposed to $K_{\mathbb{C}}$-orbits as

$$X = S_1 \sqcup S_2 \sqcup S_3$$

where $S_1 = \{i\}$ and $S_2 = \{-i\}$. X is also decomposed to $G_{\mathbb{R}}$-orbits as

$$X = S_1' \sqcup S_2' \sqcup S_3'$$

where $S_1' = \{z \in X \mid \operatorname{Im} z > 0\}$ and $S_2' = \{z \in X \mid \operatorname{Im} z < 0\}$. By the results of [M5], we can take representatives x of the double coset decomposition $G_{\mathbb{R}}\backslash G_{\mathbb{C}}/K_{\mathbb{C}}$ as follows. (Then xS_1 and xS_2 are as in the remaining columns.)

x	xS_1	xS_2	$xS_1 \subset$	$xS_2 \subset$
a_θ $(0 \leq \theta < \frac{\pi}{4})$	$\left\{i \tan\left(\theta + \frac{\pi}{4}\right)\right\}$	$\left\{-i \cot\left(\theta + \frac{\pi}{4}\right)\right\}$	S'_1	S'_2
$a_{\pi/4}$	$\{\infty\}$	$\{0\}$	S'_3	S'_3
a_θ $(\frac{\pi}{4} < \theta \leq \frac{\pi}{2})$	$\left\{i \tan\left(\theta + \frac{\pi}{4}\right)\right\}$	$\left\{-i \cot\left(\theta + \frac{\pi}{4}\right)\right\}$	S'_2	S'_1
$b_t a_{\pi/4}$ $(t > 0)$	$\{i \coth t\}$	$\{i \tanh t\}$	S'_1	S'_1
$b_t a_{\pi/4}$ $(t < 0)$	$\{i \coth t\}$	$\{i \tanh t\}$	S'_2	S'_2
$c_+ a_{\pi/4}$	$\{\infty\}$	$\{i\}$	S'_3	S'_1
$c_- a_{\pi/4}$	$\{\infty\}$	$\{-i\}$	S'_3	S'_2
$d_+ a_{\pi/4}$	$\{-i\}$	$\{0\}$	S'_2	S'_3
$d_- a_{\pi/4}$	$\{i\}$	$\{0\}$	S'_1	S'_3

Here

$$a_\theta = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix},\ b_t = \begin{pmatrix} \cosh t & i\sinh t \\ -i\sinh t & \cosh t \end{pmatrix},\ c_\pm = \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix} \text{ and } d_\pm = \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix}.$$

By the above list, we can see

$$C = C(S_1) \cap C(S_2) = \bigsqcup_{0 \leq \theta < \pi/4} G_\mathbb{R} a_\theta K_\mathbb{C} = \widetilde{D_0}.$$

On the other hand, $C(S_3)$ consists of four connected components

$$\bigsqcup_{0 \leq \theta < \pi/4} G_\mathbb{R} a_\theta K_\mathbb{C}, \qquad \bigsqcup_{\pi/4 < \theta \leq \pi/2} G_\mathbb{R} a_\theta K_\mathbb{C}, \qquad \bigsqcup_{t>0} G_\mathbb{R} b_t a_{\pi/4} K_\mathbb{C} \quad \text{and} \quad \bigsqcup_{t<0} G_\mathbb{R} b_t a_{\pi/4} K_\mathbb{C}.$$

Note that there are four non-closed $G_\mathbb{R}$-$K_\mathbb{C}$ double cosets represented by $c_\pm a_{\pi/4}$ and $d_\pm a_{\pi/4}$ which have common boundary represented by $a_{\pi/4}$.

Acknowledgement: The authors are grateful to H. Ochiai who pointed out an elementary mistake in our first draft. While we were correcting it, the proofs were very much improved.

2. Hermitian cases

Suppose that $G_\mathbb{R}$ is simple and of Hermitian type. Then $\mathfrak{k} = \mathrm{Lie}\,(K)$ has a one-dimensional center and a nontrivial central element in $i\mathfrak{k}$ defines a maximal parabolic subgroup

$$P = K_\mathbb{C} \exp \mathfrak{n}$$

of $G_\mathbb{C}$ where

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\mathbb{C}(\mathfrak{j}, \alpha)$$

is the nilpotent radical of $\mathrm{Lie}\,(P)$. Here $\mathfrak{j}$ is a maximal torus of $\mathfrak{k}$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j},\alpha)$ denote the root space for a root α in the root system $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}},\mathfrak{j})$. Let $x \mapsto \overline{x}$ denote the conjugation of $G_{\mathbb{C}}$ with respect to $G_{\mathbb{R}}$. Then we have

$$\overline{P} = K_{\mathbb{C}} \exp \overline{\mathfrak{n}} \quad \text{and} \quad \overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, -\alpha).$$

Note that

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{n} \oplus \overline{\mathfrak{n}}.$$

Let $\{\beta_1, \dots, \beta_\ell\}$ be a maximal strongly othogonal system in Δ_n^+ and $X_1, \dots, X_\ell$ be nonzero elements of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j},\beta_1), \dots, \mathfrak{g}_{\mathbb{C}}(\mathfrak{j},\beta_\ell)$, respectively. Then we can define maximal abelian subspaces

$$\mathfrak{a} = \mathbb{R}(X_1 + \overline{X_1}) \oplus \cdots \oplus \mathbb{R}(X_\ell + \overline{X_\ell})$$

of $\mathfrak{p}$ and

$$\mathfrak{t} = \mathbb{R}(X_1 - \overline{X_1}) \oplus \cdots \oplus \mathbb{R}(X_\ell - \overline{X_\ell})$$

of $i\mathfrak{p}$. Put $A = \exp \mathfrak{a}$, $T = \exp \mathfrak{t}$ and $T^+ = \exp \mathfrak{t}^+$. Though the following fact seems to be known, we will give a proof for the sake of completeness.

Lemma 2.1. (1) $AP = T^+P$.
(2) $G_{\mathbb{R}}P = KT^+P$.

Proof. (1) Let $\mathfrak{l}_{\mathbb{C}}$ be the complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\{X_1, \overline{X_1}, X_2, \overline{X_2}, \dots, X_\ell, \overline{X_\ell}\}$. Then $\mathfrak{l}_{\mathbb{C}}$ is isomorphic to the direct sum of ℓ-copies of $\mathfrak{s}l(2,\mathbb{C})$. Note that the set $\mathfrak{t}^+$ is characterized by the subset $\Delta(\mathfrak{l}_{\mathbb{C}},\mathfrak{t})$ of $\Sigma = \Delta(\mathfrak{g}_{\mathbb{C}},\mathfrak{t})$ because Σ is of type C_ℓ or BC_ℓ and $\Delta(\mathfrak{l}_{\mathbb{C}},\mathfrak{t})$ is the set of longest roots in Σ. Let $L_{\mathbb{C}}$ denote the analytic subgroup of $G_{\mathbb{C}}$ for $\mathfrak{l}_{\mathbb{C}}$. Then by a computation in $SL(2,\mathbb{C})$, we have

$$A(L_{\mathbb{C}} \cap P) = T^+(L_{\mathbb{C}} \cap P).$$

Hence we have

$$AP = T^+P.$$

(2) is clear by the Cartan decomposition $G_{\mathbb{R}} = KAK$. q.e.d.

Consider the natural embedding

$$\iota : G_{\mathbb{C}}/K_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/P \times G_{\mathbb{C}}/\overline{P}$$

given by

$$xK_{\mathbb{C}} \mapsto (xP, x\overline{P})$$

([WZ]).

Proposition 2.2. $\iota(D_0) = G_{\mathbb{R}}P/P \times G_{\mathbb{R}}\overline{P}/\overline{P}$. *(Here* $D_0 = \widetilde{D_0}/K_{\mathbb{C}}$*.)*

Proof. By Lemma 2.1, we have

$$\iota(T^+K_{\mathbb{C}}/K_{\mathbb{C}}) \subset G_{\mathbb{R}}P/P \times G_{\mathbb{R}}\overline{P}/\overline{P}$$

and hence

$$\iota(D_0) \subset G_{\mathbb{R}}P/P \times G_{\mathbb{R}}\overline{P}/\overline{P}.$$

Conversely, let g and h be elements of $G_{\mathbb{R}}$. Then we have only to find an $x \in D_0$ such that $\iota(x) = (gP, h\overline{P})$. By $G_{\mathbb{R}}$-action, we have only to consider $(h^{-1}gP, \overline{P})$. Write

$h^{-1}g = k_1ak_2$ with $k_1, k_2 \in K$ and $a \in A$. Then we have only to consider $(aP, \overline{P})$ since

$$(k_1^{-1}h^{-1}gP, k_1^{-1}\overline{P}) = (aP, \overline{P}).$$

Furthermore we have only to consider $(a^{1/2}P, a^{-1/2}\overline{P})$. Let τ denote the conjugation in $G_{\mathbb{C}}$ with respect to the compact real form $U = K \exp i\mathfrak{p}$. Then we have $\tau(P) = \overline{P}$ since P is invariant by the holomorphic involution defining $K_{\mathbb{C}}$ and we also have $\tau(a^{1/2}) = a^{-1/2}$. Write

$$a^{1/2} = tp$$

with some $t \in T^+$ and $p \in P$ by Lemma 2.1 (1). Then applying τ on the both sides, we get

$$a^{-1/2} = t\tau(p).$$

Hence we have

$$\iota(tK_{\mathbb{C}}) = (a^{1/2}P, a^{-1/2}\overline{P}).$$

Thus we have proved $\iota(D_0) = G_{\mathbb{R}}P/P \times G_{\mathbb{R}}\overline{P}/\overline{P}$. q.e.d.

Let Δ^+ be a positive system of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j})$ containing Δ_n^+. Then Δ^+ defines a Borel subgroup B of $G_{\mathbb{C}}$ such that $B \subset P$. Let w_0 be the longest element (with respect to Δ^+) of the Weyl group $W = W(\mathfrak{j})$. Then the two closed sets

$$S_1 = P = K_{\mathbb{C}}P = K_{\mathbb{C}}B \quad \text{and} \quad S_2 = \overline{P}w_0 = K_{\mathbb{C}}\overline{P}w_0 = K_{\mathbb{C}}(w_0Bw_0^{-1})w_0 = K_{\mathbb{C}}w_0B$$

are contained in the open sets

$$S_1' = G_{\mathbb{R}}P = G_{\mathbb{R}}B \quad \text{and} \quad S_2' = G_{\mathbb{R}}\overline{P}w_0 = G_{\mathbb{R}}w_0B,$$

respectively (c.f. [M2]). Let x be an element of $G_{\mathbb{C}}$. Then Proposition 2.2 implies that

$$x \in \widetilde{D_0} \Longleftrightarrow xS_1 \subset S_1' \text{ and } xS_2 \subset S_2'$$

Let W_K be the subgroup of W defined by $W_K = N_{K_{\mathbb{C}}}(\mathfrak{j})/Z_{K_{\mathbb{C}}}(\mathfrak{j})$. Then the set of closed $K_{\mathbb{C}}$-B double cosets in $G_{\mathbb{C}}$ is parametrized by $W_K\backslash W$ ([M1], [R]).

Lemma 2.3. *For $w \in W$, we have*

$$K_{\mathbb{C}}(BwB)^{cl} \cap K_{\mathbb{C}}w_0(Bw_0^{-1}wB)^{cl} = K_{\mathbb{C}}wB.$$

Proof. We can choose a $w' = w_Kw \in W_Kw$ such that

$$K_{\mathbb{C}}(BwB)^{cl} = (Bw'B)^{cl}$$

since $P = K_{\mathbb{C}}B$ and $(BwB)^{cl}$ are both expressed as products of parabolic subgoups $P_\alpha = B \cup Bw_\alpha B$ with $\alpha \in \Psi$ where Ψ denote the set of simple roots in Δ^+. It is known that

$$(Bw'B)^{cl} \cap (B^-w'B)^{cl} = w'B$$

where $B^- = w_0 B w_0^{-1}$. So we have

$$\begin{aligned} K_{\mathbb{C}}(BwB)^{cl} \cap K_{\mathbb{C}} w_0 (B w_0^{-1} w B)^{cl} &= (Bw'B)^{cl} \cap K_{\mathbb{C}}(B^- wB)^{cl} \\ &= (Bw'B)^{cl} \cap K_{\mathbb{C}}(B^- w'B)^{cl} \\ &= K_{\mathbb{C}} w' B \\ &= K_{\mathbb{C}} w B \end{aligned}$$

since $K_{\mathbb{C}}(B^- wB)^{cl} = (\overline{P}wB)^{cl} = (\overline{P}w'B)^{cl} = K_{\mathbb{C}}(B^- w'B)^{cl}$ and since w_K is contained in $K_{\mathbb{C}}$. q.e.d.

Proposition 2.4. *Let x be an element of $G_{\mathbb{C}}$. If $xS_1 \subset S_1'$ and $xS_2 \subset S_2'$, then $xS \subset S'$ for all closed $K_{\mathbb{C}}$-B double cosets S in $G_{\mathbb{C}}$.*

Proof. Let $w \in W$. Then by Lemma 2.3, we have

$$S_1(BwB)^{cl} \cap S_2(Bw_0^{-1}wB)^{cl} = K_{\mathbb{C}} wB.$$

We also have

$$S_1'(BwB)^{cl} \cap S_2'(Bw_0^{-1}wB)^{cl} = G_{\mathbb{R}} wB$$

since the correspondence between $K_{\mathbb{C}}$-orbits and $G_{\mathbb{R}}$-orbits is compatible with the right P_α-action for $\alpha \in \Psi$ ([M2]). Since $xS_1 \subset S_1'$ and $xS_2 \subset S_2'$, we have

$$xS_1(BwB)^{cl} \subset S_1'(BwB)^{cl} \quad \text{and} \quad xS_2(Bw_0^{-1}wB)^{cl} \subset S_2'(Bw_0^{-1}wB)^{cl}.$$

Hence we have

$$xK_{\mathbb{C}} wB \subset G_{\mathbb{R}} wB. \qquad \text{q.e.d.}$$

3. $SL(n, \mathbb{R})$-case

Let $G_{\mathbb{C}} = SL(n, \mathbb{C})$, $K_{\mathbb{C}} = SO(n, \mathbb{C})$ and $G_{\mathbb{R}} = SL(n, \mathbb{R})$ with $n \geq 3$. First consider the flag manifold $X_0 = P^{n-1}(\mathbb{C}) = (\mathbb{C}^n - \{0\})/\mathbb{C}^\times$. Then there are two $K_{\mathbb{C}}$-orbits and two $G_{\mathbb{R}}$-orbits on X_0. The closed $K_{\mathbb{C}}$-orbit is defined by

$$S_0 : z_1^2 + \cdots + z_n^2 = 0$$

on X_0 and the open $G_{\mathbb{R}}$-orbit is

$$S_0' = X_0 - P^{n-1}(\mathbb{R}).$$

We can take

$$\mathfrak{t}^+ = \left\{ \begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \;\middle|\; |\theta_j - \theta_k| < \frac{\pi}{2} \text{ for all } j \text{ and } k \right\}.$$

Put

$$y = \begin{pmatrix} e^{i\pi/n} I_{n-1} & 0 \\ 0 & -e^{i\pi/n} \end{pmatrix} = e^{i\pi/n} \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

and let Y denote the subgroup of $G_{\mathbb{C}}$ generated by y. Note that y normalizes the sets $G_{\mathbb{R}}$, $K_{\mathbb{C}}$ and T^+ and that y^2 generates the center Z of $G_{\mathbb{C}}$. (If n is odd, then $Y = Z$.) Though the following fact was already proved in Example 1 of [AG], we will give a detailed proof here because an elementary calculus used in this proof is the key idea of this paper.

Proposition 3.1. $xS_0 \subset S_0' \Longleftrightarrow x \in G_{\mathbb{R}}T^+YK_{\mathbb{C}}$.

Proof. Note that the projective cone $xS_0 \subset X_0$ is defined by

$$^t zmz = 0$$

where $m = {}^t x^{-1}x^{-1}$. So the condition $xS_0 \subset S_0'$ implies that

$$^t zmz \neq 0 \quad \text{for all } z \in \mathbb{R}^n - \{0\}. \tag{3.1}$$

Assume (3.1). Since $F = \mathbb{R}^n - \{0\}$ is simply connected, we can define a smooth function $\xi : F \to \mathbb{R}$ by

$$\xi(z) = \arg({}^t zmz)$$

taking some branch of arg. Since $P^{n-1}(\mathbb{R}) = F/\mathbb{R}^\times$ is compact, the image R of ξ is a closed interval in $\mathbb{R}$. So we can take a $v_1 \in F$ such that $\xi(v_1)$ is on the boundary of R and that $|{}^t v_1 m v_1| = 1$. Let u be an element of $\mathbb{R}^n$ and consider the function

$$\eta(t) = \xi(v_1 + tu) = \arg({}^t v_1 m v_1 + 2t({}^t v_1 m u) + t^2({}^t umu))$$

of $t \in \mathbb{R}$. Then we have

$$^t v_1 mu \in \mathbb{R}\, {}^t v_1 m v_1 \quad \text{for all } u \in \mathbb{R}^n$$

since $\eta(0) = \xi(v_1)$ is on the boundary of R. Put

$$U_1 = \{u \in \mathbb{R}^n \mid {}^t v_1 mu = 0\}.$$

Then U_1 is an $n-1$-dimensional subspace of $\mathbb{R}^n$ which does not contain v_1.

Next consider the restriction of ξ on $U_1 - \{0\}$ and take a $v_2 \in U_1 - \{0\}$ such that $\xi(v_2)$ is on the boundary of $\xi(U_1 - \{0\})$ and that $|{}^t v_2 m v_2| = 1$. Repeating this procedure, we can take basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ such that

$$^t v_j m v_k = 0 \quad \text{if } j \neq k$$

and that $|{}^t v_j m v_j| = 1$ for $j = 1, \ldots, n$. Hence if we put $g = (v_1 \cdots v_n) \in G_{\mathbb{R}}$, then we have

$$^t gmg = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_j = {}^t v_j m v_j \in U(1)$. Write $z = gc$ with $c \in \mathbb{R}^n$. Then we have

$$^t zmz = {}^t c\, {}^t gmgc = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2.$$

Considering the convex real cone in $\mathbb{C}$ generated by $\lambda_1, \ldots, \lambda_n$, we see that

$$(3.1) \Longleftrightarrow \{\lambda_1, \ldots, \lambda_n\} \subset \{z \in \mathbb{C} \mid \operatorname{Re} \mu z > 0\} \text{ for some } \mu \in \mathbb{C}. \tag{3.2}$$

Hence we can write

$$^t g\, {}^t x^{-1} x^{-1} g = {}^t gmg = t^2 = {}^t tt$$

with some $t \in T^+Y$. Thus we have

$$x \in gt^{-1}K_{\mathbb{C}} \subset G_{\mathbb{R}}T^+YK_{\mathbb{C}}.$$

Converse assertion is clear by the above arguments. q.e.d.

Now consider the full flag manifold X of $G_{\mathbb{C}}$ consisting of full flags

$$V_1 \subset \cdots \subset V_{n-1} \tag{3.3}$$

where $\dim_{\mathbb{C}} V_j = j$. Then such a flag is contained in a closed $K_{\mathbb{C}}$-orbit on X if and only if

$$Q(V_j, V_{n-j}) = \{0\} \quad \text{for all } j.$$

Here $Q(\ ,\ )$ denote the canonical quadratic form on $\mathbb{C}^n$ defined by $Q(z,w) = z_1w_1 + \cdots + z_nw_n$. On the other hand, a flag (3.3) is contained in an open $G_{\mathbb{R}}$-orbit on X if and only if

$$V_j \cap \overline{V_{n-j}} = \{0\} \quad \text{for all } j.$$

Let S be a closed $K_{\mathbb{C}}$-orbit on X. Then every flag (3.3) contained in xS satisfies

$$Q_x(V_j, V_{n-j}) = \{0\} \quad \text{for all } j \tag{3.4}$$

where $Q_x(z,w) = Q(x^{-1}z, x^{-1}w) = {}^t zmw$.

Assume (3.1). Then we can show that xS is contained in an open $G_{\mathbb{R}}$-orbit as follows. Let (3.3) be an element of xS and z an element of $V_j \cap \overline{V_{n-j}}$. Then by (3.4), we have

$$Q_x(z, \overline{z}) = 0.$$

Write $z = c_1v_1 + \cdots + c_nv_n$ with the basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ given in the proof of Proposition 3.1. Then we have

$$0 = Q_x(z, \overline{z}) = {}^t zm\overline{z} = |c_1|^2\lambda_1 + \cdots + |c_n|^2\lambda_n.$$

By (3.2), this implies $c_1 = \cdots = c_n = 0$ and $z = 0$. Hence xS is contained in an open $G_{\mathbb{R}}$-orbit on X.

Let $\mathfrak{j}$ be a maximal torus of $\mathfrak{k} = \mathfrak{o}(n)$. Then the set of closed $K_{\mathbb{C}}$-orbits on X are parametrized by the left coset

$$W_{K_{\mathbb{C}}}(\mathfrak{j})\backslash W_{G_{\mathbb{C}}}(\mathfrak{j})$$

where $W_*(\mathfrak{j}) = N_*(\mathfrak{j})/Z_*(\mathfrak{j})$ ([M1], [R]). First suppose n is odd. Then we have $W_{K_{\mathbb{C}}}(\mathfrak{j}) = W_{G_{\mathbb{C}}}(\mathfrak{j})$ ($\mathrm{B}_{[n/2]}$-type) and hence X has only one closed $K_{\mathbb{C}}$-orbit. Since $Y = Z$ in this case, we have $G_{\mathbb{R}}T^+YK_{\mathbb{C}} = \widetilde{D_0}Z$.

Next suppose that n is even. Then we have $|W_{K_{\mathbb{C}}}(\mathfrak{j})\backslash W_{G_{\mathbb{C}}}(\mathfrak{j})| = 2$ since $W_{K_{\mathbb{C}}}(\mathfrak{j})$ is $\mathrm{D}_{n/2}$-type and $W_{G_{\mathbb{C}}}(\mathfrak{j})$ is $\mathrm{B}_{n/2}$-type. But since $W_{O(n)}(\mathfrak{j}) = W_{G_{\mathbb{C}}}(\mathfrak{j})$ and since y can be replaced by an element

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

of $O(n)$, the two closed $K_{\mathbb{C}}$-orbits on X are written as

$$S \quad \text{and} \quad yS.$$

Since $y^2 \in Z$, we have $Y = Z \sqcup Zy$ and hence

$$G_{\mathbb{R}}T^+YK_{\mathbb{C}} = \widetilde{D_0}Z \sqcup \widetilde{D_0}Zy.$$

Thus we have

$$xS \subset S' \text{ and } xyS \subset yS' \quad \text{if } x \in \widetilde{D_0}Z$$

and

$$xS \subset yS' \text{ and } xyS \subset S' \quad \text{if } x \in \widetilde{D_0}Zy.$$

Conversely if $xS \subset S'$, then we have $xS_0 \subset S_0'$ by a general argument as in Section 8.

By the above arguments, we get:

Proposition 3.2. $x \in \widetilde{D_0}Z \Longleftrightarrow xS \subset S'$ *for a closed* $K_{\mathbb{C}}$*-orbit* S *on* X.

4. $SO(p,q)$-case

Let Q denote the quadratic form on $\mathbb{C}^n$ $(n = p + q)$ defined by

$$Q(z,w) = z_1w_1 + \cdots + z_pw_p - z_{p+1}w_{p+1} - \cdots - z_nw_n$$

and put

$$\begin{gathered}G_{\mathbb{C}} = \{g \in SL(n,\mathbb{C}) \mid Q(gz,gw) = Q(z,w) \text{ for all } z,w \in \mathbb{C}^n\} \cong SO(n,\mathbb{C}),\\ \widetilde{K_{\mathbb{C}}} = \{g \in G_{\mathbb{C}} \mid gV_+ = V_+\} = \{g \in G_{\mathbb{C}} \mid gV_- = V_-\}\\ \text{and } \widetilde{G_{\mathbb{R}}} = G_{\mathbb{C}} \cap SL(n,\mathbb{R}) = SO(p,q)\end{gathered}$$

where $V_+ = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_p$ and $V_- = \mathbb{C}e_{p+1} \oplus \cdots \oplus \mathbb{C}e_n$ with the canonical basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Let $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ denote the connected components of $\widetilde{K_{\mathbb{C}}}$ and $\widetilde{G_{\mathbb{R}}}$, respectively, containing the identity. Let F be the cone in $\mathbb{C}^n$ defined by

$$F = \{z \in \mathbb{C}^n \mid Q(z,z) = 0,\ z \neq 0\}.$$

Then $X_0 = F/\mathbb{C}^\times$ is a flag manifold of $G_{\mathbb{C}}$.

Suppose that $p \geq 3$ and $q \geq 1$. Then

$$S_0 = (V_+ \cap F)/\mathbb{C}^\times$$

is a closed $\widetilde{K_{\mathbb{C}}}$-orbit on X_0. Since S_0 is connected, it is also a closed $K_{\mathbb{C}}$-orbit on X_0. Put

$$S_0' = \{z \in F \mid Q(z,\overline{z}) > 0\}/\mathbb{C}^\times.$$

Then S_0' is an open $\widetilde{G_{\mathbb{R}}}$-orbit ($G_{\mathbb{R}}$-orbit) on X_0 containing S_0.

Let $\ell = \min(p,q)$ and write

$$Y(\theta_1,\ldots,\theta_\ell) = i\theta_1(E_{1,n} + E_{n,1}) + \cdots + i\theta_\ell(E_{\ell,n-\ell+1} + E_{n-\ell+1,\ell})$$

where $E_{j,k}$ are the matrix units. Then

$$\mathfrak{t} = \{Y(\theta_1,\ldots,\theta_\ell) \mid \theta_1,\ldots,\theta_\ell \in \mathbb{R}\}$$

is a maximal abelian subspace of $i\mathfrak{p}$ and the set $\mathfrak{t}^+$ is written as

$$\mathfrak{t}^+ = \{Y(\theta_1,\ldots,\theta_\ell) \mid |\theta_j| + |\theta_k| < \frac{\pi}{2} \text{ for all } j \neq k\}.$$

Put

$$y = \begin{pmatrix} -1 & & 0 \\ & I_{n-2} & \\ 0 & & -1 \end{pmatrix}.$$

Then we have

$$\widetilde{K_{\mathbb{C}}} = K_{\mathbb{C}} \sqcup K_{\mathbb{C}} y \quad \text{and} \quad \widetilde{G_{\mathbb{R}}} = G_{\mathbb{R}} \sqcup G_{\mathbb{R}} y.$$

Since $yT^+y^{-1} = T^+$, we have

$$\widetilde{G_{\mathbb{R}}} T^+ \widetilde{K_{\mathbb{C}}} = G_{\mathbb{R}} T^+ K_{\mathbb{C}} \sqcup G_{\mathbb{R}} T^+ K_{\mathbb{C}} y = \widetilde{D_0} \sqcup \widetilde{D_0} y.$$

Proposition 4.1. *Let x be an element of $G_{\mathbb{C}}$. Then*

$$x \in \widetilde{G_{\mathbb{R}}} T^+ \widetilde{K_{\mathbb{C}}} \Longleftrightarrow x S_0 \subset S_0'.$$

Proof. Suppose that $xS_0 \subset S_0'$. This implies that

$$Q(z, \overline{z}) > 0 \text{ for any } z \in xV_+ \cap F.$$

We will find good basis $v_1, \dots, v_p$ of $V = xV_+$ as follows.

Let $f(z)$ be the restriction of the function $Q(z, \overline{z})$ on the quadratic hypersurface $H = \{z \in V \mid Q(z, z) = 1\}$ in V. Then by the above assumption, $f(z)$ goes to $+\infty$ when z tends to the boundary of H. So $f(z)$ has its minimum at some point v_1 in H. Hence by elementary calculus, we have

$$v \in V,\ Q(v, v_1) = 0 \Longrightarrow \operatorname{Re} Q(v, \overline{v_1}) = 0.$$

Since $Q(v, v_1) = 0$ implies $Q(iv, v_1) = 0$, we also have

$$v \in V,\ Q(v, v_1) = 0 \Longrightarrow \operatorname{Re} Q(iv, \overline{v_1}) = 0.$$

Hence we get

$$v \in V,\ Q(v, v_1) = 0 \Longrightarrow Q(v, \overline{v_1}) = 0.$$

Next consider the orthogonal compliment U_1 of v_1 in V and take a $v_2 \in U_1 \cap H$ by the same procedure. Repeating this procedure, we get basis $v_1, \dots, v_p$ of V such that

$$Q(v_j, v_k) = \delta_{jk} \quad \text{and that} \quad Q(v_j, \overline{v_k}) = 0 \text{ if } j \neq k. \tag{4.1}$$

Write $v_j = u_j + iu_j'$ with $u_j, u_j' \in \mathbb{R}^n$. Then the $2p$-vectors $u_1, \dots, u_p, u_1', \dots, u_p'$ in $\mathbb{R}^n$ are mutually orthogonal by (4.1) and

$$Q(u_j, u_j) - Q(u_j', u_j') = Q(v_j, v_j) = 1,$$

$$Q(u_j, u_j) + Q(u_j', u_j') = Q(v_j, \overline{v_j}).$$

If $Q(v_2, \overline{v_2}) > 1$, then since $Q(v_2, \overline{v_2}) \leq \cdots \leq Q(v_p, \overline{v_p})$, we have

$$Q(u_j, u_j) > 0 \quad \text{and} \quad Q(u_j', u_j') > 0 \quad \text{for } j = 2, \dots, p,$$

a contradiction to the signature of Q. Hence we have

$$Q(v_2, \overline{v_2}) \leq 1.$$

Since $v_1 + iv_2 \in F$, we have

$$0 < Q(v_1 + iv_2, \overline{v_1} - i\overline{v_2}) = Q(v_1, \overline{v_1}) + Q(v_2, \overline{v_2}) \tag{4.2}$$

by the assumption. So we have

$$Q(v_1, \overline{v_1}) > -1$$

and hence

$$Q(u_1, u_1) > 0.$$

Since

$$Q(v_1, \overline{v_1}) \leq \cdots \leq Q(v_p, \overline{v_p}), \tag{4.3}$$

we have

$$0 < Q(u_1, u_1) \leq \cdots \leq Q(u_p, u_p).$$

Hence by the signature of Q, we have $Q(u'_p, u'_p) \leq 0$ and therefore $Q(u_p, u_p) \leq 1$. Hence we can write

$$Q(u_j, u_j) = \cos^2\theta_j \quad \text{and} \quad Q(u'_j, u'_j) = -\sin^2\theta_j$$

with

$$\frac{\pi}{2} > \theta_1 \geq \cdots \geq \theta_p \geq 0 \quad \text{and} \quad \theta_1 + \theta_2 < \frac{\pi}{2} \tag{4.4}$$

by (4.2). If $q < p$, then we have

$$\theta_{q+1} = \cdots = \theta_p = 0$$

by the signature of Q. Clearly we can choose an element g of $\widetilde{G_{\mathbb{R}}} = SO(p, q)$ such that

$$\begin{aligned} gv_1 &= (\cos\theta_1)e_1 + (i\varepsilon\sin\theta_1)e_n \quad \text{and} \\ gv_j &= (\cos\theta_j)e_j + (i\sin\theta_j)e_{n-j+1} \quad \text{for } j = 2, \ldots, p \end{aligned} \tag{4.5}$$

with $\varepsilon = \pm 1$. This implies

$$gxV_+ = gV = \exp Y(\varepsilon\theta_1, \theta_2, \ldots, \theta_\ell)V_+$$

and therefore

$$x \in g^{-1}\exp Y(\varepsilon\theta_1, \theta_2, \ldots, \theta_\ell)\widetilde{K_{\mathbb{C}}} \subset \widetilde{G_{\mathbb{R}}}T^+\widetilde{K_{\mathbb{C}}}.$$

Conversely, suppose that $x \in \widetilde{G_{\mathbb{R}}}T^+\widetilde{K_{\mathbb{C}}}$. Then by the action of the Weyl group $W_{\widetilde{K}}(T)$ (B_ℓ-type or D_ℓ-type, $\widetilde{K} = \widetilde{K_{\mathbb{C}}} \cap \widetilde{G_{\mathbb{R}}}$), we have

$$x \in g^{-1}\exp Y(\varepsilon\theta_1, \theta_2, \ldots, \theta_\ell)\widetilde{K_{\mathbb{C}}}$$

with some $g \in \widetilde{G_{\mathbb{R}}}$ and $\theta_1, \ldots, \theta_p$ satisfying the conditions in (4.4). So we can take basis $v_1, \ldots, v_p$ of $V = xV_+$ given by (4.5) which satisfy the conditions (4.2) and (4.3). Let z be an element of $V \cap F$. Take an element v in the plane spanned by z and v_1 such that $Q(v, v_1) = 0$ and that $Q(v, v) = 1$. Then we can write

$$v = a_2v_2 + \cdots + a_pv_p$$

with $a_j \in \mathbb{C}$ such that

$$a_2^2 + \cdots + a_p^2 = 1.$$

We have

$$\begin{aligned} Q(v,\overline{v}) &= |a_2|^2 Q(v_2,\overline{v_2}) + \cdots + |a_p|^2 Q(v_p,\overline{v_p}) \\ &\geq (|a_2|^2 + \cdots + |a_p|^2) Q(v_2,\overline{v_2}) \\ &\geq Q(v_2,\overline{v_2}) \end{aligned}$$

since $0 < Q(v_2,\overline{v_2}) \leq \cdots \leq Q(v_p,\overline{v_p})$ by (4.2) and (4.3). Since $z = a(v_1 \pm iv)$ for some $a \in \mathbb{C}^\times$, we have

$$\begin{aligned} Q(z,\overline{z}) &= |a|^2 Q(v_1 \pm iv, \overline{v_1 \pm iv}) \\ &= |a|^2 (Q(v_1,\overline{v_1}) + Q(v,\overline{v})) \\ &\geq |a|^2 (Q(v_1,\overline{v_1}) + Q(v_2,\overline{v_2})) \\ &> 0 \end{aligned}$$

by (4.2). q.e.d.

Now let us consider the full flag manifold. The full flag manifold $X = G_{\mathbb{C}}/B$ consists of full flags

$$V_1 \subset \cdots \subset V_{[n/2]}$$

where V_j are j-dimensional complex linear subspaces V_j contained in $F \cup \{0\}$. Such a flag is contained in a closed $\widetilde{K_{\mathbb{C}}}$-orbit if and only if

$$V_j = (V_j \cap V_+) \oplus (V_j \cap V_-) \quad \text{for all } j = 1, \ldots, n'$$

where $n' = [p/2] + [q/2]$. It is clear that

$$p_j = \dim(V_j \cap V_+) \quad \text{and} \quad q_j = \dim(V_j \cap V_-) \quad (j = 1, \ldots, n')$$

are the invariants of closed $\widetilde{K_{\mathbb{C}}}$-orbits. Hence there are $\binom{n'}{[p/2]}$ closed $\widetilde{K_{\mathbb{C}}}$-orbits on X.

On the other hand, a flag is contained in an open $\widetilde{G_{\mathbb{R}}}$-orbit if and only if the hermitian form $Q(z,\overline{z})$ is nondegenerate on each V_j $(j = 1, \ldots, n')$. So the signatures (p_j, q_j) of $Q(z,\overline{z})$ on V_j $(j = 1, \ldots, n')$ are the invariants of open $\widetilde{G_{\mathbb{R}}}$-orbits on X. We can see that closed $\widetilde{K_{\mathbb{C}}}$-orbits and open $\widetilde{G_{\mathbb{R}}}$-orbits on X are in one-to-one correspondence by these numbers (p_j, q_j) $(j = 1, \ldots, n')$.

Suppose that $x \in \widetilde{G_{\mathbb{R}}} T^+ \widetilde{K_{\mathbb{C}}}$. Then we want to show that $xS \subset S'$ for all closed $\widetilde{K_{\mathbb{C}}}$-orbits S on X. By Proposition 4.1, we have

$$Q(z,\overline{z}) > 0 \text{ for any } z \in xV_+ \cap F.$$

We also have

$$Q(z,\overline{z}) < 0 \text{ for any } z \in xV_- \cap F.$$

by the same arguments as in the proof of Proposition 4.1. Let

$$V_1 \subset \cdots \subset V_{[n/2]}$$

be a flag contained in S. Then since $x(V_j \cap V_+) = xV_j \cap xV_+ \subset xV_+ \cap F$, we have

$$Q(z,\overline{z}) > 0 \quad \text{for all } z \in x(V_j \cap V_+).$$

Similarly, we also have

$$Q(z,\overline{z}) < 0 \quad \text{for all } z \in x(V_j \cap V_-).$$

Since $xV_j = x(V_j \cap V_+) \oplus x(V_j \cap V_-)$ for $j = 1, \ldots, n'$, the signature of $Q(z,\overline{z})$ on xV_j is $(\dim(V_j \cap V_+), \dim(V_j \cap V_-))$ for $j = 1, \ldots, n'$. Thus we have proved

$$xS \subset S'.$$

Now let us consider $K_{\mathbb{C}}$-orbits on X. Let $\mathfrak{j}$ be a maximal torus of $\mathfrak{k}$. Then the set of closed $K_{\mathbb{C}}$-orbits and $\widetilde{K_{\mathbb{C}}}$-orbits on X are parametrized by the left cosets

$$W_{K_{\mathbb{C}}}(\mathfrak{j})\backslash W_{G_{\mathbb{C}}}(\mathfrak{j}) \quad \text{and} \quad W_{\widetilde{K_{\mathbb{C}}}}(\mathfrak{j})\backslash W_{G_{\mathbb{C}}}(\mathfrak{j}),$$

respectively, where $W_*(\mathfrak{j}) = N_*(\mathfrak{j})/Z_*(\mathfrak{j})$. First suppose that either p or q is even. Then we can see that $W_{\widetilde{K_{\mathbb{C}}}}(\mathfrak{j})$ is bigger than $W_{K_{\mathbb{C}}}(\mathfrak{j})$. Hence in this case, every closed $\widetilde{K_{\mathbb{C}}}$-orbit S on X is decomosed into two closed $K_{\mathbb{C}}$-orbits as

$$S = S_1 \sqcup yS_1.$$

Since $xS \subset S'$, we see that

$$xS_1 \subset S_1' \quad \text{and} \quad xyS_1 \subset yS_1' \quad \text{if } x \in \widetilde{D_0},$$

and that

$$xS_1 \subset yS_1' \quad \text{and} \quad xyS_1 \subset S_1' \quad \text{if } x \in \widetilde{D_0}y.$$

On the other hand, if p and q are both odd, then we can replace y by the central element $-I_n$. So the $K_{\mathbb{C}}$-orbits and $\widetilde{K_{\mathbb{C}}}$-orbits on X coincide.

Conversely, suppose that $xS_1 \subset S_1'$ for some closed $K_{\mathbb{C}}$-orbit S_1 on X. Let

$$V_1 \subset \cdots \subset V_{[n/2]}$$

be a flag contained in S_1. Then every flag contained in S_1 is of the form

$$kV_1 \subset \cdots \subset kV_{[n/2]}$$

with some $k \in K_{\mathbb{C}}$ and

$$V_1 \subset V_+ \quad \text{or} \quad V_1 \subset V_-.$$

If $V_1 \subset V_+$, then by the property of S_1', we have

$$Q(z,\overline{z}) \text{ is positive definite on } xkV_1 \text{ for every } k \in K_{\mathbb{C}}$$

and therefore

$$Q(z,\overline{z}) > 0 \quad \text{for all } z \in xV_+ \cap F$$

since $V_+ \cap F$ is $K_{\mathbb{C}}$-homogeneous ($p \geq 3$). Hence by Proposition 4.1, we have

$$x \in \widetilde{G_{\mathbb{R}}}T^+\widetilde{K_{\mathbb{C}}}$$

and

$$x \in \widetilde{D_0}Z$$

by the preceding arguments. When $V_1 \subset V_-$ and $q \geq 3$, we also get $x \in \widetilde{D_0}Z$ by the same arguments.

Summarizing the above arguments, we have:

Proposition 4.2. (1) *Suppose $p \neq 2$ and $q \neq 2$. Then the following three conditions on $x \in G_{\mathbb{C}}$ are equivalent:*

(a) $x \in \widetilde{D_0} Z$.

(b) *$xS \subset S'$ for a closed $K_{\mathbb{C}}$-orbit S on X.*

(c) *$xS \subset S'$ for all closed $K_{\mathbb{C}}$-orbits S on X.*

(2) *Suppose $p \geq 3$ and $q = 2$. Then the following three conditions on $x \in G_{\mathbb{C}}$ are equivalent:*

(a) $x \in \widetilde{D_0} Z$.

(b) *$xS \subset S'$ for a closed $K_{\mathbb{C}}$-orbit S on X consisting of flags $V_1 \subset \cdots \subset V_{[n/2]}$ such that $V_1 \subset V_+$.*

(c) *$xS \subset S'$ for all closed $K_{\mathbb{C}}$-orbits S on X.*

Remark 4.3. When $p \geq 3$ and $q = 2$, $G_{\mathbb{R}}$ is a Lie group of Hermitian type. Open $G_{\mathbb{R}}$-orbits S' on X in the above condition (b) are called "nonholomorphic type" in [WZ]. On the other hand, the two open $G_{\mathbb{R}}$-orbits S'_1 and S'_2 on X which we considered in Section 2 are called "holomorphic type".

5. Cases of complex classical Lie groups

Let G be a connected complex simple Lie group and $G_{\mathbb{C}} = G \times G$. Take a compact real form K of G and let τ be the conjugation of G with respect to K. Then

$$G_{\mathbb{R}} = \{(g, \tau(g)) \mid g \in G\} \cong G$$

is a real form of $G_{\mathbb{C}}$. We can take

$$K_{\mathbb{C}} = \{(g, g) \mid g \in G\} \cong G.$$

Let $X = G/B$ be the full flag manifold of G. Then by the correspondence $(g_1, g_2) \mapsto g_1^{-1} g_2$, we have a natural bijection

$$K_{\mathbb{C}} \backslash G_{\mathbb{C}} / B \times B \cong B \backslash G / B.$$

Since the right hand side has only one closed double coset $B = BeB$ by the Bruhat decomposition, the full flag manifold $X \times X$ of $G_{\mathbb{C}}$ has only one closed $K_{\mathbb{C}}$-orbit

$$S = \{(m, m) \mid m \in X\}.$$

When G is classical, we can take a minimal flag manifold $X_0 = F/\mathbb{C}^\times$ of G where F is given as follows.

G	F
$SL(n, \mathbb{C})$	$\mathbb{C}^n - \{0\}$
$SO(n, \mathbb{C})$	$\{z \in \mathbb{C}^n - \{0\} \mid z_1^2 + \cdots + z_n^2 = 0\}$
$Sp(n, \mathbb{C})$	$\mathbb{C}^{2n} - \{0\}$

If we write $X_0 = G/P$, then we have a natural bijection

$$K_{\mathbb{C}} \backslash G_{\mathbb{C}} / P \times P \cong P \backslash G / P \cong W_P \backslash W / W_P.$$

When W is $*_n$-type ($*=$ A, B, C or D), we can see that W_P is $*_{n-1}$-type. Hence we have

$$|W_P\backslash W/W_P| = \begin{cases} 2 & \text{if } G=SL(n,\mathbb{C}), \\ 3 & \text{if } G=SO(n,\mathbb{C})\ (n\ge 5) \text{ or } G=Sp(n,\mathbb{C})\ (n\ge 2), \\ 4 & \text{if } G=SO(4,\mathbb{C}). \end{cases}$$

The flag manifold $X_0\times X_0$ of $G_\mathbb{C}$ has only one closed $K_\mathbb{C}$-orbit

$$S_0=\{(m,m)\mid m\in X_0\}=p\{(z,z)\mid z\in F\}$$

where $p: F\times F\to X_0\times X_0$ is the canonical projection. When $G=SO(n,\mathbb{C})$ with $n\ge 5$, the $K_\mathbb{C}$-orbit decomposition of $X_0\times X_0$ is

$$X_0\times X_0=S_0\sqcup\{p(z,w)\mid \mathbb{C}z\ne\mathbb{C}w,\ Q(z,w)=0\}\sqcup\{p(z,w)\mid Q(z,w)\ne 0\}$$

where $Q(z,w)=z_1w_1+\cdots+z_nw_n$. When $G=Sp(n,\mathbb{C})$ with $n\ge 2$, the $K_\mathbb{C}$-orbit decomposition of $X_0\times X_0$ is

$$X_0\times X_0=S_0\sqcup\{p(z,w)\mid \mathbb{C}z\ne\mathbb{C}w,\ {}^tzJw=0\}\sqcup\{p(z,w)\mid {}^tzJw\ne 0\}$$

where J is the canonical alternating matrix defining G.

We will prove the following proposition in this section.

Proposition 5.1. *Let $G=SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$. Then the following three conditions on $x\in G_\mathbb{C}$ are equivalent:*
(i) $x\in\widetilde{D_0}Z$, (ii) $xS_0\subset S_0'$, (iii) $xS\subset S'$.
(Remark. (iii) implies (ii) by a general argument as in Section 8. Since $SO(3,\mathbb{C})$ and $SO(4,\mathbb{C})$ are locally isomorphic to $SL(2,\mathbb{C})$ and $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$, respectively, we may assume $n\ge 5$ for $SO(n,\mathbb{C})$-case.)

Proof. Since $G=SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$, we can take

$$\tau(g)={}^t\overline{g}^{-1}\quad\text{for } g\in G$$

as usual. Then the open $G_\mathbb{R}$-orbit on $X_0\times X_0$ is

$$S_0'=p\{(w,z)\in F\times F\mid {}^t\overline{z}w\ne 0\}.$$

The space $G_\mathbb{C}/K_\mathbb{C}$ is identified with G by the map

$$\varphi:(g,h)K_\mathbb{C}\mapsto gh^{-1}.$$

By this identification, the left $G_\mathbb{R}$-action on $G_\mathbb{C}/K_\mathbb{C}$ corresponds to the $G_\mathbb{R}$-action on G given by

$$((h,{}^t\overline{h}^{-1}),g)\mapsto hg\,{}^t\overline{h}. \tag{5.1}$$

Write $x=(g_1,g_2)$ with $g_1,g_2\in G$. Then we have

$$xS_0=(g_1,g_2)S_0=(g,e)S_0=p\{(gz,z)\mid z\in F\}$$

where $g=\varphi(x)=g_1g_2^{-1}$. So the condition $xS_0\subset S_0'$ implies that

$${}^t\overline{z}gz\ne 0\quad\text{for all } z\in F. \tag{5.2}$$

A. $SL(n,\mathbb{C})$**-case**

We can take

$$\mathfrak{t}^+ = \left\{ \left(\begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{pmatrix}, -\begin{pmatrix} i\theta_1 & & 0 \\ & \ddots & \\ 0 & & i\theta_n \end{pmatrix} \right) \;\middle|\; |\theta_j - \theta_k| < \frac{\pi}{2} \text{ for all } j \text{ and } k \right\}.$$

(ii) $\Longrightarrow$ (i). Assume (5.2). Then we can diagonalize g by the $G_{\mathbb{R}}$-action (5.1) by an argument similar to Proposition 3.1 as follows. Since $F = \mathbb{C}^n - \{0\}$ is simply connected, we can define a smooth function $\xi : F \to \mathbb{R}$ by

$$\xi(z) = \arg({}^t\overline{z}gz)$$

for $z \in F$. Then the image R of ξ is a bounded closed interval in $\mathbb{R}$ since $X_0 = P^{n-1}(\mathbb{C}) = F/\mathbb{C}^\times$ is compact. Take v_1 so that $\xi(v_1)$ is on the boundary of R and that $|{}^t\overline{v_1}gv_1| = 1$. Let u be an element of $\mathbb{C}^n$ and consider the function

$$\eta(t) = \xi(v_1 + tu) = \arg({}^t\overline{v_1}gv_1 + t({}^t\overline{v_1}gu + {}^t\overline{u}gv_1) + t^2({}^t\overline{u}gu))$$

of $t \in \mathbb{R}$. Then we have

$$(5.3) \qquad {}^t\overline{v_1}gu + {}^t\overline{u}gv_1 \in \mathbb{R}\,{}^t\overline{v_1}gv_1 \quad \text{for all } u \in \mathbb{C}^n$$

since $\eta(0) = \xi(v_1)$ is on the boundary of R. Put

$$U_1 = \{u \in \mathbb{C}^n \mid {}^t\overline{u}gv_1 = 0\}.$$

Then U_1 is an $n-1$-dimensional complex subspace of $\mathbb{C}^n$ which does not contain v_1. By (5.3), we have

$${}^t\overline{v_1}gU_1 \subset \mathbb{R}\,{}^t\overline{v_1}gv_1.$$

Hence we have

$${}^t\overline{v_1}gU_1 = \{0\}$$

because the left hand side is a complex subspace of $\mathbb{C}$.

Next consider the restriction of ξ on $U_1 - \{0\}$ and take a $v_2 \in U_1 - \{0\}$ such that $\xi(v_2)$ is on the boundary of $\xi(U_1 - \{0\})$ and that $|{}^t\overline{v_2}gv_2| = 1$. Repeating this procedure, we can take basis $v_1, \dots, v_n$ of $\mathbb{C}^n$ such that

$${}^t\overline{v_j}gv_k = 0 \quad \text{if } j \neq k$$

and that $|{}^t\overline{v_j}gv_j| = 1$ for $j = 1, \dots, n$. Hence if we put $h = (v_1 \cdots v_n) \in G$, then we have

$${}^t\overline{h}gh = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_j = {}^t\overline{v_j}gv_j \in U(1)$. By (5.2), $\lambda_1, \dots, \lambda_n$ satisfy the condition in (3.2). Hence we have

$${}^t\overline{h}gh \in \varphi(t)Z' = \varphi(tZ)$$

with some $t \in T^+$. Here $Z' = \varphi(Z)$ is the center of G. This implies that

$$x = (g_1, g_2) \in ({}^t\overline{h}^{-1}, h)tZK_{\mathbb{C}} \subset G_{\mathbb{R}}T^+K_{\mathbb{C}}Z = \widetilde{D_0}Z.$$

(i) $\Longrightarrow$ (ii) is clear by the above arguments.

(ii) $\Longrightarrow$ (iii). Elements of the full flag manifold $X \times X$ of $G_{\mathbb{C}}$ are pairs of sequences

$$(V_1 \subset \cdots \subset V_{n-1},\ V_1' \subset \cdots \subset V_{n-1}') \tag{5.4}$$

where V_j and V_j' are j-dimensional complex subspaces of $\mathbb{C}^n$. The open $G_{\mathbb{R}}$-orbit S' on $X \times X$ consists of flags (5.4) such that the canonical hermitian form is nondegenerate on the pair (V_j, V_j') for every $j = 1, \ldots, n-1$. (i.e. For every $v \in V_j$, there exists a $u \in V_j'$ such that ${}^t\overline{u}v \neq 0$.) For $x = (g_1, g_2) \in G_{\mathbb{C}}$, we can see that the set xS consists of flags of the form

$$(gV_1 \subset \cdots \subset gV_{n-1},\ V_1 \subset \cdots \subset V_{n-1})$$

$(g = \varphi(x) = g_1 g_2^{-1})$.

Suppose that $xS_0 \subset S_0'$. Then by (5.2), ${}^t\overline{z}gz \neq 0$ for every $z \in \mathbb{C}^n - \{0\}$. So the canonical hermitian form is clearly nondegenerate on the pair (gV_j, V_j) for $j = 1, \ldots, n-1$. Hence we have $xS \subset S'$.

B. $SO(n, \mathbb{C})$**-case** $(n \geq 5)$

We can take

$$\mathfrak{t}^+ = \{(Y(\theta_1, \ldots, \theta_m), -Y(\theta_1, \ldots, \theta_m)) \mid |\theta_j| + |\theta_k| < \frac{\pi}{2} \text{ if } j \neq k\}.$$

Here

$$Y(\theta_1, \ldots, \theta_m) = \begin{cases} \begin{pmatrix} \theta_1 J_1 & & 0 \\ & \ddots & \\ 0 & & \theta_m J_1 \end{pmatrix} & \text{if } n = 2m, \\ \begin{pmatrix} \theta_1 J_1 & & & 0 \\ & \ddots & & \\ & & \theta_m J_1 & \\ 0 & & & 1 \end{pmatrix} & \text{if } n = 2m+1 \end{cases}$$

with

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let Q denote the canonical quadratic form on $\mathbb{C}^n$ defined by

$$Q(z, w) = z_1 w_1 + \cdots + z_n w_n.$$

Then (5.2) is written as

$$Q(gz, \overline{z}) \neq 0 \quad \text{for all } z \in F. \tag{5.2$'$}$$

(ii) $\Longrightarrow$ (i). Assume (5.2'). Since $F = \{z \in \mathbb{C}^n - \{0\} \mid Q(z, z) = 0\}$ is simply connected, we can define a smooth function $\xi_g : F \to \mathbb{R}$ by

$$\xi_g(z) = \arg Q(gz, \overline{z}) \quad \text{for } z \in F.$$

Since $X_0 = F/\mathbb{C}^\times$ is compact, the image R of ξ_g is a bounded closed interval in $\mathbb{R}$. If $g' = \overline{h}^{-1} gh$ with an $h \in G$, then we can put

$$\xi_{g'}(z) = \xi_g(hz) \tag{5.5}$$

because $Q(g'z, \overline{z}) = Q(\overline{h}^{-1} ghz, \overline{z}) = Q(ghz, \overline{hz})$.

Let v be an element of F such that $\xi_g(v) = \theta_1$ is on the boundary of R. Replacing g by a twisted conjugate $\overline{h_0}^{-1} gh_0$ with an element $h_0 \in G$ such that $v = h_0(e_1 + ie_2)$, we may assume

$$v = e_1 + ie_2,$$

by (5.5), where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Since $Q(v, \overline{v}) = 2 \neq 0$ and since $Q(gv, \overline{v}) \neq 0$, two points $(\mathbb{C}v, \mathbb{C}\overline{v})$ and $(\mathbb{C}gv, \mathbb{C}\overline{v})$ are in the open $K_{\mathbb{C}}$-orbit on $X_0 \times X_0$. So we can take an $h \in G$ such that

$$\overline{h}\mathbb{C}\overline{v} = \mathbb{C}\overline{v} \quad \text{and that} \quad \overline{h}\mathbb{C}v = \mathbb{C}gv.$$

Hence

$$\overline{h}^{-1} ghv \in \overline{h}^{-1} g\mathbb{C}v = \mathbb{C}v$$

and so we can write

$$\overline{h}^{-1} ghv = \mu v$$

with some $\mu \in \mathbb{C}^\times$. Take an element h' of G such that

$$h'v = |\mu|^{-1/2} v \quad \text{and that} \quad h'\overline{v} = |\mu|^{1/2}\overline{v}.$$

Then we have

$$\overline{h'}^{-1}\overline{h}^{-1} ghh'v = \frac{\mu}{|\mu|} v.$$

Rewrite $\overline{h'}^{-1}\overline{h}^{-1} ghh'$ as g. Then we have

$$\xi_g(v) = \theta_1$$

by (5.5) since $hh'v \in \mathbb{C}v$. Hence we have

$$gv = \frac{\mu}{|\mu|} v = e^{i\theta_1} v. \tag{5.6}$$

Let u be an element of $U_1 = \mathbb{C}e_3 \oplus \cdots \oplus \mathbb{C}e_n$ and consider the function

$$\eta(t) = \xi_g(v + tu) = \arg(Q(gv, \overline{v}) + tQ(gv, \overline{u}) + tQ(gu, \overline{v}) + t^2 Q(gu, \overline{u}))$$

of $t \in \mathbb{R}$. Then we have

$$\frac{d\eta}{dt}(0) = 0$$

since $\eta(0) = \xi_g(v)$ is on the boundary of R and since the line $v + tu$ is tangent to F at $t = 0$. Since $gv = e^{i\theta_1} v$, we have

$$Q(gv, \overline{u}) = 0.$$

Hence

$$Q(gu, \overline{v}) \in \mathbb{R}Q(gv, \overline{v})$$

for all $u \in U_1$. Since U_1 is a complex subspace of $\mathbb{C}^n$, this implies

$$Q(gu, \overline{v}) = 0$$

and therefore

$$g^{-1}\overline{v} \in \mathbb{C}v \oplus \mathbb{C}\overline{v}.$$

Since $Q(v, g^{-1}\overline{v}) = Q(gv, \overline{v}) \neq 0$ and since $g^{-1}\overline{v} \in F$, we have

$$g^{-1}\overline{v} \in \mathbb{C}\overline{v}.$$

Hence

$$g\overline{v} = \beta\overline{v}$$

with some $\beta \in \mathbb{C}^\times$. Since $2e^{i\theta_1}\beta = Q(gv, g\overline{v}) = Q(v, \overline{v}) = 2$, we have $\beta = e^{-i\theta_1}$. Thus we get

$$g\overline{v} = e^{-i\theta_1}\overline{v}. \tag{5.7}$$

By (5.6) and (5.7), g is of the form

$$\begin{pmatrix} B_{\theta_1} & 0 \\ 0 & g' \end{pmatrix}$$

with $g' \in SO(n-2, \mathbb{C})$ where

$$B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

So by induction, we can show that a twisted conjugate g_0 of g is of the form

$$g_0 = a(\theta_1, \dots, \theta_m) = \begin{cases} \begin{pmatrix} B_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & B_{\theta_m} \end{pmatrix} & \text{if } n = 2m, \\ \begin{pmatrix} B_{\theta_1} & & & 0 \\ & \ddots & & \\ & & B_{\theta_m} & \\ 0 & & & 1 \end{pmatrix} & \text{if } n = 2m+1. \end{cases}$$

By the action of $K = SO(n)$, we may assume

$$\pi \geq |\theta_1| \geq \cdots \geq |\theta_m| \geq 0.$$

First consider the case where n is odd. Suppose $|\theta_1| + |\theta_2| \geq \pi$. Then we have

$$Q(g_0(e_1 + i\alpha e_3 + i\beta e_n), e_1 - i\alpha e_3 - i\beta e_n) = \cos\theta_1 + \alpha^2\cos\theta_2 + \beta^2 = 0$$

for some $(\alpha, \beta) \in [0, 1] \times [0, 1]$ satisfying $\alpha^2 + \beta^2 = 1$ because

$$\cos\theta_1 + \cos\theta_2 \leq 0 \quad \text{and} \quad \cos\theta_1 + 1 \geq 0.$$

Since $e_1 + i\alpha e_3 + i\beta e_n \in F$, this contradicts to (5.2'). Hence

$$|\theta_1| + |\theta_2| < \pi$$

and so we have

$$g_0 \in \varphi(T^+),$$

which implies (i).

Next consider the case where n is even ($n = 2m \geq 6$). Suppose $|\theta_1| + |\theta_m| \leq \pi$. Then we can show $|\theta_1| + |\theta_2| < \pi$ as follows. If $|\theta_1| + |\theta_2| \geq \pi$, then we have

$$Q(g_0(e_1 + i\alpha e_3 + i\beta e_{2m-1}), e_1 - i\alpha e_3 - i\beta e_{2m-1}) = \cos\theta_1 + \alpha^2\cos\theta_2 + \beta^2\cos\theta_m = 0$$

for some $(\alpha, \beta) \in [0,1] \times [0,1]$ satisfying $\alpha^2 + \beta^2 = 1$ because

$$\cos\theta_1 + \cos\theta_2 \leq 0 \quad \text{and} \quad \cos\theta_1 + \cos\theta_m \geq 0.$$

Since $e_1 + i\alpha e_3 + i\beta e_{2m-1} \in F$, this contradicts to (5.2'). Hence

$$|\theta_1| + |\theta_2| < \pi$$

and so we have $g_0 \in \varphi(T^+)$.

On the other hand, suppose $|\theta_1| + |\theta_m| \geq \pi$. Then we can show $|\theta_{m-1}| + |\theta_m| > \pi$ by the same argument as above. Hence we have

$$g_0 = -a(\theta_1 + \pi, \ldots, \theta_m + \pi) \in -\varphi(T^+) \subset \varphi(T^+Z).$$

(i) $\Longrightarrow$ (ii). Let $g = \varepsilon a(\theta_1, \ldots, \theta_m)$ with

$$\varepsilon = \pm 1, \quad |\theta_j| + |\theta_k| < \pi \text{ if } j \neq k.$$

Then we have only to show (5.2'). Clearly we may assume $\varepsilon = 1$. By the action of $K = SO(n)$, we may assume

$$|\theta_1| \geq \cdots \geq |\theta_m| \geq 0.$$

Let $z = z_1 e_1 + \cdots + z_n e_n$ be an element of F. Then we have

$$\begin{aligned} Q(gz, \overline{z}) &= \sum_{j=1}^{m}((z_{2j-1}\cos\theta_j + z_{2j}\sin\theta_j)\overline{z_{2j-1}} + (-z_{2j-1}\sin\theta_j + z_{2j}\cos\theta_j)\overline{z_{2j}}) + \delta z_n\overline{z_n} \\ &= \sum_{j=1}^{m}(|z_{2j-1}|^2 + |z_{2j}|^2)\cos\theta_j + \delta|z_n|^2 + 2i\sum_{j=1}^{m}\mathrm{Im}(\overline{z_{2j-1}}z_{2j})\sin\theta_j \end{aligned}$$

where $\delta = n - 2m$.

If $|\theta_1| < \pi/2$, then $\mathrm{Re}\, Q(gz, \overline{z}) > 0$. So we may assume that $|\theta_1| \geq \pi/2$.

Suppose $\mathrm{Im}\, Q(gz, \overline{z}) = 0$. Since $|\sin\theta_1| > |\sin\theta_2| \geq \cdots \geq |\sin\theta_m| \geq 0$, we have

$$|\mathrm{Im}(\overline{z_1}z_2)| \leq |\mathrm{Im}(\overline{z_3}z_4)| + \cdots + |\mathrm{Im}(\overline{z_{2m-1}}z_{2m})|.$$

On the other hand, since $z \in F$, we have

$$|z_1^2 + z_2^2| = |z_3^2 + \cdots + z_n^2| \leq |z_3^2 + z_4^2| + \cdots + |z_{2m-1}^2 + z_{2m}^2| + \delta|z_n^2|.$$

Since $|z_{2j-1}^2 + z_{2j}^2|^2 + 4|\mathrm{Im}(\overline{z_{2j-1}}z_{2j})|^2 = (|z_{2j-1}|^2 + |z_{2j}|^2)^2$, we get

$$|z_1|^2 + |z_2|^2 \leq |z_3|^2 + \cdots + |z_n|^2.$$

Hence

$$\begin{aligned} \mathrm{Re}\, Q(gz, \overline{z}) &\geq (|z_1|^2 + |z_2|^2)\cos\theta_1 + (|z_3|^2 + \cdots + |z_n|^2)\cos\theta_2 \\ &= (|z_1|^2 + |z_2|^2)(\cos\theta_1 + \cos\theta_2) + (|z_3|^2 + \cdots + |z_n|^2 - |z_1|^2 - |z_2|^2)\cos\theta_2 \\ &> 0 \end{aligned}$$

because $\cos\theta_2 \geq \cos\theta_1 + \cos\theta_2 > 0$ and $z \neq 0$. Thus the assertion is proved.

(ii) $\Longrightarrow$ (iii). Elements of the full flag manifold $X\times X$ of $G_{\mathbb{C}}$ are pairs of sequences

$$(V_1\subset\cdots\subset V_{[n/2]},\ V_1'\subset\cdots\subset V_{[n/2]}') \tag{5.8}$$

where V_j and V_j' are j-dimensional complex subspaces contained in $F\cup\{0\}$. The open $G_{\mathbb{R}}$-orbit S' on $X\times X$ consists of flags (5.8) such that the canonical hermitian form is nondegenerate on the pair (V_j,V_j') for every $j=1,\ldots,[n/2]$. So we can prove (ii) $\Longrightarrow$ (iii) by the same argument as in the case of $SL(n,\mathbb{C})$.

C. $Sp(n,\mathbb{C})$**-case**

Define $G=Sp(n,\mathbb{C})$ by

$$G=\{g\in GL(2n,\mathbb{C})\mid {}^tgJg=J\}$$

where

$$J=\begin{pmatrix}J_1 & & 0\\ & \ddots & \\ 0 & & J_1\end{pmatrix}\quad\text{and}\quad J_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Then we can take

$$\mathfrak{t}^+=\{(Y(\theta_1,\ldots,\theta_n),-Y(\theta_1,\ldots,\theta_n))\mid |\theta_j|<\frac{\pi}{4}\text{ for }j=1,\ldots,n\}$$

where

$$Y(\theta_1,\ldots,\theta_n)=\begin{pmatrix}\theta_1 & & & & 0\\ & -\theta_1 & & & \\ & & \ddots & & \\ & & & \theta_n & \\ 0 & & & & -\theta_n\end{pmatrix}.$$

(ii) $\Longrightarrow$ (i). Assume (5.2). Since $F=\mathbb{C}^{2n}-\{0\}$ is simply connected, we can define a smooth function $\xi_g:F\to\mathbb{R}$ by

$$\xi_g(z)=\arg{}^t\overline{z}gz\quad\text{for }z\in F.$$

Since $X_0=F/\mathbb{C}^\times$ is compact, the image R of ξ_g is a bounded closed interval in $\mathbb{R}$. If $g'={}^t\overline{h}gh$ with an $h\in G$, then we can put

$$\xi_{g'}(z)=\xi_g(hz). \tag{5.9}$$

Let v be an element of F such that $\xi_g(v)=\theta_1$ is on the boundary of R. Replacing g by a conjugate ${}^t\overline{h_0}gh_0$ with an element $h_0\in G$ such that $v=h_0e_1$, we may assume

$$v=e_1$$

by (5.9).

Since ${}^te_2Je_1=-1\neq 0$ and since

$${}^te_2Jge_1=-{}^te_1ge_1=-{}^t\overline{e_1}ge_1\neq 0,$$

two points $(\mathbb{C}e_1,\mathbb{C}e_2)$ and $(\mathbb{C}ge_1,\mathbb{C}e_2)$ are in the open $K_{\mathbb{C}}$-orbit on $X_0\times X_0$. So we can take an $h\in G$ such that

$${}^t\overline{h}\mathbb{C}e_2=\mathbb{C}e_2\quad\text{and that}\quad{}^t\overline{h}\mathbb{C}ge_1=\mathbb{C}e_1.$$

Hence we have

$$he_1 = J^{-1}{}^t h^{-1} J e_1 = -J^{-1}{}^t h^{-1} e_2 \in J^{-1}\mathbb{C}e_2 = \mathbb{C}e_1$$

and therefore

$${}^t\overline{h} g h e_1 \in {}^t\overline{h} g \mathbb{C}e_1 = \mathbb{C}e_1.$$

Write ${}^t\overline{h} g h e_1 = \mu e_1$ with $\mu \in \mathbb{C}^\times$ and take an element

$$h' = \begin{pmatrix} |\mu|^{-1/2} & & 0 \\ & |\mu|^{1/2} & \\ 0 & & I_{2n-2} \end{pmatrix}$$

of G. Then we have

$${}^t\overline{h'}\,{}^t\overline{h} g h h' e_1 = \frac{\mu}{|\mu|} e_1.$$

Rewrite ${}^t\overline{h'}\,{}^t\overline{h} g h h'$ as g. Then we have

$$\xi_g(e_1) = \theta_1$$

by (5.9) since $hh'e_1 \in \mathbb{C}e_1$. Hence we have

$$ge_1 = \frac{\mu}{|\mu|} e_1 = e^{i\theta_1} e_1. \tag{5.10}$$

Let u be an element of $U_1 = \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{2n}$ and consider the function

$$\eta(t) = \xi_g(e_1 + tu) = \arg({}^t\overline{e_1} g e_1 + t\,{}^t\overline{u} g e_1 + t\,{}^t\overline{e_1} g u + t^2\,{}^t\overline{u} g u)$$

of $t \in \mathbb{R}$. Then we have

$$\frac{d\eta}{dt}(0) = 0$$

since $\eta(0) = \xi_g(e_1)$ is on the boundary of R. Since $ge_1 = e^{i\theta_1} e_1$, we have

$${}^t\overline{u} g e_1 = 0.$$

Hence

$${}^t\overline{e_1} g u \in \mathbb{R}\,{}^t\overline{e_1} g e_1$$

for all $u \in U_1$. Since U_1 is a complex subspace of $\mathbb{C}^n$, this implies

$${}^t\overline{e_1} g u = 0$$

and therefore

$${}^t g e_1 \in \mathbb{C}e_1.$$

Hence we have

$$g^{-1} e_2 = J^{-1}{}^t g J e_2 = J^{-1}{}^t g e_1 \in J^{-1}\mathbb{C}e_1 = \mathbb{C}e_2.$$

So we can write $ge_2 = \beta e_2$ and we can compute β as

$$\beta = {}^t e_1 J g e_2 = {}^t e_1\,{}^t g^{-1} J e_2 = {}^t(g^{-1} e_1) J e_2 = e^{-i\theta_1}.$$

Thus we have

$$ge_2 = e^{-i\theta_1} e_2. \tag{5.11}$$

By (5.10) and (5.11), g is of the form

$$\begin{pmatrix} e^{i\theta_1} & & 0 \\ & e^{-i\theta_1} & \\ 0 & & g' \end{pmatrix}$$

with $g' \in Sp(n-1,\mathbb{C})$. So by induction, we can show that g is conjugate to a diagonal matrix

$$g_0 = a(\theta_1,\dots,\theta_n) = \exp Y(\theta_1,\dots,\theta_n).$$

Since

$${}^t\overline{z}g_0z = e^{i\theta_1}|z_1|^2 + e^{-i\theta_1}|z_2|^2 + \cdots + e^{i\theta_n}|z_{2n-1}|^2 + e^{-i\theta_n}|z_{2n}|^2,$$

it is clear that ${}^t\overline{z}g_0z \neq 0$ for all $z \in F$ if and only if

$$\operatorname{Re} e^{i\theta_1},\dots,\operatorname{Re} e^{i\theta_n} \text{ are all positive or all negative.}$$

Hence it is equivalent to

$$g_0 \in \varphi(T^+Z).$$

(i) $\Longrightarrow$ (ii) is included in the above argument.

(ii) $\Longrightarrow$ (iii). Elements of the full flag manifold $X \times X$ of $G_\mathbb{C}$ are pairs of sequences

$$(V_1 \subset \cdots \subset V_n,\ V_1' \subset \cdots \subset V_n') \tag{5.12}$$

where V_j and V_j' are j-dimensional complex subspaces on which the canonical symplectic form for $G = Sp(n,\mathbb{C})$ vanish. The open $G_\mathbb{R}$-orbit S' on $X \times X$ consists of flags (5.12) such that the canonical hermtian form is nondegenerate on the pair (V_j, V_j') for every $j = 1,\dots,n$. So we can prove (ii) $\Longrightarrow$ (iii) by the same argument as in the case of $SL(n,\mathbb{C})$. q.e.d.

6. $GL(n,\mathbb{H})$-case

Let $G_\mathbb{C} = SL(2n,\mathbb{C})$, $K_\mathbb{C} = \{g \in G_\mathbb{C} \mid {}^tgJg = J\} \cong Sp(n,\mathbb{C})$ and $G_\mathbb{R} = \{g \in G_\mathbb{C} \mid J\overline{g}J^{-1} = g\}$ ($\cong GL(n,\mathbb{H})$ modulo center). Here

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_1 \end{pmatrix} \quad \text{and} \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a flag manifold X_0, we consider the Grassmann manifold

$$X_0 = \{V \subset \mathbb{C}^{2n} \mid \dim_\mathbb{C} V = 2\}.$$

(Note that $P^{2n-1}(\mathbb{C})$ is $K_\mathbb{C}$-homogeneous.) There are two $K_\mathbb{C}$-orbits and two $G_\mathbb{R}$-orbits on X_0. The closed $K_\mathbb{C}$-orbit S_0 on X_0 is

$$S_0 = \{V \in X_0 \mid {}^tuJv = 0 \text{ for all } u,v \in V\}$$

and the open $G_\mathbb{R}$-orbit S_0' on X_0 is

$$S_0' = \{V \in X_0 \mid J\overline{V} \cap V = \{0\}\}.$$

We can take

$$\mathfrak{t}^+ = \left\{ \begin{pmatrix} i\theta_1 & & & & 0 \\ & i\theta_1 & & & \\ & & \ddots & & \\ & & & i\theta_n & \\ 0 & & & & i\theta_n \end{pmatrix} \in \mathfrak{sl}(2n,\mathbb{C}) \;\middle|\; |\theta_j - \theta_k| < \frac{\pi}{2} \text{ for all } j \text{ and } k \right\}.$$

Proposition 6.1. $xS_0 \subset S_0' \Longleftrightarrow x \in \widetilde{D_0}Z$.

Proof. Suppose that $V \in S_0$ and $xV \in X_0 - S_0'$ (= the closed $G_{\mathbb{R}}$-orbit). Note that

$$xV \in X_0 - S_0' \Longleftrightarrow J\overline{xV} = xV \Longleftrightarrow xV = \mathbb{C}v \oplus \mathbb{C}J\overline{v} \text{ for } v \in xV - \{0\}.$$

Since $x^{-1}v,\ x^{-1}J\overline{v} \in V \in S_0$, we have

$${}^t(x^{-1}J\overline{v})Jx^{-1}v = {}^t\overline{v}\,{}^tJ\,{}^tx^{-1}Jx^{-1}v = 0.$$

Hence the condition $xS_0 \subset S_0'$ is equivalent to

$$(6.1) \qquad {}^t\overline{v}gv \neq 0 \quad \text{for all } v \in \mathbb{C}^{2n} - \{0\}$$

where $g = {}^tJ\,{}^tx^{-1}Jx^{-1}$. Note that gJ and ${}^tJg$ are skew symmetric.

Assume (6.1). Since $\mathbb{C}^{2n} - \{0\}$ is simply connected, we can define a smooth function

$$\xi(v) = \arg {}^t\overline{v}gv$$

of $v \in \mathbb{C}^{2n} - \{0\}$. The range R of ξ is a bounded closed interval in $\mathbb{R}$ since $P^{2n-1}(\mathbb{C}) = (\mathbb{C}^{2n} - \{0\})/\mathbb{C}^\times$ is compact. Take a $v_1 \in \mathbb{C}^{2n} - \{0\}$ such that $\xi(v_1)$ is on the boudary of R and that ${}^t\overline{v_1}gv_1 = \lambda_1 \in U(1)$. Put

$$U_1 = \{u \in \mathbb{C}^{2n} \mid {}^t\overline{v_1}gu = {}^t\overline{v_1}gJ\overline{u} = 0\}.$$

Then U_1 is a complex subspace of $\mathbb{C}^{2n}$ such that

$$(6.2) \qquad J\overline{U_1} = U_1.$$

Since ${}^t\overline{v_1}gv_1 \neq 0$ by (6.1), U_1 is a $2n-2$-dimensional complex subspace of $\mathbb{C}^{2n}$ satisfying

$$U_1 \cap (\mathbb{C}v_1 \oplus \mathbb{C}J\overline{v_1}) = \{0\}.$$

Let u be an element of U_1 and define a function

$$\eta(t) = \xi(v_1 + tu) = \arg({}^t\overline{v_1}gv_1 + t\,{}^t\overline{v_1}gu + t\,{}^t\overline{u}gv_1 + t^2\,{}^t\overline{u}gu)$$

of $t \in \mathbb{R}$. Since $\eta(0) = \xi(v_1)$ is on the boundary of R and since ${}^t\overline{v_1}gu = 0$, we have

$${}^t\overline{u}gv_1 \in \mathbb{R}\,{}^t\overline{v_1}gv_1$$

for all $u \in U_1$. Since U_1 is a complex subspace of $\mathbb{C}^{2n}$, this implies

$${}^t\overline{u}gv_1 = 0.$$

By (6.2), we also have

$${}^t\overline{J\overline{u}}gv_1 = 0.$$

Since ${}^tJg$ is skew symmetric, we have

$${}^t\overline{J\overline{v_1}}gu = {}^tv_1\,{}^tJgu = -{}^tu\,{}^tJgv_1 = -{}^t\overline{J\overline{u}}gv_1 = 0.$$

On the other hand, since gJ is skew symmetric, we also have

$$^t\overline{u}gJ\overline{v_1} = -{}^t\overline{v_1}gJ\overline{u} = 0$$

by the definition of U_1. Thus we have proved

$$^t\overline{v_1}gu = {}^t\overline{J\overline{v_1}}gu = {}^t\overline{u}gv_1 = {}^t\overline{u}gJ\overline{v_1} = 0$$

for all $u \in U_1$.

Since gJ and ${}^tJg$ are skew symmetric, we have

$$^t\overline{v_1}gJ\overline{v_1} = 0, \quad {}^t\overline{J\overline{v_1}}gv_1 = {}^tv_1\,{}^tJgv_1 = 0$$

and

$$^t\overline{J\overline{v_1}}gJ\overline{v_1} = {}^tv_1\,{}^tJgJ\overline{v_1} = -{}^tv_1\,{}^tgJ^2\overline{v_1} = {}^tv_1\,{}^tg\overline{v_1} = {}^t\overline{v_1}gv_1 = \lambda_1.$$

Hence we have

$$\begin{pmatrix} {}^t\overline{v_1}gv_1 & {}^t\overline{v_1}gJ\overline{v_1} \\ {}^t\overline{J\overline{v_1}}gv_1 & {}^t\overline{J\overline{v_1}}gJ\overline{v_1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Next we take an element $v_2 \in U_1 - \{0\}$ such that $\xi(v_2)$ is on the boundary of $\xi(U_1 - \{0\})$ and that ${}^t\overline{v_2}gv_2 = \lambda_2 \in U(1)$. Repeating this procedure, we can take basis $v_1, J\overline{v_1}, \ldots, v_n, J\overline{v_n}$ of $\mathbb{C}^{2n}$ such that

$$^t\overline{v_j}gv_k = {}^t\overline{v_j}gJ\overline{v_k} = {}^t(\overline{J\overline{v_j}})gv_k = {}^t(\overline{J\overline{v_j}})gJ\overline{v_k} = 0 \quad \text{if } j \neq k$$

and that

$$\begin{pmatrix} {}^t\overline{v_j}gv_j & {}^t\overline{v_j}gJ\overline{v_j} \\ {}^t\overline{J\overline{v_j}}gv_j & {}^t\overline{J\overline{v_j}}gJ\overline{v_j} \end{pmatrix} = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}$$

for $j = 1, \ldots, n$.

Let h be an element of $GL(2n, \mathbb{C})$ such that

$$he_1 = v_1, he_2 = -J\overline{v_1}, \ldots, he_{2n-1} = v_n, he_{2n} = -J\overline{v_n}.$$

Then h is an element of $\widetilde{G_\mathbb{R}} = \{h \in GL(2n, \mathbb{C}) \mid J\overline{h}J^{-1} = h\} \cong GL(n, \mathbb{H})$ and

$$^t\overline{h}gh = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ 0 & & & & \lambda_n \end{pmatrix}.$$

We can see that $\det h \in \mathbb{R}$ and moreover that $\det h > 0$ since $\widetilde{G_\mathbb{R}}$ is connected. Since $g = {}^tJ\,{}^tx^{-1}Jx^{-1} \in SL(n, \mathbb{C})$, we have

$$(\det h)^2 = \lambda_1^2 \cdots \lambda_n^2 \in U(1).$$

So we have $\det h = 1$ which implies $h \in G_\mathbb{R}$ and we also have

$$\lambda_1^2 \cdots \lambda_n^2 = 1.$$

We have

$$ {}^t h \, {}^t x^{-1} J x^{-1} h = J \, {}^t \overline{h} \, {}^t J \, {}^t x^{-1} J x^{-1} h = J \, {}^t \overline{h} g h = \begin{pmatrix} \lambda_1 J_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n J_1 \end{pmatrix}. $$

We can take the pfaffian as a polynomial function of degree n on the space of skew symmetric matrices such that

$$ \mathrm{Pf}\, J = 1. $$

Then we have

$$ \mathrm{Pf} \begin{pmatrix} \lambda_1 J_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n J_1 \end{pmatrix} = \lambda_1 \cdots \lambda_n = 1 \tag{6.3} $$

because $\mathrm{Pf}({}^t m J m) = 1$ for any $m \in SL(2n, \mathbb{C})$.

Let c be an element of $\mathbb{C}^{2n}$. Then we have

$$ {}^t \overline{c} \, {}^t \overline{h} g h c = \lambda_1 (|c_1|^2 + |c_2|^2) + \cdots + \lambda_n (|c_{2n-1}|^2 + |c_{2n}|^2). $$

Hence the condition (6.1) is equivalent to the condition (3.2). So we can write

$$ \begin{pmatrix} \lambda_1 J_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n J_1 \end{pmatrix} = t^2 z^2 J = {}^t (tz) J t z $$

with some $t \in T^+$ and $z \in Z$ by using the condition (6.3). Hence $x^{-1} h z^{-1} t^{-1} \in K_{\mathbb{C}}$ and therefore

$$ x \in h z^{-1} t^{-1} K_{\mathbb{C}} \subset \widetilde{D_0} Z. $$

Converse assertion is clear by the above arguments.

Now let us consider the full flag manifold X consisting of full flags

$$ V_1 \subset \cdots \subset V_{2n-1} $$

where V_j is a j-dimensional complex subspace in $\mathbb{C}^{2n}$. There is only one closed $K_{\mathbb{C}}$-orbit S on X consisting of flags satisfying

$$ u \in V_j, \ v \in V_{2n-j} \Longrightarrow {}^t u J v = 0 $$

for all j. On the other hand, the corresponding open $G_{\mathbb{R}}$-orbit S' consists of flags satisfying

$$ J \overline{V_j} \cap V_{2n-j} = \{0\} $$

for all j.

Suppose $x S_0 \subset S_0'$. Then we can show $xS \subset S'$ as follows. Any flag contained in xS is written as

$$ x V_1 \subset \cdots \subset x V_{2n-1} $$

with some

$$ V_1 \subset \cdots \subset V_{2n-1} $$

in S. Let v be an element of $J\overline{xV_j} \cap xV_{2n-j}$. Then since $x^{-1}\overline{J^{-1}v} \in V_j$ and $x^{-1}v \in V_{2n-j}$, we have

$$0 = {}^t(x^{-1}\overline{J^{-1}v})Jx^{-1}v = {}^t\overline{v}\,{}^tJ\,{}^tx^{-1}Jx^{-1}v = {}^t\overline{v}gv.$$

Hence $v = 0$ by (6.1). Thus we have proved $S_0 \subset S_0' \Longrightarrow xS \subset S'$.

The assertion $xS \subset S' \Longrightarrow xS_0 \subset S_0'$ is proved by a general argument as in Section 8. Thus we have shown:

Proposition 6.2. $xS_0 \subset S_0' \Longleftrightarrow xS \subset S'$.

7. $Sp(p,q)$-case

Let $G_{\mathbb{C}} = \{g \in GL(2n,\mathbb{C}) \mid {}^tgJg = J\} \cong Sp(n,\mathbb{C})$ where J is defined as in Section 5 and 6. Put $V_+ = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{2p}$ and $V_- = \mathbb{C}e_{2p+1} \oplus \cdots \oplus \mathbb{C}e_n$ with the canonical basis $e_1,\ldots,e_{2n}$ of $\mathbb{C}^{2n}$ and let Q be the hermitian form on $\mathbb{C}^{2n}$ defined by

$$Q(z,w) = {}^tzI_{2p,2q}\overline{w} = z_1\overline{w_1} + \cdots + z_{2p}\overline{w_{2p}} - z_{2p+1}\overline{w_{2p+1}} - \cdots - z_{2n}\overline{w_{2n}}.$$

Then we define subgroups

$$K_{\mathbb{C}} = \{g \in G_{\mathbb{C}} \mid gV_+ = V_+\} = \{g \in G_{\mathbb{C}} \mid gV_- = V_-\} \cong Sp(p,\mathbb{C}) \times Sp(q,\mathbb{C})$$

$(q = n-p)$ and

$$G_{\mathbb{R}} = \{g \in G_{\mathbb{C}} \mid Q(gz,gw) = Q(z,w) \text{ for all } z,w \in \mathbb{C}^{2n}\} \cong Sp(p,q)$$

of $G_{\mathbb{C}}$. We can take a flag manifold

$$X_0 = P^{2n-1}(\mathbb{C}) = (\mathbb{C}^{2n} - \{0\})/\mathbb{C}^\times$$

of $G_{\mathbb{C}}$. There are three $K_{\mathbb{C}}$-orbits on X_0 consisting of two closed orbits $S_\pm = (V_\pm - \{0\})/\mathbb{C}^\times$ and the compliment S_0. There are also three $G_{\mathbb{R}}$-orbits $S_\pm'$ and S_0' on X_0 defined by the signature of Q. Let $\ell = \min(p,q)$ and write

$$\begin{aligned}Y(\theta_1,\ldots,\theta_\ell) = {} & \theta_1(E_{2p+1,1} + E_{2p+2,2} - E_{1,2p+1} - E_{2,2p+2}) + \cdots \\ & + \theta_\ell(E_{2p+2\ell-1,2\ell-1} + E_{2p+2\ell,2\ell} - E_{2\ell-1,2p+2\ell-1} - E_{2\ell,2p+2\ell}).\end{aligned}$$

Then we can take

$$\mathfrak{t}^+ = \{Y(\theta_1,\ldots,\theta_\ell) \mid |\theta_j| < \pi/4 \text{ for all } j = 1,\ldots,\ell\}.$$

Proposition 7.1. $x \in G_{\mathbb{R}}T^+K_{\mathbb{C}} \Longleftrightarrow xS_+ \subset S_+' \Longleftrightarrow xS_- \subset S_-'$.

Proof. The condition $xS_+ \subset S_+'$ is equivalent to

(7.1) $$Q \text{ is positive definite on } V = xV_+$$

Assume (7.1). Let H denote the hypersurface in $V \times V$ defined by

$$H = \{(u,v) \in V \times V \mid {}^tuJv = 1\}$$

and consider the function

$$f(u,v) = Q(u,u)Q(v,v)$$

of $(u,v) \in V \times V$. Then the range of $f|_H$ is of the form

$$[M,\infty)$$

with some $M > 0$ because we can write

$$\{|{}^t uJv| \mid u, v \in V,\ Q(u,u) = Q(v,v) = 1\} = \left[0, \frac{1}{M}\right]$$

by (7.1).

Let (u_1, v_1) be an element of H such that

$$f(u_1, v_1) = M.$$

Clearly, we may choose u_1 and v_1 so that

$$Q(u_1, u_1) = Q(v_1, v_1) = \sqrt{M}.$$

Put $\lambda_1 = \sqrt{M}$. By elementary calculus, we have

$${}^t u_1 Jw = 0 \Longrightarrow \operatorname{Re} Q(v_1, w) = 0$$

for $w \in \mathbb{C}^{2n}$. Since w can be replaced by iw, we have

$${}^t u_1 Jw = 0 \Longrightarrow \operatorname{Im} Q(v_1, w) = 0$$

and therefore

$${}^t u_1 Jw = 0 \Longrightarrow Q(v_1, w) = 0.$$

In the same way, we also have

$${}^t v_1 Jw = 0 \Longrightarrow Q(u_1, w) = 0.$$

Put $U_1 = \{w \in V \mid {}^t u_1 Jw = {}^t v_1 Jw = 0\}$. Then the alternating form ${}^t zJw$ is nondegenerate on U_1 since it is nondegenerate on V. By the same argument as above, we can take $(u_2, v_2) \in H_1 = (U_1 \times U_1) \cap H$ such that $f|_{H_1}$ attains its minimum at (u_2, v_2) and that

$$Q(u_2, u_2) = Q(v_2, v_2) = \lambda_2 \geq \lambda_1.$$

Repeating this procedure, we can get basis $u_1, v_1, \ldots, u_p, v_p$ of V satisfying

$${}^t u_j J v_j = 1, \qquad Q(u_j, u_j) = Q(v_j, v_j) = \lambda_j, \qquad Q(u_j, v_j) = 0$$

and

$$\text{(7.2)} \quad {}^t u_j J u_k = {}^t u_j J v_k = {}^t v_j J v_k = Q(u_j, u_k) = Q(u_j, v_k) = Q(v_j, v_k) = 0 \quad \text{if } j \neq k$$

where $0 < \lambda_1 \leq \cdots \leq \lambda_p$.

Put $J' = I_{2p,2q} J$. Then we have

$$\text{(7.3)} \qquad Q(u, J'\overline{v}) = {}^t uJv, \qquad Q(J'\overline{u}, J'\overline{v}) = Q(v, u) \quad \text{and} \quad {}^t(J'\overline{u}) J J'\overline{u} = \overline{{}^t uJv}$$

for any $u, v \in \mathbb{C}^{2n}$. Put

$$w_j^{(1)} = u_j + J'\overline{v_j}, \quad w_j^{(2)} = -J'\overline{w_j^{(1)}}, \quad w_j^{(3)} = u_j - J'\overline{v_j}, \quad w_j^{(4)} = -J'\overline{w_j^{(3)}}$$

for $j = 1, \ldots, p$. Then by (7.2) and (7.3), we have

$$\text{(7.4)} \qquad Q(w_j^{(r)}, w_k^{(s)}) = {}^t w_j^{(r)} J w_k^{(s)} = 0 \quad \text{if } j \neq k$$

for every $r, s = 1, 2, 3, 4$. On the other hand, we have

$$\begin{aligned} Q(w_j^{(1)}, w_j^{(1)}) &= Q(u_j, u_j) + Q(u_j, J'\overline{v_j}) + Q(J'\overline{v_j}, u_j) + Q(J'\overline{v_j}, J'\overline{v_j}) \\ &= Q(u_j, u_j) + {}^t u_j J v_j + \overline{{}^t u_j J v_j} + Q(v_j, v_j) \\ &= 2\lambda_j + 2, \end{aligned}$$

$Q(w_j^{(3)}, w_j^{(3)}) = 2\lambda_j - 2$, $Q(w_j^{(1)}, w_j^{(3)}) = 0$ and

$$ {}^t w_j^{(1)} J w_j^{(3)} = {}^t(u_j + J'\overline{v_j}) J (u_j - J'\overline{v_j}) = -2\,{}^t u_j J J' \overline{v_j} = 2Q(u_j, v_j) = 0.$$

Hence we have

$$Q(w_j^{(r)}, w_j^{(s)}) = \begin{cases} 2\lambda_j + 2 & \text{if } r = s = 1 \text{ or } r = s = 2 \\ 2\lambda_j - 2 & \text{if } r = s = 3 \text{ or } r = s = 4 \\ 0 & \text{otherwise} \end{cases} \tag{7.5}$$

and

$${}^t w_j^{(r)} J w_j^{(s)} = \begin{cases} 2\lambda_j + 2 & \text{if } (r, s) = (1, 2) \\ 2\lambda_j - 2 & \text{if } (r, s) = (3, 4) \\ 0 & \text{if } r \leq s \text{ and } (r, s) \neq (1, 2), (3, 4) \end{cases} \tag{7.6}$$

by (7.3).

By (7.4) and (7.5), the $4p$ vectors $w_j^{(r)}$ are mutually orthogonal with respect to Q. Since $2\lambda_j + 2 > 2 > 0$ and since the signature of Q is $(2p, 2q)$, we have

$$2\lambda_j - 2 \leq 0$$

and hence $\lambda_j \leq 1$. Moreover if $\lambda_j = 1$, then

$$w_j^{(3)} = w_j^{(4)} = 0$$

since Q is negative definite on the orthogonal compliment of the $2p$-dimensional space spanned by $w_1^{(1)}, \ldots, w_p^{(1)}, w_1^{(1)}, \ldots, w_p^{(2)}$. (If $p > q$, then we have automatically $\lambda_j = 1$ and $w_j^{(3)} = w_j^{(4)} = 0$ for $j = q + 1, \ldots, p$ since $\lambda_1 \leq \cdots \leq \lambda_p$.) Since $0 < \lambda_1 \leq \cdots \leq \lambda_p \leq 1$, we can write $\lambda_j = \cos^2\theta_j - \sin^2\theta_j$ with

$$\frac{\pi}{4} \geq \theta_1 \geq \cdots \geq \theta_p \geq 0.$$

Then $2\lambda_j + 2 = 4\cos^2\theta_j$ and $2\lambda_j - 2 = -4\sin^2\theta_j$. By (7.4), (7.5) and (7.6), we can take an element $g \in G_{\mathbb{R}}$ such that

$$g w_j^{(1)} = 2\cos\theta_j e_{2j-1}, \quad g w_j^{(2)} = 2\cos\theta_j e_{2j}$$

for $j = 1, \ldots, p$ and that

$$g w_j^{(3)} = 2\sin\theta_j e_{2p+2j-1}, \quad g w_j^{(4)} = 2\sin\theta_j e_{2p+2j}$$

for $j = 1, \dots, \ell'$ where $\ell' = \max\{j \mid \theta_j \neq 0\}$. So we have

$$gu_j = \frac{1}{2}g(w_j^{(1)} + w_j^{(3)}) = \cos\theta_j e_{2j-1} + \sin\theta_j e_{2p+2j-1}$$
$$gv_j = \frac{1}{2}g(w_j^{(2)} + w_j^{(4)}) = \cos\theta_j e_{2j} + \sin\theta_j e_{2p+2j}$$

for $j = 1, \dots, p$. This implies

$$gxV_+ = gV = \exp Y(\theta_1, \dots, \theta_\ell)V_+$$

and therefore

$$x \in g^{-1}\exp Y(\theta_1, \dots, \theta_\ell)K_{\mathbb{C}} \subset G_{\mathbb{R}}T^+K_{\mathbb{C}}.$$

Conversely, let $v = \sum_{j=1}^{2p} c_j e_j \in V_+$ and $x = \exp Y(\theta_1, \dots, \theta_\ell)$ with $|\theta_j| < \pi/4$. Then we have

$$Q(xv, xv) = (|c_1|^2+|c_2|^2)(\cos^2\theta_1-\sin^2\theta_1)+\cdots+(|c_{2p-1}|^2+|c_{2p}|^2)(\cos^2\theta_p-\sin^2\theta_p) > 0$$

if $v \neq 0$. (Here $\theta_{\ell+1} = \cdots = \theta_p = 0$ if $p > q$.) Hence $xS_+ \subset S'_+$.

We also have $x \in G_{\mathbb{R}}T^+K_{\mathbb{C}} \iff xS_- \subset S'_-$ by the same argument. q.e.d.

Now let us consider the full flag manifold. The full flag manifold $X = G_{\mathbb{C}}/B$ consists of full flags

$$V_1 \subset \cdots \subset V_n$$

where V_j are j-dimensional complex linear subspaces V_j such that ${}^t zJw = 0$ for all $z, w \in V_j$. Such a flag is contained in a closed $K_{\mathbb{C}}$-orbit if and only if

$$V_j = (V_j \cap V_+) \oplus (V_j \cap V_-) \quad \text{for all } j = 1, \dots, n.$$

It is clear that

$$p_j = \dim(V_j \cap V_+) \quad \text{and} \quad q_j = \dim(V_j \cap V_-) \quad (j = 1, \dots, n)$$

are the invariants of closed $K_{\mathbb{C}}$-orbits. Hence there are $\binom{n}{p}$ closed $K_{\mathbb{C}}$-orbits on X.

On the other hand, a flag is contained in an open $G_{\mathbb{R}}$-orbit if and only if the hermitian form Q is nondegenerate on each V_j ($j = 1, \dots, n$). So the signatures (p_j, q_j) of Q on V_j ($j = 1, \dots, n$) are the invariants of open $G_{\mathbb{R}}$-orbits on X. We can see that closed $K_{\mathbb{C}}$-orbits and open $G_{\mathbb{R}}$-orbits on X are in one-to-one correspondence by these numbers (p_j, q_j) ($j = 1, \dots, n$).

Suppose that $x \in G_{\mathbb{R}}T^+K_{\mathbb{C}}$. Then we want to show that $xS \subset S'$ for all closed $K_{\mathbb{C}}$-orbits S on X. By Proposition 7.1, Q is positive defnite on xV_+ and negative definite on xV_-. Let

$$V_1 \subset \cdots \subset V_n$$

be a flag contained in S. Then Q is positive definite on $xV_j \cap xV_+ = x(V_j \cap V_+)$ and negative definite on $xV_j \cap xV_- = x(V_j \cap V_-)$ for every $j = 1, \dots, n$. So the signature of Q on xV_j is $(\dim(V_j \cap V_+), \dim(V_j \cap V_-))$ for $j = 1, \dots, n$. Thus we have proved $xS \subset S'$.

Conversely, if $xS \subset S'$ for some closed $K_{\mathbb{C}}$-orbit on X, then we have $xS_+ \subset S'_+$ or $xS_- \subset S'_-$ by a general argument as in Section 8 and hence $x \in G_{\mathbb{R}}T^+K_{\mathbb{C}}$ by Proposition 7.1.

Summarizing the above arguments, we have:

Proposition 7.2. $x \in G_{\mathbb{R}}T^+K_{\mathbb{C}} \iff xS \subset S'$ *for a closed* $K_{\mathbb{C}}$*-orbit on* X.

8. Arguments on full flag manifolds

Let B be a Borel subgroup of $G_{\mathbb{C}}$. Let Ψ denote the set of simple roots corresponding to B. For $\alpha \in \Psi$, we can take a parabolic subgroup $P_\alpha = B \cup Bw_\alpha B$ of $G_{\mathbb{C}}$.

Proposition 8.1. *Let x be an element of $G_{\mathbb{C}}$. If $xS \subset S'$ for all closed $K_{\mathbb{C}}$-B double cosets in $G_{\mathbb{C}}$, then*

$$xS \cap S' \text{ is non-empty and closed in } G_{\mathbb{C}} \text{ for all } S \in K_{\mathbb{C}}\backslash G_{\mathbb{C}}/P$$

for an arbitrary parabolic subgroup P of $G_{\mathbb{C}}$.

Proof. First assume that $P = B$. We will prove this by induction on $\dim S$. If $S \in K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B$ is not closed, then there exists an $\alpha \in \Psi$ such that
(1) $SP_\alpha = S \cup S_1 \cup S_2,\ \dim S_1 = \dim S_2 = \dim S - 2$,
or
(2) $SP_\alpha = S \cup S_1,\ \dim S_1 = \dim S - 2$.
Consider the case (1). (The case (2) is easier than (1).) It is known ([M3], [Sp]) that the closure S_1^{cl} of S_1 is written as

$$S_1^{cl} = S_0 P_{\alpha_1} \cdots P_{\alpha_\ell}$$

with some closed $K_{\mathbb{C}}$-B double coset S_0 and $\alpha_1, \ldots, \alpha_\ell \in \Psi$. So by the assumption, we have
(8.1)

$$xS_1 \subset xS_1^{cl} = xS_0P_{\alpha_1} \cdots P_{\alpha_\ell} \subset S_0'P_{\alpha_1} \cdots P_{\alpha_\ell} = \bigcup \{T' \in G_{\mathbb{R}}\backslash G_{\mathbb{C}}/B \mid T \subset S_1^{cl}\}.$$

(Note that the right hand side is open in $G_{\mathbb{C}}$ since S_0' is open.) Hence we have

$$xS_1 \cap S' = \phi \quad \text{and} \quad xS_1 \cap S_2' = \phi. \tag{8.2}$$

because S and S_2 are not contained in S_1^{cl}. In the same way, we also have

$$xS_2 \cap S' = \phi \quad \text{and} \quad xS_2 \cap S_1' = \phi. \tag{8.3}$$

By (8.2), we have

$$(xS_1 \cap S_1')P_\alpha = (xS_1 \cap (S_1' \cup S_2' \cup S'))P_\alpha = (xS_1 \cap S_1'P_\alpha)P_\alpha = xS_1P_\alpha \cap S_1'P_\alpha.$$

By the assumption of induction, this set is closed in $G_{\mathbb{C}}$ (compact in $G_{\mathbb{C}}/B$). On the other hand, we have

$$xS_1P_\alpha \cap S_1'P_\alpha = (xS_1 \cup xS_2 \cup xS) \cap (S_1' \cup S_2' \cup S') = ((xS_1 \cup xS_2 \cup xS) \cap (S_1' \cup S_2')) \cup (xS \cap S')$$

since

$$(xS_1 \cup xS_2 \cup xS) \cap S' = xS \cap S' \tag{8.4}$$

by (8.2) and (8.3). Hence the set $xS \cap S'$ is closed in $G_{\mathbb{C}}$ because it is the compliment of the open set

$$\bigcup \{T' \in G_{\mathbb{R}}\backslash G_{\mathbb{C}}/B \mid T \subset (S_1)^{cl}\} \cup \bigcup \{T' \in G_R\backslash G_{\mathbb{C}}/B \mid T \subset (S_2)^{cl}\}$$

in the closed set $xS_1P_\alpha \cap S_1'P_\alpha$.

By (8.4), we have

$$(xS \cap S')P_\alpha = (xSP_\alpha \cap S')P_\alpha = xSP_\alpha \cap S'P_\alpha = xS_1P_\alpha \cap S_1'P_\alpha.$$

This set is non-empty since $xS_1 \cap S_1'$ is non-empty by the assumption of induction. Hence $xS \cap S'$ is non-empty.

Now consider an arbitrary parabolic subgroup P. We may assume that $B \subset P$. Take a $K_{\mathbb{C}}$-B double coset S_1 contained in S such that S_1 is relatively closed in S. Then by (8.1), we have

$$xS_1 \cap T' = \phi$$

for $K_{\mathbb{C}}$-B double cosets T contained in $S - S_1$ because $T \not\subset (S_1)^{cl}$. Hence we have

$$xS_1 \cap S' = xS_1 \cap S_1'$$

since $S' = \bigcup_{T \subset S} T'$ as in [M2]. Thus the set

$$xS \cap S' = xS_1P \cap S' = (xS_1 \cap S')P = (xS_1 \cap S_1')P$$

is closed in $G_{\mathbb{C}}$ since $(xS_1 \cap S_1')/B$ is compact. q.e.d.

Remark 8.2. (More precise arguments when $P = B$) First consider the case (1). Let y be an element of $xS \cap S'$. Then we have

$$\begin{aligned} yP_\alpha &= (yP_\alpha \cap xS_1) \cup (yP_\alpha \cap xS_2) \cup (yP_\alpha \cap xS) \\ &= (yP_\alpha \cap S_1') \cup (yP_\alpha \cap S_2') \cup (yP_\alpha \cap S') \end{aligned}$$

Here $yP_\alpha/B \cong P^1(\mathbb{C})$, $(yP_\alpha \cap xS_1)/B \cong \{\text{a point}\}$, $(yP_\alpha \cap xS_2)/B \cong \{\text{a point}\}$ and $(yP_\alpha \cap S')/B \cong S^1$. By (8.2) and (8.3), we have

$$yP_\alpha \cap xS_1 \subset yP_\alpha \cap S_1' \quad \text{and} \quad yP_\alpha \cap xS_2 \subset yP_\alpha \cap S_2'.$$

We also have the same formulas when $y \in xS_1 \cap S_1'$. Thus we can construct a smooth surjection of $(xS \cap S')/B$ onto $(xS_1 \cap S_1')/B$ with fibers diffeomorphic to S^1.

Next consider the case (2). Let y be an element of $xS \cap S'$. Then we have

$$\begin{aligned} yP_\alpha &= (yP_\alpha \cap xS_1) \cup (yP_\alpha \cap xS) \\ &= (yP_\alpha \cap S_1') \cup (yP_\alpha \cap S') \end{aligned}$$

There are two cases:

(2a) $(yP_\alpha \cap xS_1)/B \cong \{\text{a point}\}$ and $(yP_\alpha \cap S')/B \cong \{\text{a point}\}$,

(2b) $(yP_\alpha \cap xS_1)/B \cong \{\text{two points}\}$ and $(yP_\alpha \cap S')/B \cong S^1$.

In (2a)-case, we get a diffeomorphism of $(xS \cap S')/B$ onto $(xS_1 \cap S_1')/B$. On the other hand, in (2b)-case, we have a S^1-to-$\{2 \text{ points}\}$ correspondence.

Proposition 8.3. *Let S be the open $K_{\mathbb{C}}$-orbit on X. Then*

$$C(S)_0 = C_0.$$

Here $*_0$ denote the connected component of $*$ containing the identity e.

Proof. Let $[0,1] \ni t \mapsto x_t$ be a continuous curve in $C(S)$ such that $x_0 = e$. Then by Proposition 8.1, we have only to show that $x_t \in C(S_1)$ for all closed $K_{\mathbb{C}}$-orbits S_1 on X. Suppose $x_t \notin C(S_1)$ for some t and S_1. Then we will get a contradiction as follows.

We may assume that $x_t S_1 \cap S_2' \neq \phi$ for some $G_{\mathbb{R}}$-orbit S_2' contained in the boundary of S_1'. Put $\ell = \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} S_2$ and let $\alpha_1, \ldots, \alpha_\ell$ be a sequence of simple roots such that

$$\dim_{\mathbb{C}} S_2 P_{\alpha_1} \cdots P_{\alpha_k} = \dim_{\mathbb{C}} S_2 + k$$

for all $k = 1, \ldots, \ell$. Since $S_2 P_{\alpha_1} \cdots P_{\alpha_k} \supset S$, we have

$$S_2' P_{\alpha_1} \cdots P_{\alpha_k} \supset S'$$

(cf. [M2]) and therefore

$$S_2' \subset S' P_{\alpha_k} \cdots P_{\alpha_1}.$$

Since $x_t \in C(S)$, we have $S' \subset x_t S$. So we have

$$S_2' \subset x_t S P_{\alpha_k} \cdots P_{\alpha_1}$$

and hence

$$\phi \neq x_t S_1 \cap x_t S P_{\alpha_k} \cdots P_{\alpha_1} = x_t (S_1 \cap S P_{\alpha_k} \cdots P_{\alpha_1}).$$

But since the set $S P_{\alpha_k} \cdots P_{\alpha_1}$ does not contain $K_{\mathbb{C}}$-orbits whose dimensions are smaller than $\dim_{\mathbb{C}} S_2$, the right hand side of the above formula must be ϕ. Thus we have got a contradiction. q.e.d.

9. Appendix for $SO(p,q)$-case

Retain the notations in Section 4. Assume $p \leq q$. The orthogonal compliment $\mathfrak{m}$ of $\mathfrak{so}(p,q) + \mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$ in $\mathfrak{su}(p,q)$ is written as

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -i\,{}^tB \\ iB & 0 \end{pmatrix} \;\middle|\; B \text{ is a } q \times p \text{ real matrix} \right\}.$$

So if we write

$$Z(t_1, \ldots, t_p) = it_1(E_{n,1} - E_{1,n}) + \cdots + it_p(E_{n-p+1,p} - E_{p,n-p+1}),$$

then we can take

$$\mathfrak{a} = \{Z(t_1, \ldots, t_p) \mid t_1, \ldots, t_p \in \mathbb{R}\}$$

as a maximal abelian subspace of $\mathfrak{m}$. By a fundamental decomposition theorem (c.f. [R] Theorem 10), we have

$$SU(p,q) = SO(p,q)_0 A S(U(p) \times U(q)) \tag{9.1}$$

where $A = \exp \mathfrak{a}$.

Let X be the Grassmann manifold consisting of p-dimensional complex subspaces of $\mathbb{C}^n$. Let D' denote the $SU(p,q)$-orbit on X containing $V_+ = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_p$. Then by (9.1), we have

$$D' = SO(p,q)_0 A V_+.$$

Let $\mathfrak{t}$ be as in Section 4 and define

$$\mathfrak{t}^{+'} = \{Y(\theta_1, \ldots, \theta_p) \mid |\theta_j| < \pi/4 \text{ for all } j = 1, \ldots, p\}$$

which is a subset of $\mathfrak{t}^+$. Put $T^{+'} = \exp \mathfrak{t}^{+'}$. Then we have

$$AV_+ = T^{+'}V_+$$

because

$$(\exp Z(t_1, \ldots , t_p))e_j = (\cosh t_j)e_j + i(\sinh t_j)e_{n-j+1}$$

and

$$(\exp Y(\theta_1, \ldots , \theta_p))e_j = (\cos \theta_j)e_j + i(\sin \theta_j)e_{n-j+1}.$$

(This is a typical example of Lemma 2.1 (1).) Hence $D_0 = SO(p, q)_0T^+V_+$ is bigger than $D' = SO(p, q)_0T^{+'}V_+$.

References

[AG] D. N. Akhiezer and S. G. Gindikin. On Stein extensions of real symmetric spaces. *Math. Ann.*, 286:1–12, 1990.

[BGW] L. Barchini, S. G. Gindikin and H. W. Wong. Determinant functions and geometry of the flag manifold for $SU(p, q)$. *J. of Lie Theory*, 6:191-206, 1996.

[G] S. Gindikin. Tube domains in Stein symmetric spaces. *Positivity in Lie theory: Open problems*, W. de Gruyter, 81–98, 1998.

[M1] T. Matsuki. The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. *J. Math. Soc. Japan*, 31:331–357, 1979.

[M2] T. Matsuki. Orbits on affine symmetric spaces under the action of parabolic subgroups. *Hiroshima Math. J.*, 12:307–320, 1982.

[M3] T. Matsuki. Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups. *Adv. Studies in Pure Math.*, 14:541–559, 1988.

[M4] T. Matsuki. Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits. *Hiroshima Math. J.*, 18:59–67, 1988.

[M5] T. Matsuki. Double coset decompositions of reductive Lie groups arising from two involutions. *J. of Algebra*, 197:49–91, 1997.

[MUV] I. Mirković, T. Uzawa and K. Vilonen. Matsuki correspondence for sheaves. *Invent. Math.*, 109:231–245, 1992.

[R] W. Rossmann. The structure of semisimple symmetric spaces. *Canad. J. Math.*, 31:157–180, 1979.

[Sp] T. A. Springer. Some results on algebraic groups with involutions. *Adv. Studies in Pure Math.*, 6:525–534, 1984.

[W] J. A. Wolf. The Stein condition for cycle spaces of open orbits on complex flag manifolds. *Ann. of Math.*, 136:541–555, 1992.

[WZ] J. A. Wolf and R. Zierau. Linear cycle spaces in flag domains. *Math. Ann.*, 316:529–545, 2000.

Simon Gindikin, Department of Mathematics, Hill Center, Rutgers University, 110 Frelinghysen Road, Piscataway, NJ 08854-8019, U.S.A.

E-mail address: gindikin@math.rutgers.edu

Toshihiko Matsuki, Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-8501, Japan

E-mail address: matsuki@math.h.kyoto-u.ac.jp