NONCLASSICAL DESCRIPTIONS OF ANALYTIC COHOMOLOGY

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There are two classical languages for analytic cohomology: Dolbeault and Čech. In some applications, however (for example, in describing the Penrose transform and certain representations), it is convenient to use some nontraditional languages. In [?] was developed a language that allows one to render analytic cohomology in a purely holomorphic fashion. In this article we indicate a more general construction, which includes a version of Čech cohomology based on a smoothly parameterised Stein cover. The idea of this language is that, usually, there are only infinite Stein coverings of the complex manifold in question but, often, we can find natural Stein coverings parameterised by an auxiliary smooth manifold. Under these circumstances, it is unnatural to work with classical Čech cohomology. Instead, it is possible to construct the analytic cohomology from the de Rham complex on the parameter space but with holomorphic dependence in the corresponding Stein subset. This switch of language is rather like replacing sums by integrals to pass from discrete to continuous.

This material was the subject of a lecture presented by one of us (MGE) at the 22nd Czech Winter School on Geometry and Physics held in Srní in January 2002. This article contains only an outline and a couple of examples. Precise proofs will appear elsewhere.

1. Introduction

We wish to compute the analytic cohomology $H^r(Z,\mathcal{O})$ of a complex manifold Z. Suppose we are given:–

(1)
$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & \Xi \end{tikzcd}$$

where

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- F and Ξ are smooth manifolds,
- η is a fibration with contractible fibres,
- τ is a fibration with Stein manifolds as fibres,
- η is holomorphic when restricted to the fibres of τ,

and a further technical condition due to Jurchescu [?] holds. In these circumstances, there is a complex of sheaves $\text{Œ}(B^\bullet)$ on F so that

$$H^p(Z,\mathcal{O}) \cong H^p(\Gamma(F,\text{Œ}(B^\bullet))). \tag{2}$$

Three special cases are included in this formulation:–

- $Z = F = \Xi$: in this case $\text{Œ}(B^\bullet)$ reduces to the usual Dolbeault resolution;
- $\Xi = \{\mathrm{pt}\}$: in this case F is a Stein manifold and we obtain the holomorphic language of [?];
- the fibres of τ are embedded by η as open subsets of Z: in this case we obtain the smoothly parameterised Čech cohomology of [?].

In fact, there is no need that the fibres of τ be embedded as open subsets of Z. Our formulation allows for them to be Stein submanifolds of Z and there are naturally occurring examples for which this in the case. Also, as in [?], it is easily possible to incorporate a holomorphic vector bundle E on Z into this language so that we compute $H^p(Z,\mathcal{O}(E))$.

2. Examples

Our first example is taken from [?] but here we make explicit its relation to the Penrose transform as in [?]. It a purely holomorphic description. Thus, $\Xi = \{\mathrm{pt}\}$ and F is a Stein manifold. The manifold Z is a homogeneous space of $\mathrm{SU}(2,1)$, namely

$$Z = \{[z_1,z_2,z_3] \in \mathbb{CP}_2 \text{ s.t. } |z_1|^2 + |z_2|^2 > |z_3|^2\}.$$

It is the complement of a ball in the projective plane. We shall take

$$F = \{(z,\zeta) \in \mathbb{CP}_2 \text{ s.t. } z \neq \zeta \text{ and the line joining } z \text{ and } \zeta \text{ lies entirely in } Z\}.$$

It is easy to check that F is Stein and $\eta : F \to Z$ defined by $(z,\zeta) \mapsto z$ has contractible fibres. In this case, $F \subset \mathbb{CP}_2 \times \mathbb{CP}_2$, an open subset of a product. Therefore, the complex $\text{Œ}(B^\bullet)$ is more easily described:–

$$\text{Œ}(B^p) \ni \omega(z,\zeta,d\zeta), \text{ a holomorphic } p\text{-form in } \zeta \text{ depending holomorphically on } z$$

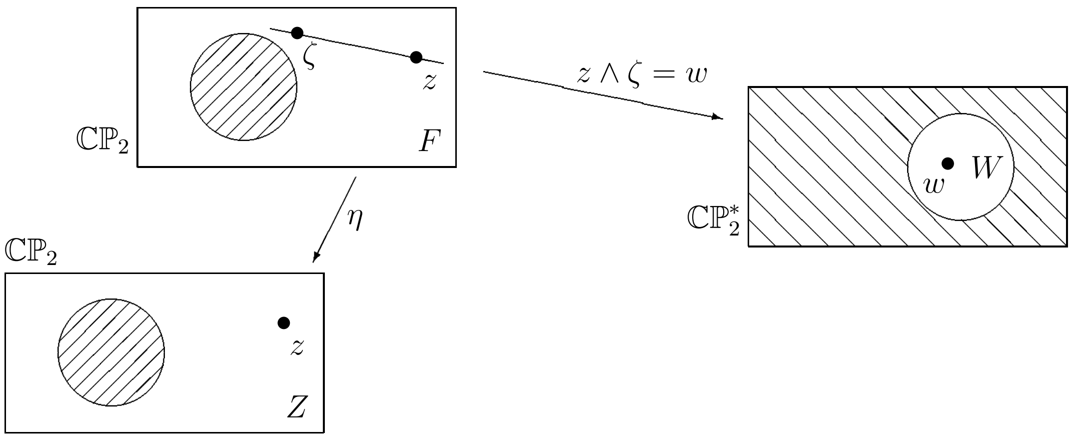
with differential $d_\eta : \text{Œ}(B^p) \to \text{Œ}(B^{p+1})$ being exterior derivative in ζ. We shall describe an easy variation on (??) in which the analytic cohomology of Z is taken with coefficients in a holomorphic line bundle. Specifically, we shall describe $H^1(Z,\mathcal{O}(-2))$. To do this, write $w = z \wedge \zeta$, considered as a point of the dual projective plane $\mathbb{CP}_2^*$. To say that the line joining z and ζ lies entirely in Z is to say that w lies in the ball

$$W = \{[w_1,w_2,w_3] \in \mathbb{CP}_2^* \text{ s.t. } |w_1|^2 + |w_2|^2 < |w_3|^2\}.$$

Perhaps the simplest Penrose transform is essentially due to Martineau [?]. It is the isomorphism

$$H^1(Z,\mathcal{O}(-2)) \xrightarrow{\cong} \Gamma(W,\mathcal{O}(-1)).$$

A proof is easily constructed along the lines of [?] or [?]. The correspondence between $\mathbb{CP}_2$ and $\mathbb{CP}_2^*$ fits well with the fibration $\eta : F \to Z$:–



For $\phi(w) \in \Gamma(W, \mathcal{O}(-1))$, write $w = z \wedge \zeta$ and, following Gelfand, Graev, and Shapiro [**?**] in the case of real integral geometry, consider

$$\kappa\phi := \sum_\alpha \frac{\partial \phi(z \wedge \zeta)}{\partial z_\alpha} d\zeta_\alpha.$$

It is a holomorphic 1-form in ζ, homogeneous of degree -2 in z. It is readily verified that $d_\eta(\kappa\phi) = 0$ so $\kappa\phi$ represents an element of $H^1(Z, \mathcal{O}(-2))$ in accordance with (**??**). From this point of view, we have constructed the inverse Penrose transform

$$\Gamma(W, \mathcal{O}(-1)) \to H^1(Z, \mathcal{O}(-2)).$$

In [**?**] it is explained, following [**?**], how to obtain Dolbeault representatives from this construction. We shall return to this point at the end of this article.

Our second example is adapted from [**?**]. We shall set up a double fibration (**??**) so that (**??**) describes $H^p(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$. It is as follows:–

$$\begin{tikzcd} & F = \{(x+iy, \xi) \text{ s.t. } \langle \xi, y \rangle > 0\} \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{C}^n \setminus \mathbb{R}^n = Z & & \Xi = S^{n-1} = \{\xi \in \mathbb{R}^n \text{ s.t. } |\xi| = 1\} \end{tikzcd}$$

where $z = x + iy \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$. Again, F is an open subset of a product; this time $F \subset Z \times \Xi$. The complex $\text{Œ}(B^\bullet)$ is easily described:–

$$\text{Œ}(B^p) \ni \omega(z, \xi, d\xi), \text{ a smooth } p\text{-form in } \xi \text{ depending holomorphically on } z$$

with differential $d_\eta : \text{Œ}(B^p) \to \text{Œ}(B^{p+1})$ being exterior derivative in ξ. The conditions under which (**??**) is valid are easily verified. For each $\xi \in \Xi$, for example, we obtain an open subset

$$Z_\xi = \{x + iy \in \mathbb{C}^n \text{ s.t. } \langle \xi, y \rangle > 0\} \subset \mathbb{C}^n \setminus \mathbb{R}^n.$$

As a tube over a half-space, it is Stein. Of course, only finitely many Z_ξ's are needed to give a Čech cover of $\mathbb{C}^n \setminus \mathbb{R}^n$. To use all of them, however, gives a more symmetrical realisation.

According to Sato's theory [**?**], the cohomology $H^{n-1}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$ may be viewed as the space of hyperfunctions on $\mathbb{R}^n$. In this theory, one views hyperfunctions as a sum of boundary values of holomorphic functions defined on tubes over cones with edges along $\mathbb{R}^n$. Formally, we are replacing this finite sum by an average over S^{n-1}. Certainly, we should be able to embed Schwartz space $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^{n-1}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$ in a natural fashion. To see this, write the Fourier inversion formula in polar coördinates:–

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle \xi, x\rangle} d\xi = \int_{\xi \in S^{n-1}} \left(\int_{r=0}^{\infty} \hat{f}(r\xi) e^{ir\langle \xi, x\rangle} r^{n-1} dr \right) d\Omega, \tag{3}$$

where $d\Omega$ is the volume form on the unit sphere. If we replace x by $z = x + iy$ in the integrand of this expression, we obtain

$$\omega(z, \xi, d\xi) = \left(\int_{r=0}^{\infty} \hat{f}(r\xi) e^{-r\langle \xi, y\rangle + ir\langle \xi, x\rangle} r^{n-1} dr \right) d\Omega,$$

which is convergent and holomorphic in z provided $\langle \xi, y\rangle > 0$. This represents our cohomology class in

$$H^{n-1}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O}) \cong \frac{\Gamma(F, \text{Œ}(B^{n-1}))}{\operatorname{im} d_\eta : \Gamma(F, \text{Œ}(B^{n-2})) \to \Gamma(F, \text{Œ}(B^{n-1}))}.$$

Formally, if we set $y = 0$ and average over S^{n-1}, then (**??**) recovers f.

3. Formulation and sketch of proof

The complex structures on the fibres of τ should vary smoothly. One possible formulation is as a smooth differentially closed sub-bundle $\Lambda^{1,0}_F \subset \Lambda^1_F$ containing the annihilator of the vertical vectors and inducing complex structures on each fibre. If we let $\Lambda^{0,1}_\tau$ denote the quotient bundle $\Lambda^1_F / \Lambda^{1,0}_F$, then exterior derivative induces a differential operator $\bar{\partial}_\tau : \Lambda^0_F \to \Lambda^{0,1}_\tau$ whose kernel is the smooth functions on F that are holomorphic along the fibres of τ. Let us write Œ for the sheaf of such functions.

Now, recall that η is supposed to be holomorphic on each fibre of τ. Precisely, this means that $\eta^* \Lambda^{1,0}_Z \subseteq \Lambda^{1,0}_F$. Define a vector bundle B^1 on F by the exact sequence

$$0 \to \eta^* \Lambda^{1,0}_Z \to \Lambda^{1,0}_F \to B^1 \to 0.$$

One may verify that B^1 is naturally holomorphic along the fibres of τ. The same is true for $B^p := \Lambda^p(B^1)$. Let us write $\text{Œ}(B^p)$ for the sheaf of smooth sections of B^p that are holomorphic along the fibres of τ. Then, there is a complex of sheaves on F:–

$$\text{Œ}(B^0) \xrightarrow{d_\eta} \text{Œ}(B^1) \xrightarrow{d_\eta} \text{Œ}(B^2) \to \cdots \to \text{Œ}(B^p) \xrightarrow{d_\eta} \text{Œ}(B^{p+1}) \to \cdots \tag{4}$$

This is the complex occurring in (**??**). For the proof, one shows that (**??**) resolves $\eta^{-1}\mathcal{O}_Z$ and then proceeds as in [**?**]. At some point in this proof one needs a cohomology vanishing result, namely that

$$H^q(F, \text{Œ}(B^p)) = 0, \ \forall q \geq 1.$$

This result is due to Jurchescu [**?**]. He supposes that F is a 'Cartan manifold'. In our context, this means that the partially holomorphic functions separate points and are sufficiently many to provide local coördinates. Jurchescu's vanishing result may be viewed as solving the Levi problem for a family of Stein manifolds. An alternative proof of this vanishing result may be found in [**?**]. We are grateful to Gennadi Henkin for drawing our attention to these articles.

Finally, we remark that Dolbeault representatives may always be obtained from (**??**). To so this, choose an arbitrary smooth section $\gamma : Z \to F$ of η in (**??**). Locally, we may represent $\omega \in \Gamma(F, Œ(B^p))$ by a smooth p-form on F and pull it back to $\gamma^*\omega$ on Z. The $(0,p)$ component $(\gamma^*\omega)^{0,p}$ is well-defined and gives a chain mapping

$$\Gamma(F, Œ(B^\bullet)) \longrightarrow \Gamma(Z, \Lambda^{0,\bullet}),$$

which induces an isomorphism on cohomology. This generalises a construction in [**?**]. In our first example, any choice of γ gives an explicit inverse to the Penrose transform.

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