(20) 1. Evaluate the indicated limits exactly. Give evidence to support your answers.

a) $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$ **Answer** $x^2+x-6=(x+3)(x-2)$. The quotient is $x+3$ which has limit 5 as $x\to 2$.

b) $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$ **Answer** Since $\frac{1}{x}+\frac{1}{2}=\frac{2+x}{2x}$, the quotient $\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\frac{2+x}{2x}\cdot\frac{1}{x+2}=\frac{1}{2x}$, and $\frac{1}{2x}$ has limit $-\frac{1}{4}$ as $x\to -2$.

c) $\lim_{x\to\frac{\pi}{2}^-}\frac{\sin x}{x-\frac{\pi}{2}}$ **Answer** As $x\to\frac{\pi}{2}$, $\sin x\to 1$. But $x-\frac{\pi}{2}\to 0$ and $x-\frac{\pi}{2}<0$. The quotient is therefore $\frac{\approx 1}{\text{small negative }\#}$ which is a large positive number, so the limit is $-\infty$.

d) $\lim_{x\to 0^+}\frac{|x|}{x}$ **Answer** If $x>0$, then $|x|=x$ and $\frac{|x|}{x}=\frac{x}{x}=1$. Therefore the limit is 1.

(14) 2. a) Suppose $f(x)$ is a function. Write the **definition** of $f'(x)$, the derivative of $f(x)$, as a limit.

**Answer** $f'(x)=\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}$.

b) Use your answer to a) and familiar properties of limits to find the derivative of $f(x)=\sqrt{x-3}$.

**Answer** $f(x+\triangle x)-f(x)=\sqrt{x+\triangle x-3}-\sqrt{x-3}$. We can get an algebraically equivalent form by multiplying top and bottom by the conjugate expression (motivated by our desire to cancel $\triangle x$'s later): $\left(\sqrt{x+\triangle x-3}-\sqrt{x-3}\right)\cdot\frac{\sqrt{x+\triangle x-3}+\sqrt{x-3}}{\sqrt{x+\triangle x-3}+\sqrt{x-3}}=\frac{(x+\triangle x-3)-(x-3)}{\sqrt{x+\triangle x-3}+\sqrt{x-3}}=\frac{\triangle x}{\sqrt{x+\triangle x-3}+\sqrt{x-3}}$.
The difference quotient $\frac{f(x+\triangle x)-f(x)}{\triangle x}$ then becomes $\frac{1}{\sqrt{x+\triangle x-3}+\sqrt{x-3}}$ because the $\triangle x$'s cancel, and the limit can be found easily: $f'(x)=\lim_{\triangle x\to 0}\frac{1}{\sqrt{x+\triangle x-3}+\sqrt{x-3}}=\frac{1}{2\sqrt{x-3}}$.

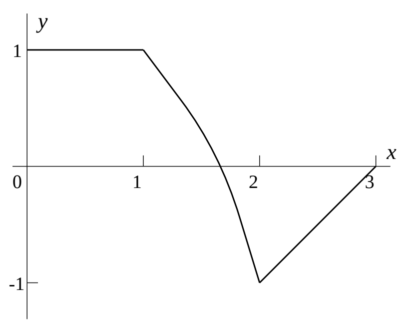
c) Use your answer to b) to write an equation for the line tangent to $y=\sqrt{x-3}$ when $x=7$.

**Answer** The slope of the tangent line when $x=7$ is $f'(7)=\frac{1}{2\sqrt{x-3}}=\frac{1}{2\sqrt{7-3}}=\frac{1}{4}$, and one point on the tangent line is $(7,f(7))$ which is $(7,\sqrt{7-3})=(7,2)$. An equation for the line is therefore $y-2=\frac{1}{4}(x-7)$, which is a fine answer. It can be converted to slope-intercept form $\left(y=\frac{1}{4}x+\frac{1}{4}\right)$ but it doesn't have to be.

(14) 3. Suppose that the function $G$ is described by $G(x)=\begin{cases}1 & \text{if } 0\le x\le 1\\ Ax^2+B & \text{if } 1<x\le 2\\ x-3 & \text{if } 2<x\le 3\end{cases}$.

a) Find $A$ and $B$ so that $G$ is continuous for all numbers in its domain. Briefly explain your answer.

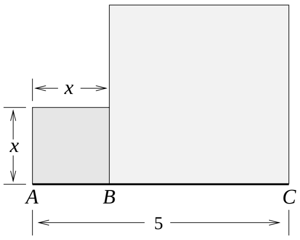
**Answer** The only discontinuities can occur at 1 and 2. The left- and right-hand limits of $G$ at 1 must agree with $G$'s value at 1. $G(1)=1$. $\lim_{x\to 1^-}G(x)=\lim_{x\to 1^-}1=1$. $\lim_{x\to 1^+}G(x)=\lim_{x\to 1^+}Ax^2+B=A+B$. Thus $\boxed{A+B=1}$. The left- and right-hand limits of $G$ at 2 must agree with $G$'s value at 2. $G(2)=2-3=-1$. $\lim_{x\to 2^-}G(x)=\lim_{x\to 2^-}Ax^2+B=A\cdot 2^2+B=4A+B$. $\lim_{x\to 2^+}G(x)=\lim_{x\to 2^+}x-3=2-3=-1$. Thus $\boxed{4A+B=-1}$. We can find $A$ and $B$ by subtracting the first equation from the second to get $3A=-2$ so $A=-\frac{2}{3}$. Substitution in either equation then gives $B=\frac{5}{3}$.




b) Graph $y=G(x)$ on the axes given for the values of $A$ and $B$ found in a).

**OVER**

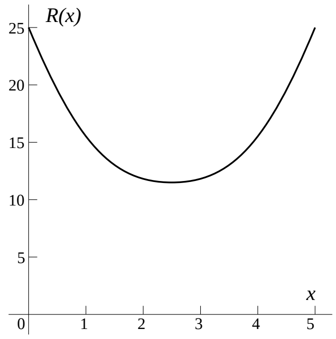
(16) 4. The line segment $\overline{AC}$ is 5 units long and is divided into two parts by the point $B$. Suppose that $x$ is the length of the segment $\overline{AB}$ and that $R(x)$ is the sum of the areas of the two squares with sides $\overline{AB}$ and $\overline{BC}$ as shown.




a) Write an algebraic formula for $R(x)$ as a function of $x$.

**Answer** The length of $\overline{BC}$ is the length of $\overline{AC}$ minus the length of $\overline{AB}$, so the length of $\overline{BC}$ is $5-x$. A square with side $x$ has area $x^2$ and a square with side $5-x$ has area $(5-x)^2$, so the total area, $R(x)$, is $x^2+(5-x)^2$.

b) Graph $R(x)$ on the axes shown. Note that the horizontal and vertical axes have different scales.




c) Use algebra to find all <u>exact</u> values of the length $x$ for which $R(x)=17$.

**Answer** We can expand $x^2+(5-x)^2$ to get $x^2+25-10x+x^2=2x^2-10x+25$. The equation $R(x)=17$ then becomes $2x^2-10x+25=17$ or $2x^2-10x+8=0$. This can be solved with the quadratic formula or by factoring: $2x^2-10x+8=(2x-2)(x-4)$. The roots of $(2x-2)(x-4)=0$ are 1 and 4. Realize again that exams are *very* artificial situations. Only rarely could a "random" quadratic polynomial with integer coefficients be written as a product of linear terms with integer coefficients!

(9) 5. Use all differentiation rules here. Please do *not* simplify the answers in this problem!

a) If $P(x)=6x^5-3x^3+4$, what is $P'(x)$? **Answer** $6\cdot 5x^4-3\cdot 3x^2$.

b) If $Q(x)=\dfrac{x^3+1}{3x-2}$, what is $Q'(x)$? **Answer** $\dfrac{3x^2(3x-2)-3(x^3+1)}{(3x-2)^2}$.

c) If $R(x)=(5+\cos x)^{100}$, what is $R'(x)$? **Answer** $100\,(5+\cos x)^{99}\,(-\sin x)$.

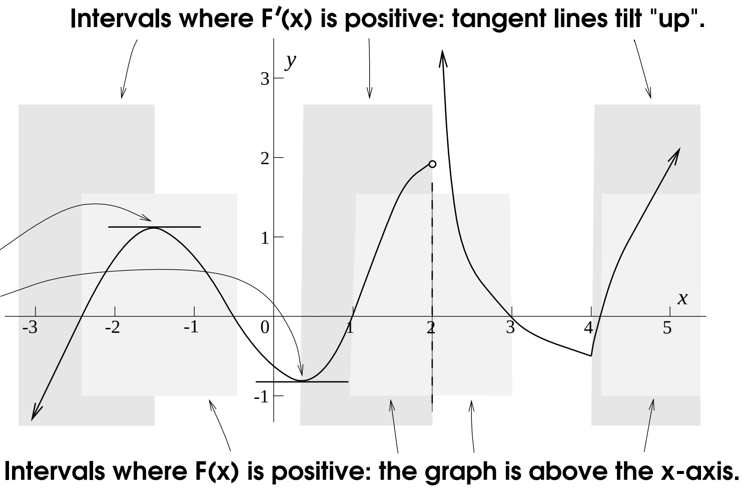
(8) 6. Suppose that $T$ is a differentiable function and the following facts are known about $T$ and its derivatives:

$$T(5)=0 \text{ and } T'(5)=-3 \text{ and } T''(5)=2\,.$$

Suppose that $S(x)=e^{T(x)}$ (this is a composition). Compute $S(5)$ and $S'(5)$ and $S''(5)$. Give the exact answer in each case.

**Answer** $S(5)=e^{T(5)}=e^0=1$. $S'(x)=e^{T(x)}T'(x)$ by the chain rule, so $S'(5)=e^{T(5)}T'(5)=e^0(-3)=-3$. Now use the chain rule *and* the product rule to get $S''(x)=e^{T(x)}T'(x)^2+e^{T(x)}T''(x)$, so $S''(5)=e^{T(5)}T'(5)^2+e^{T(5)}T''(5)=e^0(-3)^2+e^0(2)=11$.

(19) 7. Below is a graph of the function $F$. The domain of $F$ is <u>all numbers not equal to 2</u>.




The graph of $y=F(x)$

Use this graph to answer the questions as accurately as possible.

a) For which $x$ is $F(x)=0$? **Answer** $-2.4$ and $-.5$ and 1 and 3 and 4.1: this is where the graph crosses the $x$-axis.

b) For which $x$ is $F(x)>0$? **Answer** $-2.4<x<-.5$ and $1<x<2$ and $2<x<3$ and $4.1<x$: this is where the graph is above the $x$-axis, where $y$ is positive.

c) What is $\lim_{x\to 2^-} F(x)$? **Answer** 2.

d) What is $\lim_{x\to 2^+} F(x)$? **Answer** $+\infty$.

e) For which $x$ (in its domain) is $F$ <u>not</u> differentiable? **Answer** 4: where there's a corner.

f) For which $x$ is $F'(x)=0$? **Answer** $-1.6$ and $.3$: where the tangent lines are horizontal.

g) What is $F(x)$ for those $x$'s for which $F'(x)=0$? **Answer** $F(-1.6)=1.1$ and $F(.3)=-.8$

h) For which x is $F'(x)>0$? **Answer** $x<-1.6$ and $.3<x<2$ and $4<x$: where tangent lines have positive slope.